HEAT TRANSFER

V.P. Isachenko V.A. Osipova
A.S. Sukomel

MIR PUBLISHERS MOSCOW

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LIST OF SYMBOLS

A	absorptivity
a	sonic velocity; thermal diffusivity
B	intensity or brightness of thermal radiation
b	constant determined experimentally
C	integration constant
c	radiation factor of grey bodies; specific heat
c_f	friction coefficient
c_p	specific heat of liquid at constant pressure
c_v	specific heat of gas at constant volume
c_1; c_2	first and second Planck's constants
D	diffusion coefficient; transmissivity
D_{pi}	diffusion coefficient reduced to partial pressure gradient
D_{th}	thermal diffusion coefficient
d_0	separation diameter of vapour bubble
d_{cr}	critical diameter
$d_{cr\ ins}$	critical diameter of insulation
E	emissive power; fin efficiency factor
E_{out}	outgoing radiation
$e =$	2.718 ..., base of natural logarithms
F	surface; heat-transfer surface
F_1	centre of the sphere
F_l	surface per time τ
f	surface area; momentum exchange factor
f_0	averaging surface
f_h	cross-sectional area of the hydraulic channel
f_r	reflecting surface
G	mass rate of flow
g	acceleration of gravity
h	height
I	electric current
i	specific enthalpy
i_s	saturation enthalpy
J	mass flow
J_{con}	convective mass flow
J_λ	spectral intensity of radiation
j	rate of mass flow
K	proportionality factor
K	turbulence of the free stream flow
k	measure of turbulence of free-stream flow; overall heat-transfer coefficient; thickness of thermal boundary layer
k_l	linear overall heat-transfer coefficient
k_t	thermal diffusion ratio

L	$\frac{l}{d}$, dimensionless parameter
L	optical beam length
l	free gas molecular path
l_0	reference linear dimension
l_{fs}	dimension of full-size equipment
l_{sm}	dimension of model
M	molecular weight
m	mass transfer of a component in mixture; number of molecules per unit volume; rate of cooling
N	(newton) unit of force
P_t	thermocapillar motive force
p	pressure; porosity
p_l	local partial pressure
Q	amount of heat; heat flux
Q_λ	flux of monochromatic spectral radiation
q	rate of heat flow
q_d	density of rate of heat flow
q_l	rate of heat flow per unit length
$q_{max\ cr}$	critical heat flux
q_{min}	minimum heat flux
q_n	net radiant flux
q_{net}	net radiation
q_v	inner heat source; volume rate of heat liberation
q_w	rate of heat flow of the wall
q_x	projection of the rate of heat flow
R	reflectivity; thermal resistance
R_c	thermal resistance of the condensate film
$R_l = 1/k_l$	linear thermal resistance
R_φ	thermal resistance to phase transition
R°	thermal resistance of dense gas
R_v, R_g	vapour and gas constants respectively
r	heat of evaporation; radius; recovery factor
S	cooling area
s	entropy; force of friction reduced to unit area
s_1, s_2	transverse and longitudinal pitches of banks of tubes respectively
s_1/d, s_2/d	relative transverse and longitudinal pitches of banks of tubes respectively
T	absolute temperature
T_{aw}	adiabatic temperature of the wall
T_r	recovery temperature
T_{st}	stagnation temperature
t	temperature, K
$\overline{t}$	mean temperature
t_{ar}	arithmetic mean
t_f	ambient temperature
t_{log}	logarithmic mean
t_m	wet-bulb temperature
t_s, t_w	surface temperature
u	electrical potential; perimeter of straight rectangular rod
V	volume
w	flow alternating factor; fluid flow velocity

$\bar{w}$ mean travel velocity of gas molecules; mean velocity of flow
w_j velocity jump
w_m maximum probable molecular velocity
w_r rising velocity of vapour and vapour bubble
w_* reduced rate of evaporation
X dimensionless coordinate
x local coordinate; mass rate of vapour flow in two-phase system

GREEK LETTERS

α heat-transfer coefficient
α_N heat-transfer coefficient calculated by Nusselt formula
$\alpha_{x=l}$ local heat-transfer coefficient located by coordinate $x = l$
β mass-transfer coefficient; volume rate of vapour flow in two-phase system; volume expansion coefficient
β_p mass-transfer coefficient reduced to partial pressure difference
γ accommodation factor
Δ prefix indicating final increment
δ thickness of hydrodynamic boundary layer; wall thickness
$\bar{\delta}$ mean layer thickness
δt variation in fluid temperature in the direction of flow
δ/λ thermal resistance
ε emissivity of grey bodies
ε_l correction factor accounting for variation in heat transfer in the first rows of a bank of tubes
ε_0 turbulent diffusivity of mass
ε_q turbulent diffusivity of heat
ε_s turbulent diffusivity of momentum
ε_s correction factor for banks of tubes accounting for the effect of relative pitches
ε_t isothermal compressibility
ε_v correction accounting for wave motion of condensate flow
ε_ψ correction accounting for the effect of convective heat transfer and inertia forces
ζ local friction factor
η efficiency; packing utilization factor
$\varkappa$ instrument constant
Θ contact angle; dimensionless temperature
ϑ excess temperature; local temperature difference
ϑ_v volume mean excess temperature
λ thermal conductivity; wavelength
λ_{eq} equivalent thermal conductivity
λ_{ins} thermal conductivity of insulating material
λ_m integral mean thermal conductivity
λ_0 thermal conductivity at 0 °C
λ/δ thermal conductance
μ molecular weight of gas; viscosity
ν kinematic viscosity
ρ gas density
ρ_l local concentration of a given component of a two-component system

ρ_0 specific electrical resistance
σ Boltzmann's constant; surface tension coefficient
τ time
φ geometrical view factor
χ coefficient of non-uniform temperature distribution; number of molecular collisions per s
ψ coefficient of non-uniform temperature distribution
ω flow alternating factor

DIMENSIONLESS TERMS

Ar Archimedes number $= \frac{gl_0^3}{\nu^2}\frac{\rho_0-\rho}{\rho_0}$

Bi Biot number $= \frac{\alpha l}{\lambda}$

Bo Boltzmann number $= \frac{\rho c_p w}{\sigma_0 T_3}$

Eu Euler number $= \frac{\Delta p}{\rho\omega^2}$

Fo Fourier number $= \frac{a\tau}{l^2}$

Ga Galilei number $= \frac{gl_0^3}{\nu^2}$

Gr Grashof number $= \frac{g\beta\vartheta_\omega l_0^3}{\nu^2}$

Ja Jacob number $\frac{c_p\Delta t}{r}\frac{\rho t}{\rho v}$

Ki Kirpichev number $= \frac{\sigma_0 T_3}{\lambda k}$

Kn Knudsen number $= \frac{\bar{l}}{l_0}$

Ko Kondratyev number $= \mathrm{Nu}_{fd}\mathrm{Pr}_f^{-0.43}(\mathrm{Pr}_f/\mathrm{Pr}_w)^{-0.25}$

K Kutateladze number $= \frac{r}{C_p\Delta t}$

Le Lewis number $= \frac{\alpha}{D}$

M Mach number $= \frac{w}{a}$

Nu Nusselt number $= \frac{\alpha l_0}{\lambda}$

Pe Peclet number $= \mathrm{Re}\cdot\mathrm{Pr} = \frac{w_0 l_0}{a}$

Pr Prandtl number $= \frac{\nu}{a}$

Re Reynolds number $= \frac{w_0 l_0}{\nu}$

Sh Strouhal number $= \frac{fd}{w_0} = 0.2$

St Stanton number $= \frac{\alpha}{\rho c_p w_0} = \frac{\mathrm{Nu}}{\mathrm{Re}\cdot\mathrm{Pr}}$

X, Y, Z reference dimensionless terms

SUBSCRIPTS AND INDICES

ab	absorbed
ad	adiabatic
b	boundary, interphase
c	circulation, centre
cd	concentration of ordinary diffusion
cr	critical value
D	pertaining to process of mass transfer
d	diameter
eff	effective
eq	equivalent
f	fluid (liquid or gas), film
fb	film boiling
fs	full-size (in theory of models)
h	heating
in	incident
l	linear; liquid
ls	laminar sublayer
m	mean
mix	mixture
n	net
0	initial or inlet conditions
p	parallel
r	radiation; radius; reflected; rough surface
s, *sc*	saturation; screen (in thermal radiation); standard
sm	scale model (in theory of models)
su	condensate surface
t	turbulent
th	thermal
v	vapour
vg	vapour-gas mixture
w	wall
x	local value
z	counterflow
′	inlet conditions
″	outlet conditions

PREFACE

This book, now in English translation from the 3rd revised and supplemented Russian edition, is written by members of the staff of the Moscow Power Institute, and sets out the fundamentals of the theory of heat transfer, the science that is gaining ever greater importance in many engineering fields.

The book contains an analysis of the main modes of heat transfer: conduction, natural and forced convection, radiation heat transfer, processes of heat transfer in boiling and condensation. A separate part is devoted to the design of various types of heat exchangers. Outlining the physical phenomena of the processes, the authors give practical solutions of typical problems of heat transfer. The analytical treatment of the subject is well confirmed by experimental results obtained by the authors and other investigators. References to the specialist literature are given at the end of each chapter of the book.

The book is written as a text-book for students of heat-engineering departments. It is also of interest as a reference work to engineers and technicians engaged in design and operation of all kinds of heat-transfer equipment.

Chapters 1 to 3, 19 and 20 are written by A.S. Sukomel, Chapters 4 to 12 and 14 by V.P. Isachenko, Chapters 13, 16 to 18 and Sec. 3-12 by V.A. Osipova

INTRODUCTION

Heat transfer is the science of spontaneous irreversible processes of heat propagation in space.

By a process of heat propagation is meant the exchange in internal energy between individual elements, regions of the medium considered.

Heat may be transmitted in three ways, by conduction, convection or thermal radiation.

Heat conduction is identified as the process of molecular transport of heat in bodies (or between them), due to temperature variation in the medium considered.

Convection is possible only in a fluid medium. The term is applied to the transport of heat that occurs as volumes of liquid or gas fluid medium move from regions of one temperature to those of another temperature. The transport of heat is inseparably linked here with the movement of the medium itself.

Thermal radiation is the process of heat propagation by means of electromagnetic waves, depending only on the temperature and on the optical properties of an emitter, with its internal energy being converted into radiation energy. The process involving the conversion of the internal energy of a substance into radiation energy is referred to as (thermal) radiation heat transfer. The various basic processes of heat transfer—conduction, convection and radiation—are often combined both in nature and in engineering applications.

Pure conduction is found only in solids.

Heat transfer by convection is always accompanied by conduction. The combined process of heat transfer by convection and conduction is referred to as *convective heat transfer.*

The subject of engineering analysis is usually the transfer of heat by convection between a stream of liquid or gas and a solid surface; this process of convection heat exchange is referred to as *convection heat transfer or heat transfer.*

Processes of conduction and convection can be accompanied by thermal radiation heat transfer. Heat transfer between media characterized by combined transport of heat by radiation and conduction is called radiation-conduction heat transfer. If heat is transported also by convection, a process of this kind is called radiation-convective heat transfer. Radiation-conduction and radiation-convection modes of heat transport are sometimes referred to as complex heat transfer.

In engineering and daily life processes are often encountered, involving heat transfer between various fluids separated by a solid wall. The process of heat transfer from a hot fluid to a cold one through a partition separating the two fluids is also simply referred to as *heat transfer*, and it may involve heat transfer by conduction, radiation and convection. The steam generating tubes of a boiler, for instance, receive heat from the products of fuel combustion by all three modes of heat transfer: conduction, convection, and radiation. Heat is transferred by conduction through the layer of ash covering the tube, the metal wall

of the tube itself and the layer of scale covering its inner surface; from the inner surface heat is transferred to the liquid filling the tube both by convection and conduction.

Processes of heat transfer may occur in various media, in pure substances and in mixtures, with and without changes of phase of the working medium, etc. and accordingly will differ in character and be described by different equations.

Many of the processes of heat transfer are accompanied by transfer of mass. As water evaporates, for instance, heat transfer is accompanied by transport of the vapour formed through an air-vapour mixture. The transport of steam generally occurs both through molecular interaction and convection. The combined molecular and convective transport of mass is called *convection mass transfer*. With mass transfer the process of heat transfer becomes more complicated. In addition, heat may be transported together with the mass of diffusing substances.

In the general case, in a mixture of various substances, transport of heat may be provoked by non-uniform distribution of other physical quantities, apart from the temperature. A difference in the concentration of mixture components, for instance, brings about an additional molecular transport of heat (the diffusion thermoeffect). The amount of heat transported due to such effects is usually comparatively small and it can be ignored.

Theoretical studies of heat transfer necessitate the introduction of some models, giving an idea of the medium in which the investigated processes take place. The gases, liquids and solids being investigated in the book are usually considered as a continuum, i.e. as a medium whose discrete structure can be ignored.

Homogeneous and heterogeneous continuous media are distinguished. In the first, the physical properties at different points are the same at equal temperatures and pressures, and in heterogeneous media the physical properties differ. Also distinguished are isotropic and anisotropic continuous media. The physical properties of an isotropic medium at any point of the medium are independent of the direction selected and, on the contrary, at a given point some properties of an anisotropic medium may happen to be a function of direction. Heat transfer in isotropic media is often encountered in practical applications and the process has been more exhaustively studied.

There may be single-phase and multiphase continuous media. The properties of a single-phase medium, consisting of a pure substance or of a mixture of substances, change in space continuously. The properties of a multiphase medium, consisting of a number of single-phase parts, change at the interfaces in jumps. In single-phase and multiphase media heat transfer occurs in different ways.

The object of this course is to study the laws governing the development of both the simple and more complicated processes of heat transfer in various media.

I CONDUCTION

CHAPTER
1

BASIC CONCEPTS OF CONDUCTION

1-1. Investigation Methods for Physical Phenomena

On the basis of the notions of contemporary physics natural phenomena in general and that of conduction, in particular, can be described and investigated as *phenomenological* and *statistical methods.*

The phenomenological method of describing a process ignores the microscopic structure of a substance and considers it as a continuous medium (continuum).

The phenomenological investigation method makes it possible to relate in a general way the parameters, or properties, characterizing the investigated phenomenon as a whole. Phenomenological laws are of a very general nature, and the role of a particular physical medium is accounted for by coefficients determined directly by experiment.

Another way of studying physical phenomena is based on investigating the internal structure of a substance. A medium is looked upon as some physical system consisting of a great number of molecules, ions or electrons with given properties and laws of interaction. The main problem of the method referred to as statistical is to get macroscopic characteristics based on the given microscopic properties of a medium.

Both the first and the second method has its merits and limitations.

The phenomenological method permits at once the investigator to relate in a general way the parameters characterizing a process and to make use of experimental data whose accuracy predetermines the accuracy of the method itself, and this is the advantage of using the phenomenological method in investigating a phenomenon.

The mere fact, however, of conducting experimental work so as to identify the characteristics of a physical medium is at the same time a shortcoming of the method since it restricts the field of application for phenomenological laws. In addition, up-to-date experimentation is rather involved and requires great expenditures.

The statistical method permits phenomenological relations to be obtained on the basis of the given properties of the microscopic structure of a medium investigated without conducting any experimental work, and this is its advantage. The demerit of the statistical method is its intricacy, whereby, final calculation relationships can be obtained only for the simplest physical models

of a substance. In addition, application of the method requires the knowledge of a number of parameters which can be determined with the aid of special investigations.

Processes of conduction are investigated applying the phenomenological method. The analytical theory of conduction ignores the molecular structure of a substance and regards substances as continuous media. Such an approach is suitable, if the dimensions of the objects investigated are sufficiently large compared with the distances of effective intermolecular interaction.

1-2. Temperature Field

The phenomenon of heat conduction is a process of propagation of heat energy by direct contact between the separate particles of a body, or between bodies of different temperatures and results from the movement of the microparticles of the substance.

In gases heat transfer by conduction is effected by means of molecular and atomic diffusion, and in liquids and solid dielectrics, by means of elastic waves. In metals heat transfer is mainly due to the diffusion of free electrons, and elastic oscillations of the crystal lattice are of minor importance.

It should be mentioned that pure heat conduction takes place in liquids and gases only when the possibility of heat transfer by convection is excluded.

Any physical phenomenon is generally accompanied by a change in space and time of its physical properties. The process of heat conduction, like other modes of heat transfer, can only take place when different points in a body (or a system of bodies) are at different temperatures. In solids conduction is generally accompanied by variations of temperature, in both space and time.

Analytical investigation of heat conduction amounts to study of the space-time variations of temperature, i.e. to determining the equation

$$t = f(x, y, z, \tau) \tag{1-1}$$

Equation (1-1) is the mathematical expression for the *temperature field*, which is thus the whole set of temperatures at all points of the space studied at any given moment of time.

Temperature fields are of two kinds: *steady-state*, or stationary, and *transient*, or non-stationary. Equation (1-1) describes the commonest type of temperature field, in which the temperature varies in time and from point to point. This type is characteristic of transient conduction and it is referred to as a transient temperature field.

If a heat flow in which the temperature at each point is constant in time is established, its temperature field is referred to as stationary. Temperature here is a function only of the space coordinates:

$$t = f_1(x, y, z); \quad \frac{\partial t}{\partial \tau} = 0 \tag{1-2}$$

The temperature field agreeing with Eqs. (1-1) and (1-2) is a space, or three-dimensional field since temperature is a function of three coordinates. If temperature is a function of two coordinates, the field is referred to as two-dimensional, and is described by the equation

$$t = f_2(x, y, \tau); \quad \frac{\partial t}{\partial z} = 0 \tag{1-3}$$

If temperature is a function of one coordinate only, the field is called one-dimensional:

$$t = f_3(x, \tau); \quad \frac{\partial t}{\partial y} = \frac{\partial t}{\partial z} = 0 \tag{1-4}$$

The equation of a one-dimensional, steady-state temperature field is of the simplest form:

$$t = f_4(x); \quad \frac{\partial t}{\partial \tau} = 0, \quad \text{and} \quad \frac{\partial t}{\partial y} = \frac{\partial t}{\partial z} = 0 \tag{1-5}$$

1-3. Temperature Gradient

If all points of a body having the same temperature are joined, we obtain a surface of equal temperature, known as *isothermic*. Thus the isothermic surface of a body is the locus of points of identical temperature in a temperature field.

Since no one point of a body can be simultaneously at different temperatures, isothermal surfaces do not intersect. They either end at the surface of the body or are located completely within it.

Intersection of isothermal surfaces by a plane gives a family of isotherms on this plane, possessing the same properties as the isothermal surfaces themselves, i.e. they do not intersect, are continuous within the body, end at the surface, or are wholly within the body.

Figure 1-1 represents isotherms differing in temperature by Δt.

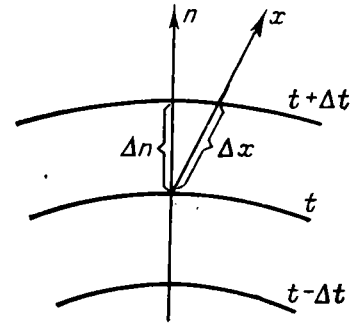

Fig. 1-1. Isotherms

The temperature in a body varies only in directions crossing isothermal surfaces; the greatest temperature difference per unit length is observed in the direction normal to the isothermal surface.

An increase in temperature in this direction is characterized by the *temperature gradient.*

The temperature gradient is a vector normal to the isothermal surface and is positive in the direction of increasing temperature. It is numerically equal to the temperature derivative in this direction, i.e.

$$\text{grad}\, t = \vec{n}_0 \frac{\partial t}{\partial n} \tag{1-6}$$

where $\vec{n}_0$ = unit vector normal to the isothermal surface and positive in the direction of increasing temperature;

$\frac{\partial t}{\partial n}$ = temperature derivative along the normal (n).

The scalar of the temperature gradient $\partial t/\partial n$ is not the same for different points of the isothermal surface, and is the larger, the smaller the distance Δn between the surfaces. The scalar quantity of the temperature gradient $\partial t/\partial n$ will also be referred to here as the magnitude of the temperature gradient.

The magnitude of $\partial t/\partial n$ is negative in the direction of the decreasing temperature.

Projections of the vector grad t on the coordinate axes Ox, Oy, Oz are

$$\left.\begin{aligned}(\text{grad } t)_x &= \frac{\partial t}{\partial n}\cos(\widehat{n, x}) = \frac{\partial t}{\partial x}\\ (\text{grad } t)_y &= \frac{\partial t}{\partial n}\cos(\widehat{n, y}) = \frac{\partial t}{\partial y}\\ (\text{grad } t)_z &= \frac{\partial t}{\partial n}\cos(\widehat{n, z}) = \frac{\partial t}{\partial z}\end{aligned}\right\} \quad (1\text{-}7)$$

1-4. Heat Flow. Fourier's Law

Propagation of heat requires that temperature distribution in a considered medium be non-uniform. Thus, heat transfer by conduction requires that the temperature gradient at various points of the body be greater than zero.

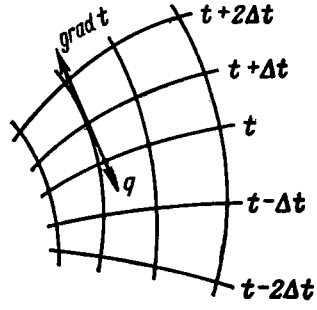

Fig. 1-2. Isotherms and lines of heat flow

In accordance with *Fourier's law* [1.1], the quantity of heat dQ_τ, J, passing through an element of an isothermal surface dF per time interval $d\tau$ is proportional to the temperature gradient $\partial t/\partial n$:

$$dQ_\tau = -\lambda \frac{\partial t}{\partial n} dF\, d\tau \quad (1\text{-}8)$$

The proportionality factor in Eq. (1-8) has been proved experimentally to be a physical property of a substance. It defines the ability of the substance to conduct heat and is called *thermal conductivity*.

The quantity of heat transferred through unit area of isothermal surface per unit time, $q = \frac{dQ_\tau}{dF\, d\tau}$, W/m², is called the *rate of heat flow*, which is a vector determined by the relation

$$\vec{q} = -\vec{n}_0 \lambda \frac{\partial t}{\partial n} \quad (1\text{-}9)$$

The vector (rate of heat flow) $\vec{q}$ is normal to the isothermal surface, and is positive in the direction of decreasing temperature, since heat always flows from the hotter sections of a body to the colder ones. Consequently, the vectors $\vec{q}$ and grad t run along the same straight line but in opposite directions, which explains the "minus" sign in the right-hand side of Eqs. (1-9) and (1-8).

The lines whose tangents run in the direction of vector $\vec{q}$ are known as *heat-flow lines*, and are orthogonal to the isothermal surfaces (Fig. 1-2).

The scalar representing the rate of heat flow q, W/m², is equal to:

$$q = -\lambda \frac{\partial t}{\partial n} \quad (1\text{-}10)$$

The validity of Fourier's law has been demonstrated by numerous experiments. Therefore, Eq. (1-8) as well as Eq. (1-9) are taken as the mathematical expression of the basic law of conduction, which is formulated as follows: *rate of heat flow is proportional to the temperature gradient.*

The quantity of heat passing through an isothermal surface F per unit time is called the *heat flux*. If the gradient of temperature is different for various

points of isothermal surface the amount of heat flowing per unit time through an isothermal surface whose points are at different temperature gradients is:

$$Q = \int_F q\,dF = -\int_F \lambda \frac{\partial t}{\partial n}\,dF \tag{1-11}$$

where dF is an element of the isothermal surface. The heat flux Q is measured in watts.

The total quantity of heat Q, J, passing through an isothermal surface F per time τ is

$$Q_\tau = -\int_0^\tau \int_F \lambda \frac{\partial t}{\partial n}\,dF\,d\tau \tag{1-12}$$

The quantity of heat flowing through an elementary surface area dF_l set at an angle φ to the plane tangential to the isothermal surface (Fig. 1-3) can be determined by the same formula, (1-12), taking into account that

$$q_l = q\cos\varphi = \frac{dQ_\tau}{d\tau}\frac{1}{dF}\cos\varphi = \frac{dQ_\tau}{d\tau}\frac{1}{dF_l} \tag{1-13}$$

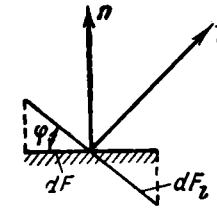

Fig. 1-3. Calculating the hea flux

Since $dF = dF_l \cos\varphi$ is a projection of the area dF_l on the isothermal surface, the heat flux flowing through an elementary surface area dF_l per time $d\tau$ is expressed by the formula

$$dQ_\tau = q_l\,dF_l\,d\tau = q\,(dF_l\cos\varphi)\,d\tau = q\,dF\,d\tau \tag{1-14}$$

The total quantity of heat flowing through the surface F_l per time τ is

$$Q_\tau = -\int_0^\tau \int_{F_l} \lambda \frac{\partial t}{\partial l}\,dF_l\,d\tau \tag{1-15}$$

From Eq. (1-13) it follows that the greatest rate of heat flow will be the rate along the lines normal to the isothermal surfaces. If this flow is projected onto the coordinate axes Ox, Oy, Oz, then, in accordance with Eq. (1-7), we have:

$$\left.\begin{aligned} q_x &= -\lambda\frac{\partial t}{\partial x} \\ q_y &= -\lambda\frac{\partial t}{\partial y} \\ q_z &= -\lambda\frac{\partial t}{\partial z} \end{aligned}\right\} \tag{1-16}$$

The heat flows expressed by Eq. (1-16) are components of the vector representing the rate of heat flow

$$\vec{q} = iq_x + jq_y + kq_z \tag{1-17}$$

It follows that in order to determine the quantity of heat passing through any surface of a solid body, it is necessary to know the temperature field established within it. The principal problem of analytical theory of heat conduction, therefore, is to determine this temperature field.

1-5. Thermal Conductivity

As already said, thermal conductivity is a physical parameter of a substance. In the general case thermal conductivity depends on temperature, pressure and nature of the substance, mostly the thermal conductivity of various materials is usually determined experimentally and a number of methods for

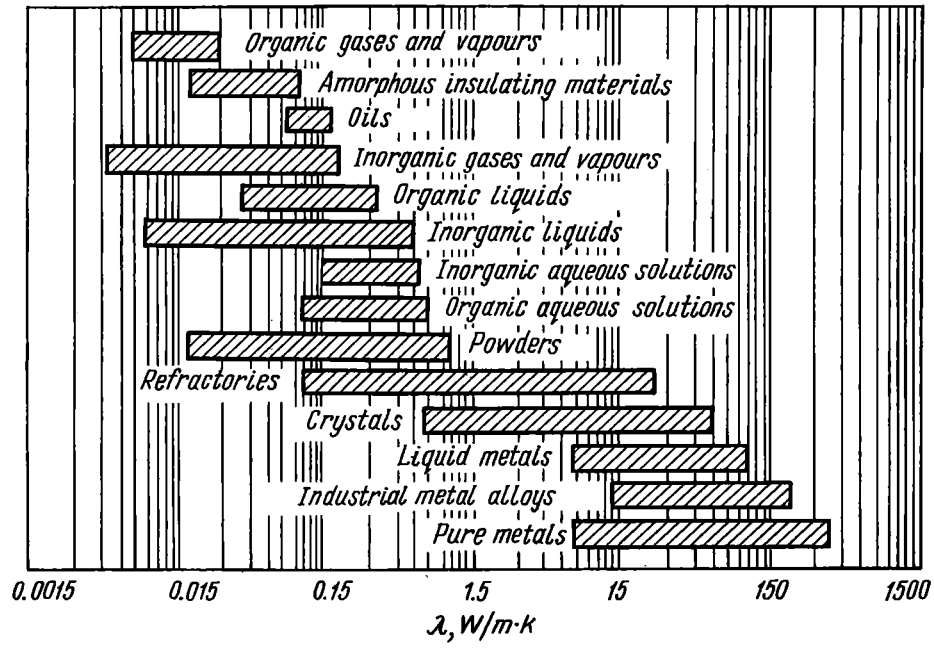

Fig. 1-4. The order of thermal conductivity coefficients for various substances

the purpose are known [1.2-1.6], most of them based on measuring rate of heat flow and temperature gradient in the given substance. Thermal conductivity λ, W/m·K, is then determined from the relation

$$\lambda = \frac{|\vec{q}|}{|\text{grad } t|} \tag{1-18}$$

From Eq. (1-18) it follows that thermal conductivity is numerically equal to the quantity of heat passing through unit area of the isothermal surface per unit time at a temperature gradient of unity.

The order of magnitudes of λ for various materials is illustrated in Fig. 1-4 [1.7, 1.8].

Tables of the results of measuring thermal conductivity λ have been published [1.9, 1.10], and are used to calculate the processes of heat conduction.

Since bodies may have various temperatures, and the temperature distribution within a body is non-uniform during heat transfer, it is of prime importance to know the dependence of thermal conductivity on temperature. As has been shown experimentally with an accuracy sufficient for practical applications, the dependence of thermal conductivity on temperature can be assumed to be linear for many materials:

$$\lambda = \lambda_0 [1 + b (t - t_0)] \tag{1-19}$$

where λ_0 = thermal conductivity at a temperature t_0
b = constant determined experimentally.

(*a*) *Thermal Conductivity of Gases*

According to the kinetic theory of matter, the heat transfer by conduction in gases at ordinary pressures and temperatures occurs through the transport of the kinetic energy of molecular motion arising from the random movement and collision of the gas molecules. The thermal conductivity of gases is determined by the relation:

$$\lambda = \frac{1}{3}\overline{w}\overline{l}c_v\rho/3 \qquad (1\text{-}20)$$

where $\overline{w}$ = mean travel velocity of gas molecules;
$\overline{l}$ = mean free path of gas molecules during the interval between collisions;
c_v = specific heat of gas at constant volume;
ρ = gas density.

With rising pressure, gas density ρ increases to the same extent while $\overline{l}$ diminishes in inverse proportion, and the product $\overline{l}\rho$ remains constant. Thermal conductivity, therefore, does not depend on pressure with the exception of very low ($<2.66 \cdot 10^3$ Pa) and very high ($>2 \cdot 10^9$ Pa) pressures.

The mean travel velocity of gas molecules depends on temperature:

$$\overline{w} = \sqrt{\frac{3R_\mu T}{\mu}}$$

where R_μ = 8314,2 J/kmole·K is the universal gas constant;
μ = molecular weight of gas
T = absolute temperature, K.

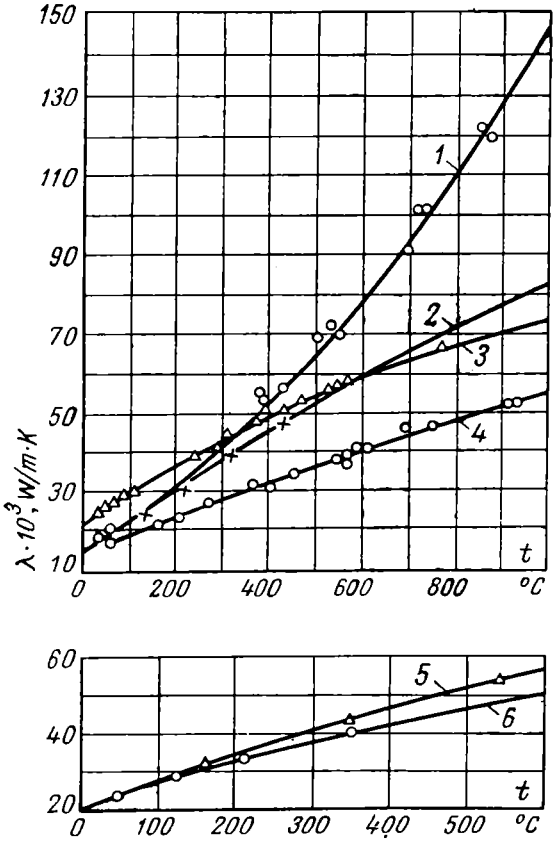

Fig. 1-5. Thermal conductivity coefficients of gases:
1—steam; *2*—carbon dioxide; *3*—air; *4*—argon; *5*—oxygen; *6*—nitrogen

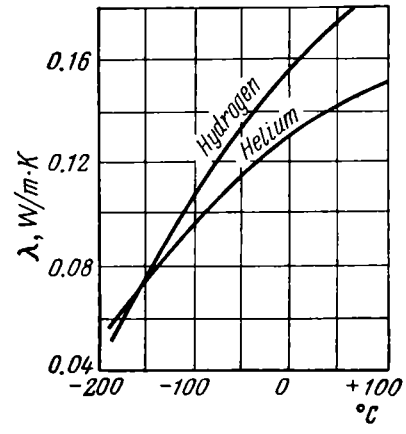

Fig. 1-6. Thermal conductivity coefficients of helium and hydrogen

The specific heat of gases increases with temperature, which explains why their thermal conductivity also increases with temperature.

The thermal conductivity of gases ranges from $\lambda = 0.006$ to $\lambda = 0.6$ W/m·K.

The results of measuring the thermal conductivity of various gases, carried out by N.B. Vargaftik, are shown in Fig. 1-5.

Helium and hydrogen differ markedly from other gases in possessing a thermal conductivity five to ten times greater than that of other gases [1.11], as may be clearly seen from Fig. 1-6. The molecules of helium and hydrogen have small mass and as a result high mean travel velocity, which in turn explains their high thermal conductivity.

The thermal conductivity of steam and other imperfect gases which differ essentially from perfect gases, also depends, very much, on pressure. The thermal conductivity of a mixture of gases cannot be determined analytically by the additivity law, and must be found experimentally.

(*b*) *Thermal Conductivity of Liquids*

The mechanism of heat propagation in dropwise liquids can be conceived as transport of energy by way of unstable elastic oscillations. This theoretical concept of the mechanism of heat transfer in liquids, put forward by the Soviet scientist A. S. Predvoditelev [1.12], has been used by N. B. Vargaftik [1.9] to describe the experimental data on thermal conductivity of various liquids. The hypothesis has been well substantiated for most liquids, and based on it is the following formula for their thermal conductivity:

$$\lambda = A \frac{c_p \rho^{4/3}}{\mu^{1/3}} \tag{1-21}$$

where c_p = specific heat of the liquid at constant pressure;
ρ = volumetric density of the liquid;
μ = molecular weight.

The factor A, which is proportional to the velocity of elastic wave propagation in a liquid, depends not on the nature of the liquid, but on temperature; Ac_p is nearly constant.

Since the density ρ of liquids diminishes with rising temperature, it follows from Eq. (1-21) that the thermal conductivity of liquids of a constant molecular weight (non-associated or slightly associated liquids) must drop with rising temperature. With heavily associated liquids (water, alcohols, etc.) an association factor, accounting for the variation in molecular weight, must be introduced in Eq. (1-21). This factor also depends on temperature its effects on thermal conductivity, therefore, may vary at different temperatures.

The thermal conductivity λ of most liquids has been shown experimentally to diminish with rising temperature, water and glycerin being important exceptions (Fig. 1-7). The thermal conductivity of liquids ranges approximately from 0.07 to 0.7 W/m·K. With rising pressure the thermal conductivity of liquids increases.

(*c*) *Thermal Conductivity of Solids*

Metals and alloys. The principal agents of heat transfer in metals are the free electrons, which may be likened to a perfect monatomic gas. Heat transfer through oscillation of atoms or by means of elastic sound waves is also possible, but this mode of heat transfer is insignificant in comparison with the transport of energy by an electron gas. The motion of free electrons brings about equalization of temperature at all points of the heated or cooled metal, as they move both from regions of higher temperature to those of lower temperature and vice versa. In the first case they give up energy to the atoms, and in the second they remove energy from the atoms. Since electrons are carriers both of heat and of

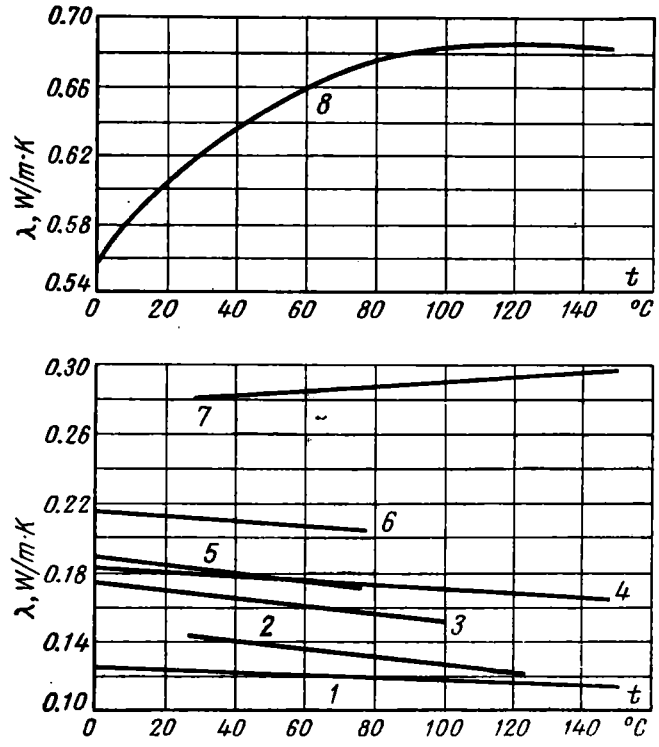

Fig. 1-7. $\lambda(t)$ of various liquids:
1—vaseline oil; *2*—benzene; *3*—acetone; *4*—castor oil; *5*—ethyl alcohol; *6*—methyl alcohol; *7*—glycerine; *8*—water

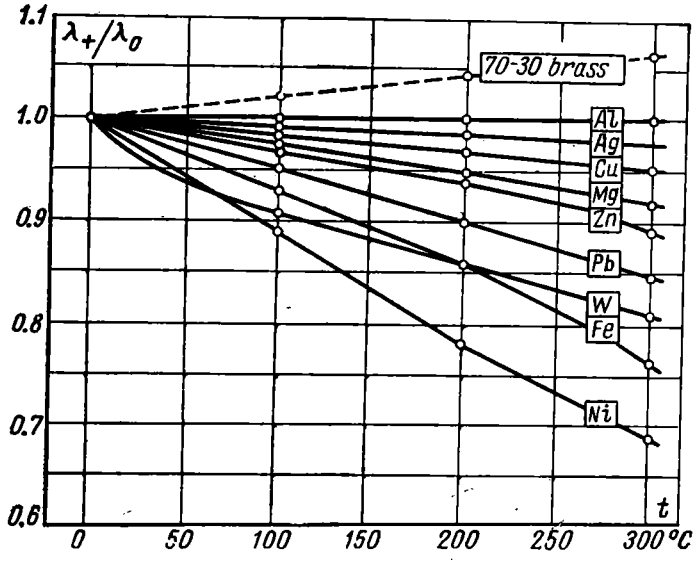

Fig. 1-8. Dependence of thermal conductivity coefficient on temperature for some pure metals

electricity in metals, the thermal conductivity of a metal is proportional to its electric conductivity. The scattering of electrons intensifies with rising temperature, owing to the increase of thermal non-uniformity. In pure metals this leads to a decline in thermal and electric conductivity (Fig. 1-8).

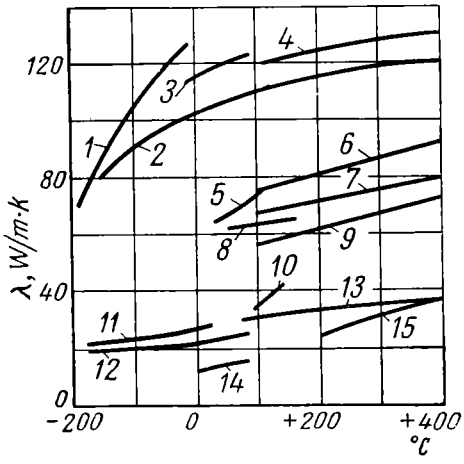

Fig. 1-9. $\lambda(t)$ of various alloys:
1—brass 18; *2*—brass 30; *3*—brass 12; *4*—Nichrome; *5*—bronze; *6*—manganese bronze; *7*—gun bronze; *8*—Sn-Zn alloy; *9*—phosphorus bronze; *10*—white metal; *11*—constantan; *12*—Monel metal; *13*—manganin; *14*—nickel steel; *15*—liquid Sn-Zn alloys

The thermal conductivity of metals drops sharply in the presence of all kinds of admixtures. This phenomenon can be explained by an increase in structural heterogeneity, which brings about electron scattering. Thus, for instance, the thermal conductivity λ of pure copper is 396 W/m·K, but for the same copper containing traces of arsenic $\lambda = 142$ W/m·K.

In contrast to pure metals the thermal conductivity of alloys increases with rising temperature (Fig. 1-9).

Solid dielectrics (non-metals). The thermal conductivity of dielectrics usually increases with temperature, as shown in Fig. 1-10. Materials with a high volumetric density, as a rule, have a high thermal conductivity, but the latter also depends on the structure of the material, its porosity, and moisture content.

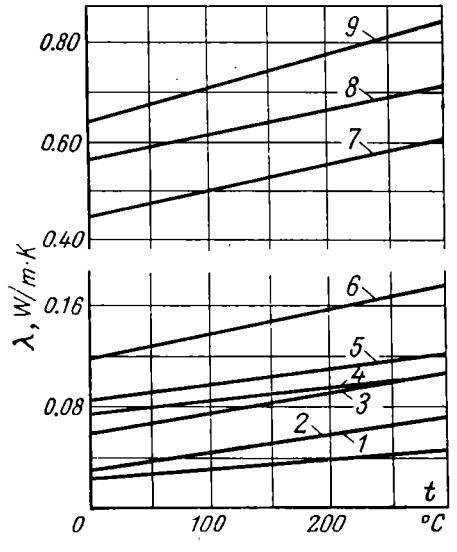

Fig. 1-10. $\lambda(t)$ of construction and heat-insulating materials:
1—air; *2*—mineral wool; *3*—slag wool; *4*—85% magnesia; *5*—sovelite; *6*—diatomaceous brick; *7*—red brick; *8*—slag-concrete brick; *9*—fireclay (chamotte) brick

Many construction and heat-insulating materials (brick, concrete, asbestos, slag, etc.) have a porous structure, and the application of Fourier's law to them is to a great extent conditional. The presence of pores makes it impossible to consider them as a continuum.

The thermal conductivity of a porous material is also conditional. It can be taken as the thermal conductivity of a solid body of the same shape and size, through which the same amount of heat is transmitted under the given boundary conditions of temperature [1.13].

The thermal conductivity of powdery and porous solids depends greatly on their volumetric density ρ [1.14]. The thermal conductivity of asbestos, for instance, increases from 0.105 to 0.248 W/m·K with an increase in density from 400 to 800 kg/m^3. This effect is explained by the fact that the thermal conductivity λ of the air filling the pores is much less than that of the hard constituents of the porous material.

The effective thermal conductivity of porous materials also depends greatly on moisture content. The thermal conductivity of damp material is considerably greater than the thermal conductivities of dry material and water taken separately. The thermal conductivity of dry brick $\lambda = 0.35$, and of water, $\lambda = 0.60$, but for damp brick $\lambda \cong 1.0$ W/m·K. This can be explained by the effect of convective heat transfer due to capillary movement of water within the porous material, and partly by the fact that the properties of combined absorbed moisture differ from those of free moisture.

The thermal conductivity of granular materials increases with temperature, and this may be explained by the fact that an increase in temperature is accompanied by an increase in heat conduction of the medium filling the spaces between the grains, and also by an intensification of radiant heat transfer within the granular material.

The thermal conductivities of construction and heat-insulating materials range from 0.023 to 2.9 W/m·K.

Materials with a low thermal conductivity (below 0.25 W/m·K), commonly used for thermal insulation, are called *heat-insulating* materials.

1-6. Differential Equation of Conduction

The aim of study of any physical phenomenon is to establish the relationship between the quantities or properties characteristic of the phenomenon considered. In complex physical processes it is very often difficult to determine these relationships when the properties vary considerably in space and time. In these cases use is then made of the method of mathematical physics, in which it is assumed that the time interval is restricted and only an elementary volume of the entire space is considered. This makes it possible to ignore, within the limits of the elementary volume and the small time interval selected, variations in certain characteristic quantities of the process, and considerably simplifies analysis.

Thus the elementary volume dv and the elementary time interval $d\tau$ selected, taken for investigation, are mathematically infinitesimal quantities, though from a physical standpoint they are sufficiently large to let us disregard the discrete structure of the medium and to consider it as a continuum. The relation obtained in this way is the general differential equation for the process in question. By integrating this equation, we can obtain analytical relationship between values for the entire range of integration and for any interval of time considered.

To resolve temperature-field problems, therefore, we require a *differential equation of conduction.*

To facilitate derivation of this differential equation, we make the following assumptions:

(a) that the solid is homogeneous and isotropic;

(b) its physical parameters are constant;

(c) the deformation of the volume considered, caused by temperature variations, is very small in relation to the volume itself;

(d) the inner heat sources [which can be given as $q_v = f(x, y, z, \tau)$, in the general case] are uniformly distributed in the solid.

The derivation of the differential equation of conduction is based on the law of conservation of energy which can be formulated for our purposes as follows: the quantity of heat dQ imparted to the elementary volume by conduction from outside and from inner heat sources during the time interval $d\tau$ is equal to the

change in internal energy or enthalpy of the substance contained in that volume:

$$dQ_1 + dQ_2 = dQ \tag{1-22}$$

where dQ_1 = quantity of heat, J, imparted by conduction to the elementary volume per time interval $d\tau$;

dQ_2 = quantity of heat liberated within the elementary volume dv by inner heat sources per time interval $d\tau$;

dQ = change in internal energy or enthalpy of the substance contained in dv per time $d\tau$.

In order to find the terms of Eq. (1-22), we may consider an elementary parallelepiped in a body with sides dx, dy, dz (Fig. 1-11) so arranged that its sides are parallel to the corresponding coordinate planes.

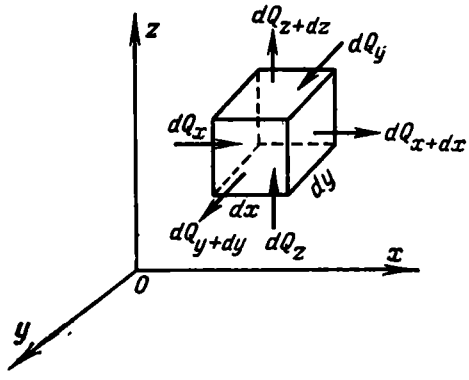

Fig. 1-11. Deriving the differential equation of conduction

The quantities of heat imparted to the sides of the elementary volume in the time interval $d\tau$ and in the direction of the Ox-, Oy-, and Oz-axes are denoted by dQ_x, dQ_y and dQ_z respectively.

The quantities of heat removed through the opposite sides of the parallelepiped, and in the same directions, are denoted respectively by dQ_{x+dx}, dQ_{y+dy}, dQ_{z+dz}. The quantity of heat imparted to side $dy\,dz$ in the direction of the Ox-axis per time $d\tau$ will be determined as

$$dQ_x = q_x\,dy\,dz\,d\tau$$

where q_x is the projection of the rate of heat flow on the direction perpendicular to the indicated side. The quantity of heat removed through the opposite side of the elementary parallelepiped in the direction of the Ox-axis is written as

$$dQ_{x+dx} = q_{x+dx}\,dy\,dz\,d\tau$$

The difference between the quantities of heat imparted to and removed from the parallelepiped in time $d\tau$ in the direction of the Ox-axis gives a quantity of heat:

$$dQ_{x1} = dQ_x - dQ_{x+dx}$$

or

$$dQ_{x1} = q_x\,dy\,dz\,d\tau - q_{x+dx}\,dy\,dz\,d\tau \tag{a}$$

The function q_{x+dx} is continuous in the considered interval dx, and can be expanded into Taylor's series:

$$q_{x+dx} = q_x + \frac{\partial q_x}{\partial x}dx + \frac{\partial^2 q_x}{\partial x^2}\frac{dx^2}{2!} + \dots$$

For the two first terms of the series equation (a) may be written in the following form:

$$dQ_{x1} = -\frac{\partial q_x}{\partial x}dx\,dy\,dz\,d\tau \tag{b}$$

In a similar manner it is possible to find the quantity of heat imparted to the elementary volume in the directions of the other two coordinate axes, Oy and Oz.

The quantity of heat dQ_1 imparted to the considered volume by conduction then is equal to:

$$dQ_1 = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) dx\, dy\, dz\, d\tau \qquad \text{(c)}$$

Let us determine the second term in the left side of Eq. (1-22). Let us denote the amount of heat evolved by the inner heat sources in unit volume of the medium per unit time, and referred to as the capacity of the inner heat sources, by q_v, W/m³. Then

$$dQ_2 = q_v dv\, d\tau \qquad \text{(d)}$$

The third term of Eq. (1-22) is found depending on the nature of the thermodynamic process which a system undergoes.

In the case of an isochoric (constant-volume) process all the heat transferred to the elementary volume will be spent to change the internal energy of the substance confined in this volume, i.e. $dQ = dU$.

Considering the internal energy per unit volume as $u = u\,(t,\ v)$, we find that

$$dU = C_v \frac{\partial t}{\partial \tau}\, d\tau\, dv = c_v \rho \frac{\partial t}{\partial \tau}\, d\tau\, dv \qquad \text{(e)}$$

where C_v = heat capacity at constant volume per unit volume, J/m³·K;
c_v = heat capacity at constant volume per unit mass, J/kg·K;
ρ = density of substance, kg/m³.

Substituting the obtained results (c), (d) and (e) into Eq. (1-22), we obtain:

$$c_v \rho \frac{\partial t}{\partial \tau} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) + q_v, \qquad \text{(1-23)}$$

or

$$c_v \rho \frac{\partial t}{\partial \tau} = -\operatorname{div} \vec{q} + q_v \qquad \text{(1-23')}$$

Expression (1-23) is the differential equation of energy for constant-volume processes of heat transfer.

In considering an isobaric, constant-pressure, process, all the heat transferred to the volume will be spent to change the enthalpy of the substance confined in this volume, and Eq. (1-22) will take the following form:

$$dQ_1 + dQ_2 = dI \qquad \text{(1-24)}$$

Assuming the enthalpy of unit volume to be $i = i\,(t,\ p)$, it can be shown that

$$dI = C_p \frac{\partial t}{\partial \tau}\, d\tau\, dv = c_p \rho \frac{\partial t}{\partial \tau}\, d\tau\, dv = \rho \frac{\partial i}{\partial \tau}\, d\tau\, dv \qquad \text{(f)}$$

where C_p = heat capacity at constant pressure per unit volume, J/m³·K;
c_p = heat capacity at constant pressure per unit mass, J/kg·K.

Substituting the obtained expressions (c), (d) and (f) into Eq. (1-24), we obtain:

$$\rho \frac{\partial i}{\partial \tau} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) + q_v \qquad \text{(1-25)}$$

or

$$\rho \frac{\partial i}{\partial \tau} = -\operatorname{div} \vec{q} + q_v \qquad \text{(1-25')}$$

Relationship (1-25) is the general differential equation for constant-pressure processes of heat transfer and it will be made use of in considering particular modes of heat transfer.

In solids the transport of heat obeys the Fourier law $q = -\lambda\, \partial t/\partial x$, the numerical value of the difference between c_p and c_v is small and it can be assumed that $c_v = c_p = c$.

It will be recalled that the projections of the vector representing the rate of heat flow on the coordinate axes Ox, Oy, Oz are determined by the expressions:

$$q_x = -\lambda \frac{\partial t}{\partial x}; \quad q_y = -\lambda \frac{\partial t}{\partial y}; \quad q_z = -\lambda \frac{\partial t}{\partial z}$$

Substituting in Eq. (1-23) the expressions obtained for the projections of the heat flow vector and omitting, in doing so, the subscript at the heat capacity c, we obtain:

$$\frac{\partial t}{\partial \tau} = \frac{1}{c\rho}\left[\frac{\partial t}{\partial x}\left(\lambda \frac{\partial t}{\partial x}\right) + \frac{\partial t}{\partial y}\left(\lambda \frac{\partial t}{\partial y}\right) + \frac{\partial t}{\partial z}\left(\lambda \frac{\partial t}{\partial z}\right)\right] + \frac{q_v}{c\rho} \tag{1-26}$$

or

$$\frac{\partial t}{\partial \tau} = \frac{1}{c\rho} \operatorname{div}(\lambda \operatorname{grad} t) + \frac{q_v}{c\rho} \tag{1-26'}$$

Expression (1-26), just as equation (1-26') is called the differential equation of conduction. It establishes the relation between the variation of temperature in time and space at any point of the heat in which heat transfer by conduction takes place.

The most general partial differential equation of conduction has the same form as Eq. (1-26), but contains variable thermal and physical characteristics λ, c and ρ which can be denoted by $\lambda\,(x, y, z, t)$, $c\,(x, y, z, t)$ and $\rho\,(x, y, z, t)$. When presented in this form the equation includes both the time-space and temperature dependencies. Eq. (1-26) describes a great number of problems in conduction which are of practical interest. So, assuming the thermal and physical characteristics to be constant, as it was done in deriving the equation, then Eq. (1-26) takes the form:

$$\frac{\partial t}{\partial \tau} = \frac{\lambda}{c\rho}\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right) + \frac{q_v}{c\rho} \tag{1-27}$$

In Eq. (1-27) we can denote

$$\frac{\lambda}{c\rho} = a \tag{g}$$

and

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = \nabla^2 t \tag{h}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the Laplacian operator in Cartesian coordinates.

In cylindrical coordinates the Laplacian operator $\nabla^2 t$ takes the following form:

$$\nabla^2 t = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \varphi^2} + \frac{\partial^2 t}{\partial z^2} \tag{i}$$

where r = radius-vector;
φ = vectorial (polar) angle;
z = z-coordinate.

In spherical coordinates the Laplacian operator $\nabla^2 t$ takes the following appearance:

$$\nabla^2 t = \frac{\partial^2 t}{\partial r^2} + \frac{2}{r}\frac{\partial t}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial t}{\partial \mu}\right] + \frac{1}{r^2(1-\mu^2)}\frac{\partial^2 t}{\partial \psi^2}, \quad \text{(j)}$$

where r = radius-vector;
$\mu = \cos\theta$;
θ and ψ = polar distance and longitude respectively.

Taking into account the foregoing, the most general form of Eq. (1-27) is as follows:

$$\frac{\partial t}{\partial \tau} = a\nabla^2 t + \frac{q_v}{c\rho} \quad (1\text{-}28)$$

The factor of proportionality a, m^2/s, in Eq. (1-28) is called *thermal diffusivity*, and represents a physical property of the substance. It is essential for transient processes of heat flow and defines the rate of change in temperature. If thermal conductivity is the ability of solids to conduct heat, thermal diffusivity is the measure of their thermal inertia. From Eq. (1-28) it follows that for any point in space the temperature variation in time $\partial t/\partial \tau$ is proportional to a. In other words, the rate of change in temperature at any point of the solid the higher, the larger its thermal diffusivity a. From this it follows, other things being equal, that equalization of temperature at all points of space will proceed at a higher rate in the solid possessing a larger thermal diffusivity. The magnitude of thermal diffusivity depends on the nature of the substance. Liquids and gases, for instance, possess a high thermal inertia and hence a small thermal diffusivity, while metals have a smaller thermal inertia and a large thermal diffusivity. Further, if the system contains no inner heat sources ($q_v = 0$), then expression (1-28) reduces into Fourier's equation:

$$\frac{\partial t}{\partial \tau} = a\nabla^2 t \quad (1\text{-}29)$$

If there are inner heat sources, but the temperature field corresponds to steady-state conduction, i.e. $t = t(x, y, z)$, the differential equation of conduction turns into Poisson's equation:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} + \frac{q_v}{\lambda} = 0 \quad (1\text{-}30)$$

Finally, for steady-state conduction in the absence of inner heat sources expression (1-27) takes the form of the Laplace equation:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0 \quad (1\text{-}31)$$

The main object of the theory of conduction consists in finding particular solutions of these partial differential equations and of some other equations describing conduction.

1-7. Conditions of Unambiguity for Heat Conduction

Since the differential equation of conduction is based on the general laws of physics, it describes the phenomenon of conduction in the most general form. It can be stated, therefore, that the derived differential equation describes a whole class of conduction phenomena. In order to single out one process from the countless number of other processes and give it full mathematical des-

cription, it is necessary to supplement the differential equation with a mathematical description of all particular peculiarities of the process concerned. These particular, or characteristic, peculiarities, which together with the differential equation give a comprehensive mathematical description of the process of conduction being investigated, are called the *conditions of unambiguity* (*single-valuedness*).

The conditions of unambiguity include:

(a) geometric conditions characteristic of the form and size of the body in which the process of conduction takes place;

(b) physical conditions, i.e. the physical properties of the medium and solid;

(c) time (initial) conditions which describe temperature distribution in the body at the initial moment of time;

(d) boundary conditions defining the interaction of the investigated body with its surroundings.

The *geometric conditions* are determined by the shape and size of the body in which the process develops.

The *physical conditions* are given by its physical properties, such as λ, c, ρ, etc. and may express the law governing the distribution of inner heat sources.

Initial conditions must be given for the study of transient processes; they state the law of temperature distribution within the body at the initial moment of time. In the general case they may be described analytically as follows:

with $\tau = 0$

$$t = f(x, y, z) \tag{1-32}$$

If temperature distribution is uniform the initial condition is simpler:

with $\tau = 0$

$$t = t_0 = \text{const} \tag{1-33}$$

Boundary conditions may be given in a number of ways.

(a) *Boundary conditions of the first kind.* The temperature distribution on the surface of the body is given for each moment of time:

$$t_s = f(x, y, z, \tau) \tag{1-34}$$

where t_s is the surface temperature; x, y, z are the surface coordinates.

Equation (1-34) is simplified in the particular case when surface temperature remains constant during the whole process of heat transfer, and acquires the following form:

$$t_s = \text{const}$$

(b) *Boundary conditions of the second kind.* Here the rates of heat flow are given for each point on the surface of the body and for any moment of time, which can be expressed analytically as follows:

$$q_d = f(x, y, z, \tau) \tag{1-35}$$

where q_d = density of heat flow on the surface of the body;

x, y, z = coordinates on the surface of the body as in Eq. (1-34).

In the simplest case the rate of heat flow remains constant in time over the whole surface of the body:

$$q_d = q_0 = \text{const} \tag{1-36}$$

Heat transfer takes place in this way, for instance, when metal objects are heated in high-temperature furnaces.

(c) *Boundary conditions of the third kind.* Here the ambient temperature t_f and the law governing heat transfer between the surface of the body and its surroundings are given. This type of boundary condition describes the law of heat transfer between the surface and surroundings during the heating and cooling of a body. The process of heat transfer between the surface of a body and the surroundings is described by Newton's law.

The process of heat exchange between the surface of a body and its surroundings is a very complex one depending on a great number of variables. These problems will be considered in greater detail in Parts Two and Three.

In accordance with Newton's law the quantity of heat removed from unit surface of a body in unit time is proportional to the difference between its temperature t_s and that of its surroundings t_f, $(t_s > t_f)$:

$$q = \alpha (t_s - t_f) \tag{1-37}$$

where α is a proportionality factor called the *heat-transfer coefficient.* Its units are $W/m^2 \cdot K$.

The *heat-transfer coefficient* is the measure of the intensity of heat transfer between the surface of a body and its surroundings, and is numerically equal to the quantity of heat removed from, or imparted to, unit area in unit time at a temperature difference of one degree between the surface and surroundings.

In accordance with the law of conservation of energy, the amount of heat removed from unit area in unit time (Eq. 1-37) is equal to the amount of heat imparted to the unit area of the surface by conduction from inner volumes of the body (Eq. 1-10)*, i.e.

$$\alpha (t_s - t_f) = -\lambda \left(\frac{\partial t}{\partial n}\right)_w \tag{1-38}$$

where n is the normal to the surface of the body, and the subscript w indicates that the temperature and temperature gradient pertain to the surface (at $n = 0$).

Boundary conditions of the third kind can be written in a final form as follows:

$$\left(\frac{\partial t}{\partial n}\right)_w = -\frac{\alpha}{\lambda} (t_w - t_f) \tag{1-39}$$

which virtually expresses the law of conservation of energy for the particular case of the surface of a body.

The heat-transfer coefficient α depends on a great number of variables. It may however be regarded as invariable in many cases, and we shall, therefore, assume it further on to be constant, when solving problems of conduction.

(d) *Boundary conditions of the fourth kind* define the process of conduction between a single body, or system of bodies and surroundings. The bodies are assumed to be in perfect contact (the surfaces in contact are at the same temperature).

In the conditions considered the heat fluxes through the surfaces in contact are equal, i.e.

$$\lambda_1 \left(\frac{\partial t_1}{\partial n}\right)_w = \lambda_2 \left(\frac{\partial t_2}{\partial n}\right)_w \tag{1-40}$$

* This is also true for heat flow in the opposite direction.

In cases governed by the boundary conditions of the fourth kind the ratio of the slopes of the tangents to the temperature curves at the points of contact of two bodies or a body and surroundings* is given (Fig. 1-12) as:

$$\frac{\tan \varphi_1}{\tan \varphi_2} = \frac{\lambda_2}{\lambda_1} = \text{const} \qquad (1\text{-}41)$$

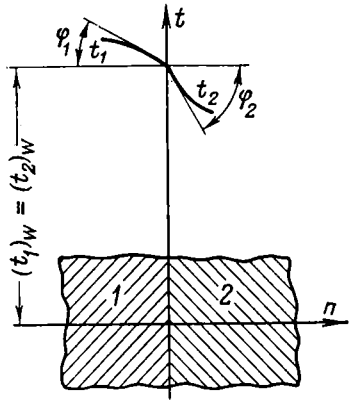

Fig. 1-12. To boundary conditions of the fourth kind

Since, with perfect contact of two bodies, their contacting surfaces are at the same temperature, the tangents at the interface pass through the same point (Fig. 1-12).

The differential equation (1-28) taking along the conditions of unambiguity gives a full mathematical statement of a particular problem of conduction. The problem thus formulated may be solved analytically, or by numerical or experimental methods. Experimental solution involves the use of methods of physical simulation or heat analogies (see Chapters 5 and 6).

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* Boundary conditions of the fourth kind in essence give the rule for conjugating the temperature fields of the investigated volume and the external body to which heat is transferred by conduction. A commensurate statement of this problem naturally requires additional information concerning the development of the process in the external body.

CHAPTER

2

STEADY-STATE CONDUCTION

2-1. Heat Transfer Through a Plain Wall ($q_v = 0$)

With steady-state, or stationary, conduction the temperature of a body remains constant in time, i.e. $\partial t/\partial \tau = 0$.

The differential equation of steady-state conduction has the following form:

$$a\nabla^2 t + \frac{q_v}{c\rho} = 0 \tag{2-1}$$

or

$$\nabla^2 t + \frac{q_v}{\lambda} = 0 \tag{2-1'}$$

In the absence of inner heat sources, i.e. ($q_v = 0$), Eq. (2-1) acquires a simpler form:

$$\nabla^2 t = 0 \tag{2-2}$$

or

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = 0 \tag{2-2'}$$

This chapter is concerned with heat transfer by conduction through bodies of the simplest geometrical form. The two variants of bodies, without inner heat sources ($q_v = 0$) and those with such sources ($q_v \neq 0$), are considered separately. Let us first consider the transfer of heat through a plane wall with $q_v = 0$.

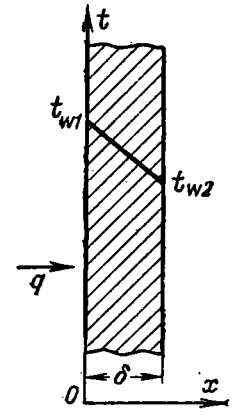

Fig. 2-1. Plain homogeneous wall

(a) *Boundary Conditions of the First Kind*

Let us consider a homogeneous, isotropic wall of thickness δ with constant thermal conductivity λ. The external surfaces of the wall are maintained at constant temperatures t_{w1} and t_{w2}.

With the given conditions temperature will change only in the direction normal to the plane wall. Should the Ox-axis be directed as shown in Fig. 2-1, temperature will remain constant in the direction of the Oy- and Oz-axes, i.e,

$$\frac{\partial t}{\partial y} = \frac{\partial t}{\partial z} = 0$$

Accordingly the temperature shall be a function of only one coordinate x and for the case being considered the differential equation of conduction acquires

the following appearance:

$$\frac{d^2t}{dx^2} = 0 \tag{2-3}$$

The boundary conditions for this problem may be formulated as follows:

$$\left.\begin{array}{ll} \text{with} & x = 0, \quad t = t_{w1} \\ \text{with} & x = \delta \quad t = t_{w2} \end{array}\right\} \tag{2-4}$$

Equation (2-3) and the conditions (2-4) give a full mathematical description of the problem. Its solution consists in finding the temperature distribution in the wall, i.e. the function $t = f(x)$, and in deriving the formula for determining the heat flow through the wall per unit time.

The law of temperature distribution in wall thickness may be found by double integration of Eq. (2-3).

The first integration gives:

$$\frac{dt}{dx} = C_1 \tag{2-5}$$

The second integration yields:

$$t = C_1 x + C_2 \tag{2-6}$$

From Eq. (2-6) it follows that with constant thermal conductivity variation in wall temperature is linear.

The constants C_1 and C_2 of Eq. (2-6) are determined from the boundary conditions:

$$\text{with } x = 0, \quad t = t_{w1} \text{ and } C_2 = t_{w1}$$

$$\text{with } x = \delta, \quad t = t_{w2} \text{ and } C_1 = -\frac{t_{w1} - t_{w2}}{\delta}$$

Substituting the constants C_1 and C_2 in Eq. (2-6), we get the temperature distribution or temperature profile in a plane wall:

$$t = t_{w1} - \frac{t_{w1} - t_{w2}}{\delta} x \tag{2-7}$$

If the reference temperature is taken as the smallest given temperature of the wall t_{w2}, we can present Eq. (2-7) in dimensionless form.

Let us denote:

$\Delta t = t - t_{w2}$ as the current temperature difference or excess temperature;

$\Delta t_0 = t_{w1} - t_{w2}$ as the full temperature difference or maximum excess temperature.

Upon introduction of these terms in Eq. (2-7), we get

$$\Delta t = \Delta t_0 - \frac{\Delta t_0}{\delta} x \tag{2-8}$$

or

$$\frac{\Delta t}{\Delta t_0} = 1 - \frac{x}{\delta}$$

If we then denote $\frac{\Delta t}{\Delta t_0} = \Theta$ as the dimensionless temperature difference or dimensionless excess temperature, and $\frac{x}{\delta} = X$ as the dimensionless coordinate, we obtain:

$$\Theta = 1 - X \tag{2-8'}$$

The equation of temperature field (2-8′) is universally applicable, i.e. it makes it possible to present the temperature distribution across the wall as a single straight line for any given value of t_{w1}, t_{w2} and δ (Fig. 2-2). The dimensionless equations are often very convenient in problems of conduction.

The quantity of heat flowing through unit area of a wall per unit time in the direction of the Ox-axis is determined by Fourier law according to which

$$q = -\lambda \frac{\partial t}{\partial x}$$

Since $\frac{\partial t}{\partial x} = C_1 = \frac{-(t_{w1}-t_{w2})}{\delta}$, then after substituting the value of $\frac{\partial t}{\partial x}$ into the expression of the Fourier law we get:

$$q = \frac{\lambda}{\delta}(t_{w1} - t_{w2}) \tag{2-9}$$

From Eq. (2-9) it follows that the quantity of heat passing through unit area of the wall per unit time is directly proportional to the thermal conductivity λ and the difference between the temperatures of the external surfaces of the wall $(t_{w1} - t_{w2})$, and inversely proportional to the wall thickness δ. It should be noted here that the rate of heat flow depends not on the absolute values of the temperatures, but on their difference, $t_{w1} - t_{w2} = \Delta t$ which is generally referred to as the temperature difference.

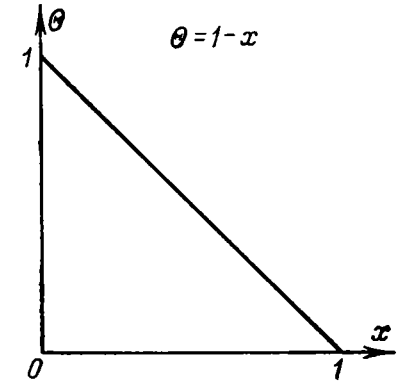

Fig. 2-2. Dimensionless temperature field in a plane wall $\Theta = 1 - X$

The ratio λ/δ W/m^2·K is called the thermal conductance of the wall and its reciprocal δ/λ (m^2·K/W) the thermal resistance of the wall. The latter represents the temperature drop across the wall related to the unit rate of heat flow. This rate of heat flow being known, we can easily calculate the total quantity of heat Q_τ transferred through the surface of the wall F during a time interval τ:

$$Q_\tau = qF\tau = \frac{\lambda}{\delta}(t_{w1} - t_{w2})F\tau \tag{2-10}$$

From Eq. (2-9) we find:

$$\frac{t_{w1}-t_{w2}}{\delta} = \frac{q}{\lambda}$$

Introducing this expression into the temperature field equation (2-7), we obtain

$$t = t_{w1} - \frac{q}{\lambda}x \tag{2-11}$$

It follows from Eq. (2-11) that, other conditions being equal, the temperature drop within the wall is the larger, the greater the rate of heat flow.

Expressions (2-7) and (2-9) are based on the assumption of constant thermal conductivity λ. In fact λ is variable.

Let us now consider the case where thermal conductivity is a function of temperature only:

$$\lambda = \lambda(t)$$

For many materials thermal conductivity has a nearly linear dependence on temperature

$$\lambda = \lambda_0 (1 + bt)$$

where λ_0 is the thermal conductivity at 0 °C.

From Fourier law

$$q = -\lambda (t) \frac{dt}{dx} = -\lambda_0 (1 + bt) \frac{dt}{dx} \tag{a}$$

Separating the variables and integrating (a) within the limits from $x = 0$ and $x = \delta$ and in the temperature interval from t_{w1} to t_{w2}, we obtain:

$$q\delta = \lambda_0 \left[1 + b \frac{t_{w1} + t_{w2}}{2}\right] (t_{w1} - t_{w2}) \tag{b}$$

The factor

$$\lambda_0 \left(1 + b \frac{t_{w1} + t_{w2}}{2}\right)$$

in expression (b) is the integral mean thermal conductivity, i.e.

$$\lambda_m = \frac{1}{t_{w1} - t_{w2}} \int_{t_{w2}}^{t_{w1}} \lambda (t)\, dt \tag{2-12}$$

Thus the rate of heat flow q, W/m^2, at the wall surface is

$$q = \frac{\lambda_m}{\delta} (t_{w1} - t_{w2}) \tag{2-13}$$

It follows from Eq. (2-13) that when λ is temperature-dependent, the rate of heat flow q may be found approximately as an integral mean value between temperatures t_{w1} and t_{w2} on the assumption $\lambda = \text{const}$.

Integrating (a) between $x = 0$ and any given coordinate x and in the temperature interval from t_{w1} to t we obtain the expression for the temperature field:

$$t = \sqrt{\left(\frac{1}{b} + t_{w1}\right)^2 - \frac{2qx}{\lambda_0 b}} - \frac{1}{b} \tag{2-14}$$

It follows from this equation that the variation in wall temperature is not linear but follows a curve whose shape depends on the sign and numerical value of factor b.

Let us now examine heat conduction through a composite plane wall built up of n homogeneous layers, assuming perfect contact between layers and equal temperatures at the surfaces in contact.

In steady-state conduction the quantity of heat passing through any isothermal surface of a heterogeneous wall is identical, i.e.

$$\frac{\partial q}{\partial x} = 0$$

Given the temperatures of the external wall surfaces, and thickness and heat conductivity of each layer, we can obtain the following system of equations:

$$\left.\begin{aligned} q &= \frac{\lambda_1}{\delta_1} (t_{w1} - t_{w2}) \\ q &= \frac{\lambda^2}{\delta_2} (t_{w2} - t_{w3}) \\ &\cdots\cdots\cdots\cdots \\ q &= \frac{\lambda_n}{\delta_n} [t_{wn} - t_{w(n+1)}] \end{aligned}\right\} \tag{c}$$

From equations (c) we determine the temperature difference for each layer and summation of the left and right sides of these equations gives:

$$t_{w1} - t_{w(n+1)} = q\left(\frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2} + \ldots + \frac{\delta_n}{\lambda_n}\right)$$

Hence, the rate of heat flow is

$$q = \frac{t_{w1} - t_{w(n+1)}}{\frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2} + \ldots + \frac{\delta_n}{\lambda_n}} = \frac{t_{w1} - t_{w(n+1)}}{\sum\limits_{i=1}^{i=n} \frac{\delta_i}{\lambda_i}} \tag{2-15}$$

The value $\sum\limits_{i=1}^{i=n} \frac{\delta_i}{\lambda_i}$ equalling the sum of the thermal resistances of each of n layers, is called the full thermal resistance of conduction of the composite wall.

Comparison of the processes of heat transfer through composite and plane homogeneous walls is made more convenient by introducing the concept of equivalent thermal conductivity λ_{eq}. It is equal to the thermal conductivity coefficient of a homogeneous wall the thickness of which Δ equals the thickness of a composite wall $\sum\limits_{i=n}^{i=n} \delta_i$ and the thermal resistance of the composite wall concerned, i. e.

$$\frac{\sum\limits_{i=1}^{i=n} \delta_i}{\lambda_{eq}} = \sum_{i=1}^{i=n} \frac{\delta_i}{\lambda_i}$$

Hence

$$\lambda_{eq} = \frac{\sum\limits_{i=1}^{i=n} \delta_i}{\sum\limits_{i=1}^{i=n} \frac{\delta_i}{\lambda_i}} \tag{2-16}$$

It follows from Eq. (2-16) that equivalent thermal conductivity λ_{eq} depends not only on the thermal and physical properties of the layers, but also on their thickness.

The interface temperatures between any two neighbouring layers are:

$$\left.\begin{aligned} t_{w2} &= t_{w1} - q\frac{\delta_1}{\lambda_1} \\ t_{w3} &= t_{w2} - q\left(\frac{\delta_1}{\lambda_1} + \frac{\delta^2}{\lambda_2}\right) \\ &\text{and} \\ t_{w(i+1)} &= t_{w1} - q\sum_{i=1}^{i} \frac{\delta_i}{\lambda_i} \end{aligned}\right\} \tag{2-17}$$

Within each layer temperature changes according to either Eq. (2-7) or Eq. (2-14), and the temperature profile of a composite wall as a whole is a broken line.

(b) Boundary Conditions of the Third Kind (Heat Transfer)

The transport of heat from one moving medium (liquid or gas) to another through a plane or composite solid wall of any form separating the two media is referred to as *heat transfer*. The process involves the transfer of heat from the hotter fluid to the wall, transfer of heat by conduction through the wall and transfer of heat from the separating wall to the colder moving medium.

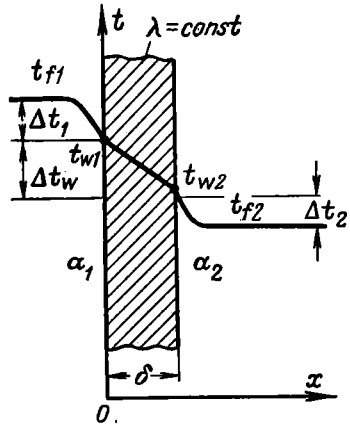

Fig. 2-3. Heat transfer through a plane homogeneous wall

Let us consider heat transfer through homogeneous and composite plane walls.

Let a plane homogeneous wall be of thickness δ (Fig. 2-3), and the given data include: thermal conductivity λ of the wall; ambient temperatures t_{f1} and t_{f2}; and local heat-transfer coefficients α_1 and α_2. Let us further assume that t_{f1}, t_{f2}, α_1 and α_2 are constant and do not change along the surface; this allows us to consider temperature variations of the fluids and wall solely in the direction normal to the wall.

It is necessary to find the amount of heat flow in the given conditions from the hot fluid to the cold, and the temperatures at the wall surfaces.

The rate of heat flow from the hot fluid to the wall is determined by the equation

$$q = \alpha_1 (t_{f1} - t_{w1}) \tag{2-18}$$

In steady-state conditions the same amount of heat will be transferred through the wall by conduction:

$$q = \frac{\lambda}{\delta} (t_{w1} - t_{w2}) \tag{2-19}$$

The same amount of heat is transferred from the surface of the wall to the cold fluid

$$q = \alpha_2 (t_{w2} - t_{f2}) \tag{2-20}$$

Equations (2-18), (2-19) and (2-20) may be presented in the form

$$\left.\begin{aligned} q\frac{1}{\alpha_1} &= t_{f1} - t_{w1} \\ q\frac{\delta}{\lambda} &= t_{w1} - t_{w2} \\ q\frac{1}{\alpha_2} &= t_{w2} - t_{f2} \end{aligned}\right\} \tag{2-21}$$

Putting together equalities (2-21) termwise, we obtain

$$q\left(\frac{1}{\alpha_1} + \frac{\delta}{\lambda} + \frac{1}{\alpha_2}\right) = t_{f1} - t_{f2}$$

Hence, the rate of heat flow, W/m²

$$q = \frac{t_{f1} - t_{f2}}{\frac{1}{\alpha_1} + \frac{\delta}{\lambda} + \frac{1}{\alpha_2}} \tag{2-22}$$

Denote:

$$\frac{1}{\frac{1}{\alpha_1} + \frac{\delta}{\lambda} + \frac{1}{\alpha_2}} = k \tag{2-23}$$

This quantity is in W/m²·K.

Taking into account Eq. (2-23), we can write Eq. (2-22) in the following form:

$$q = k(t_{f1} - t_{f2}), \text{ W/m}^2 \tag{2-24}$$

The factor k has the same dimension as α, and is called the *overall heat-transfer coefficient.* It defines the intensity of heat transfer from one fluid to another through a wall separating them, and is numerically equal to the quantity of heat passing through unit area of wall surface in unit time at a temperature difference of 1 °C.

The reciprocal of this coefficient is called the *overall thermal resistance to heat transfer.*

The overall thermal resistance of a single-layer homogeneous wall is

$$R = \frac{1}{k} = \frac{1}{\alpha_1} + \frac{\delta}{\lambda} + \frac{1}{\alpha_2} \tag{2-25}$$

It will be seen from Eq. (2-25) that the overall thermal resistance is the sum of separate thermal resistances $\frac{1}{\alpha_1}$, $\frac{\delta}{\lambda}$, and $\frac{1}{\alpha_2}$ where

$\frac{1}{\alpha_1} = R_1$ is the thermal resistance to heat transfer from the hot fluid to the wall;

$\frac{\delta}{\lambda} = R_w$ is the thermal resistance to heat transfer by conduction across the wall;

$\frac{1}{\alpha_2} = R_2$ is the thermal resistance to heat transfer from the wall to the cold fluid.

Since the overall thermal resistance of a composite wall is the sum of the separate thermal resistances of the component layers, it is quite clear that in this case the thermal resistance of each component layer should be taken into account. If the wall consists of layers the overall thermal resistance to heat transfer through such a wall will be:

$$R = \frac{1}{k} = \frac{1}{\alpha_1} + \frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2} + \ldots + \frac{\delta_n}{\lambda_n} + \frac{1}{\alpha_2}$$

or

$$R = \frac{1}{\alpha_1} + \sum_{i=1}^{i=n} \frac{\delta_i}{\lambda_i} + \frac{1}{\alpha_2}$$

Hence

$$k = \frac{1}{\frac{1}{\alpha_1} + \sum_{i=1}^{i=n} \frac{\delta_i}{\lambda_i} + \frac{1}{\alpha_2}} \tag{2-26}$$

The rate of heat flow through a composite wall consisting of n layers is:

$$q = \frac{1}{\frac{1}{\alpha_1} + \sum_{i=1}^{i=n} \frac{\delta_i}{\lambda_i} + \frac{1}{\alpha_2}} = k(t_{f1} - t_{f2}) \qquad (2\text{-}27)$$

Equation (2-27) for a composite wall is similar to Eq. (2-24) for a plane homogeneous wall, differing only in the expression of the overall heat-transfer coefficient k. Comparison of equations (2-26) and (2-23) shows that the latter is a particular case of the former when $n = 1$.

The heat flow Q, W, through surface F of a wall is

$$Q = qF = k\Delta tF \qquad (2\text{-}28)$$

The surface temperatures of a homogeneous wall can be found from Eq. (2-21). It follows that

$$t_{w1} = t_{f1} - q\frac{1}{\alpha_1}$$

$$t_{w2} = t_{f1} - q\left(\frac{1}{\alpha_1} + \frac{\delta}{\lambda}\right)$$

or

$$t_{w2} = t_{f2} + q\frac{1}{\alpha_2}$$

Comparison of Eqs. (2-15) and (2-27) shows that heat transfer through a composite wall with boundary conditions of the first kind is a special case of heat transfer with boundary conditions of the third kind.

From the foregoing, the interface temperature between any two layers of a composite wall i and $i + 1$ with boundary conditions of the third kind, may be determined by the equation

$$t_{w(i+1)} = t_{f1} - q\left(\frac{1}{\alpha_1} + \sum_{i=1}^{i} \frac{\delta_i}{\lambda_i}\right) \qquad (2\text{-}29)$$

Along with Eq. (2-29) interface temperatures can also be determined graphically. Let us consider the graphical method for determining interface temperatures of a non-homogeneous composite wall based on the linear relationship between the temperature difference in a wall and thermal resistance:

$$t_{f1} - t_{f2} = q\frac{1}{k}$$

or, for any layer of the system

$$t_{wi} - t_{w(i+1)} = q\frac{\delta_i}{\lambda_i}$$

This relationship allows us to construct an imaginary wall in which the thickness of the separate layers is proportional to the respective thermal resistance, and the external thermal resistances to heat transfer, $1/\alpha_1$ and $1/\alpha_2$ are accounted for by introducing two conditional boundary layers of corresponding thickness. The essence of the method will be explained on the model of a three-layer wall

The overall thermal resistance to heat flow through this wall is:

$$R = \frac{1}{k} = \frac{1}{\alpha_1} + \frac{\delta_1}{\lambda_1} + \frac{\delta_2}{\lambda_2} + \frac{\delta_3}{\lambda_3} + \frac{1}{\alpha_2}$$

We first plot sections O_1A_1, A_1A_2, A_2A_3, A_3A_4 and A_4O_2 along the horizontal, equal each respectively to the thermal resistances $1/\alpha_1$, δ_1/λ_1, δ_2/λ_2, δ_3/λ_3, and $1/\alpha_2$ (Fig. 2-4), then raise perpendiculars at points O_1, A_1, A_2, A_3, A_4 and O_2, and on O_1K_1 and O_2K_2 plot the temperatures of the moving media t_{f1} and t_{f2} at any convenient scale. These points C_1 and B_2 are joined by a straight line. The sections A_1E_1, A_2E_2, A_3E_3 and A_4E_4 are equal to the unknown temperatures t_{w1}, t_{w2}, t_{w3} and t_{w4}. Since the triangles $C_1B_1B_2$ and $C_1C_2E_1$ are similar, it follows that

$$\frac{C_1C_2}{C_1B_1}=\frac{C_2E_1}{B_1B_2} \quad \text{or} \quad \frac{C_1C_2}{t_{f1}-t_{f2}}=\frac{\frac{1}{\alpha_1}}{\frac{1}{k}} \qquad (2\text{-}30)$$

It also follows from Eq. (2-30) that $C_1C_2 = t_{f1} - t_{w1}$, hence

$$A_1E_1 = O_1C_1 - C_1C_2 = t_{w1}$$

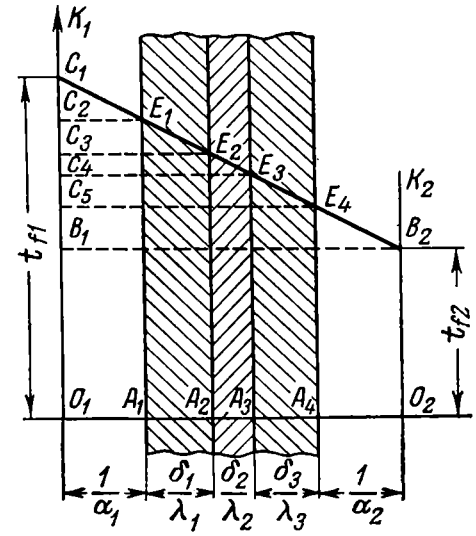

Fig. 2-4. Graphical method for determining temperature

By analogy we prove that lines A_2E_2, A_3E_3 and A_4E_4 are equal respectively to temperatures t_{w2}, t_{w3} and t_{w4}.

c) Boundary Conditions of the Second and Third Kind

Let us now consider the case of heat transfer through a homogeneous isotropic wall with boundary conditions of the second kind expressed as q_w = const (at $x = 0$) for one of the wall surfaces. For the second surface we know the heat-transfer coefficient α_2 and ambient temperature t_{f2}, i.e. boundary conditions of the third kind (Fig. 2-5). There are no inner heat sources within the wall ($q_v = 0$).

Solution of this problem consists in determining the temperature distribution across the wall and the surface temperatures. The process of heat transfer being steady, we can write the following system of equations:

$$\left.\begin{aligned} q_w &= (t_{w1}-t_{w2})\frac{\lambda}{\delta} \\ q_w &= \alpha_2 (t_{w2}-t_{f2}) \end{aligned}\right\} \qquad (2\text{-}31)$$

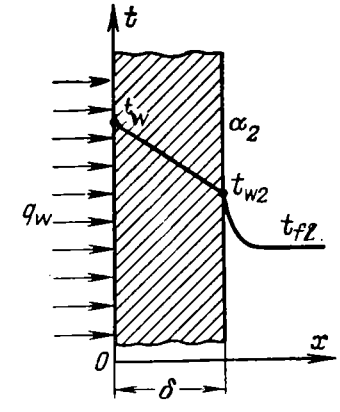

Fig. 2-5. Heat transfer through a plane wall (composite boundary conditions)

It follows from Eqs. (2-31) that for any given value of q_w

$$\left.\begin{aligned} t_{w2} &= t_{f2} + q_w \frac{1}{\alpha_2} \\ t_{w1} &= t_{f2} + q_w \left(\frac{1}{\alpha_2}+\frac{\delta}{\lambda}\right) \end{aligned}\right\} \qquad (2\text{-}32)$$

For a composite wall consisting of n homogeneous layers, the temperature of wall surfaces and interface temperatures can be determined by the following equations:
on the right external surface

$$t_{w(n+1)} = t_{f2} + q_w \frac{1}{\alpha_2}$$

on the left external surface

$$t_{w1} = t_{f2} + q_w \left(\frac{1}{\alpha_2} + \sum_{i=1}^{i=n} \frac{\delta_i}{\lambda_i} \right)$$

on the interface between the $m - 1$ and mth layer

$$t_{wm} = t_{f2} + q_w \left(\frac{1}{\alpha_2} + \sum_{i=m}^{i=n} \frac{\delta_i}{\lambda_i} \right) \tag{2-33}$$

The temperature distribution across any layer may be found by Eqs. (2-7) or (2-14).

2-2. Heat Transfer Through a Cylindrical Wall ($q_v = 0$)

(*a*) *Boundary Conditions of the First Kind*

Consider the case of steady-state conduction through a cylindrical wall (tube) of inside diameter $d_1 = 2r_1$ and outside diameter $d_2 = 2r_2$ (Fig. 2-6).

The surfaces of the wall are at constant temperatures t_{w1} and t_{w2}. The thermal conductivity λ of the wall material is constant in the given temperature range. It is necessary to find the temperature distribution in the cylindrical wall and the heat flow through it.

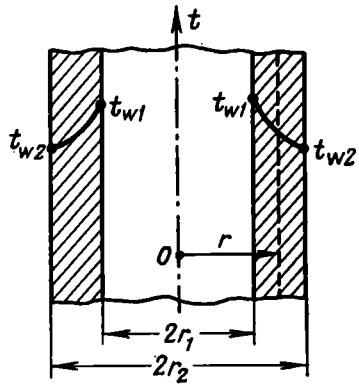

Fig. 2-6. Heat conduction through a cylindrical wall

In this case it is convenient to write the differential equation of conduction in the cylindrical system of coordinates:

$$\nabla^2 t = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t}{\partial \varphi^2} + \frac{\partial^2 t}{\partial z^2} = 0 \tag{2-34}$$

The Oz-axis coincides with the axis of the tube.

In the given conditions temperature changes only in radial direction, and the temperature field is one-dimensional. Consequently

$$\frac{\partial t}{\partial z} = 0 \quad \text{and} \quad \frac{\partial^2 t}{\partial z^2} = 0 \tag{a}$$

In addition, since the temperatures of the outside and inside wall surfaces are constant, the isothermal surfaces are cylindrical and their axis is that of the tube. Temperature then must also not change along φ, i.e.

$$\frac{\partial t}{\partial \varphi} = 0 \quad \text{and} \quad \frac{\partial^2 t}{\partial \varphi^2} = 0 \tag{b}$$

Taking (a) and (b) into account, Eq. (2-34) becomes:

$$\frac{d^2t}{dr^2}+\frac{1}{r}\frac{dt}{dr}=0 \tag{2-35}$$

Boundary conditions are as follows:

$$\left.\begin{aligned}&\text{with } r=r_1, \quad t=t_{w1}\\&\text{with } r=r_2, \quad t=t_{w2}\end{aligned}\right\} \tag{2-36}$$

Solving Eq. (2-35) together with Eq. (2-36), we obtain the equation of the temperature field in a cylindrical wall.

If a new variable

$$u=\frac{dt}{dr} \tag{c}$$

is introduced, then we have:

$$\frac{d^2t}{dr^2}=\frac{du}{dr}; \quad \frac{1}{r}\frac{dt}{dr}=\frac{u}{r} \tag{d}$$

Substituting (c) and (d) in Eq. (2-35), we get

$$\frac{du}{dr}+\frac{1}{r}u=0 \tag{2-37}$$

Integration of Eq. (2-37) gives

$$\ln u + \ln r = \ln C_1 \tag{e}$$

Taking the antilogarithms of (e) and returning to the initial variables we get

$$dt=C_1\frac{dr}{r} \tag{f}$$

After integration we obtain:

$$t = C_1 \ln r + C_2 \tag{2-38}$$

The constants C_1 and C_2 can be determined, if the given boundary conditions are substituted in Eq. (2-38)

$$\left.\begin{aligned}&\text{with } r=r_1, \quad t=t_{w1}, \quad \text{hence } t_{w1}=C_1\ln r_1+C_2\\&\text{with } r=r_2, \quad t=t_{w2}, \quad \text{hence } t_{w2}=C_1\ln r_2+C_2\end{aligned}\right\} \tag{g}$$

Solving Eqs. (g) with respect to C_1 and C_2, we get:

$$C_1=\frac{t_{w1}-t_{w2}}{\ln\frac{r_1}{r_2}}$$

$$C_2=t_{w1}-(t_{w1}-t_{w2})\frac{\ln r_1}{\ln\frac{r_1}{r_2}}$$

Substituting these values of C_1 and C_2 in Eq. (2-38), we obtain:

$$t=t_{w1}-(t_{w1}-t_{w2})\frac{\ln\frac{r}{r_1}}{\ln\frac{r_2}{r_1}}$$

or

$$t=t_{w1}-(t_{w1}-t_{w2})\frac{\ln\frac{d}{d_1}}{\ln\frac{d_2}{d_1}} \tag{2-39}$$

The derived expression is the equation of a logarithmic curve. A curvilinear distribution of temperature in a cylindrical wall may be explained as follows.

In case of a plane wall the rate of heat flow q is the same for all isothermal surfaces, and the temperature gradient therefore remains constant for all isothermal surfaces. With a cylindrical wall the rate of heat flow through any isothermal surface depends on the radius.

The quantity of heat passing through a cylindrical surface F per unit time can be found by Fourier's law:

$$Q = -\lambda \frac{dt}{dr} F$$

Substituting the value of the temperature gradient given by equation (f) into Fourier's formula and taking into account that $F = 2\pi rl$, we obtain:

$$Q = \frac{2\pi\lambda l (t_{w1} - t_{w2})}{\ln \frac{d_2}{d_1}} \tag{2-40}$$

Here Q is in W.

From Eq. (2-40) it follows that the quantity of heat passing through a cylindrical wall per unit time is completely defined by the given boundary conditions and does not depend on the radius.

Heat flow (2-40) can be related either to unit tube length or to unit area of inner or outer surface. We then obtain the following calculation formulae for the rate of heat flow, W/m²:

$$\frac{Q}{\pi d_1 l} = q_1 = \frac{2\lambda (t_{w1} - t_{w2})}{d_1 \ln \frac{d_2}{d_1}} \tag{2-41}$$

(heat flow through unit area of inner surface);

$$\frac{Q}{\pi d_2 l} = q_2 = \frac{2\lambda (t_{w1} - t_{w2})}{d_2 \ln \frac{d_2}{d_1}} \tag{2-42}$$

(heat flow through unit area of outer surface);

$$\frac{Q}{l} = q_l = \frac{\pi (t_{w1} - t_{w2})}{\frac{1}{2\lambda} \ln \frac{d_2}{d_1}} \tag{2-43}$$

(heat flow through unit tube length, W/m).

Heat flow reduced to unit tube length is measured in W/m, and is called the *linear rate of heat flow*. Equation (2-43) shows it to be independent of the surface area of the cylindrical wall at a constant ratio d_2/d_1. With heat transfer through a tube the rates of heat flow q_1 and q_2 via the inner and outer surfaces of the cylindrical wall are not equal, q_1 being always larger than q_2, as may be clearly seen from Eqs. (2-41) and (2-42).

The relation between q_l, q_1 and q_2 can be easily found from Eqs. (2-41), (2-42) and (2-43):

$$q_l = \pi d_1 q_1 = \pi d_2 q_2 \tag{2-44}$$

When thermal conductivity is a function of temperature of the kind $\lambda (t) = \lambda_0 (1 + bt)$ it can be shown that the rate of heat flow can be calculated by the

formula used when $\lambda = \text{const}$:

$$q_l = \frac{\pi (t_{w1} - t_{w2})}{\frac{1}{2\lambda_m} \ln \frac{d_2}{d_1}} \tag{2-45}$$

bearing in mind that in formula (2-45) the term λ_m is the integral mean thermal conductivity:

$$\lambda_m = \frac{1}{(t_{w1} - t_{w2})} \int_{t_{w2}}^{t_{w1}} \lambda (t)\, dt$$

To find the temperature fields for the case when $\lambda = \lambda (t) = \lambda_0 (1 + bt)$, we can use the Fourier equation applied to a cylindrical wall:

$$q_l = -\lambda (t) \frac{dt}{dr} 2\pi r \tag{2-46}$$

Separating the variables and integrating Eq. (2-46) within the limits $r = r_1$ and r, $t = t_{w1}$ and t and finding t from the integral obtained, we get the following expression for the temperature field:

$$t = \sqrt{\left(\frac{1}{b} + t_{w1}\right)^2 - \frac{q_l \ln \frac{d}{d_1}}{\pi b \lambda_0}} - \frac{1}{b} \tag{2-47}$$

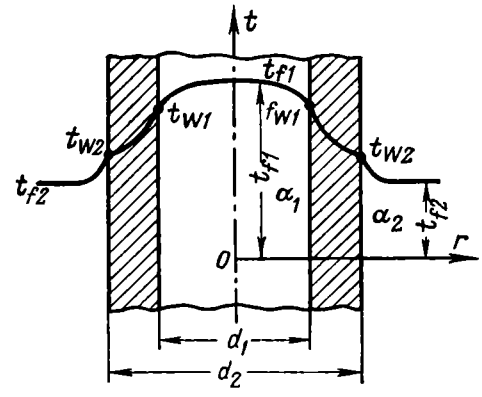

Fig. 2-7. Heat transfer through a cylindrical homogeneous wall

(b) Boundary Conditions of the Third Kind (Heat Transfer)

Consider a homogeneous cylindrical wall (tube) of a material with constant thermal conductivity λ. The given quantities include the temperatures of the moving media, t_{f1} and t_{f2}, and the heat-transfer coefficients for the inner and outer surfaces of the tube α_1 and α_2 (Fig. 2-7), which are all constant.

It is necessary to find q_l and t_w. Let us assume the tube to be quite long as compared with its wall thickness. Heat loss through the tube ends can then be ignored, and in the steady-state conduction, the quantity of heat transferred from the hot fluid to the surface of the wall, transmitted across the wall, and given up to the cold fluid will be the same throughout. Thus, we can write:

$$\left.\begin{aligned} q_l &= \alpha_1 \pi d_1 (t_{f1} - t_{w1}) \\ q_l &= \frac{\pi (t_{w1} - t_{w2})}{\frac{1}{2\lambda} \ln \frac{d_2}{d_1}} \\ q_l &= \alpha_2 \pi d_2 (t_{w2} - t_{f2}) \end{aligned}\right\} \tag{2-48}$$

Let us rearrange these equations as follows:

$$\left.\begin{aligned} t_{f1} - t_{w1} &= \frac{q_l}{\pi} \frac{1}{\alpha_1 d_1} \\ t_{w1} - t_{w2} &= \frac{q_l}{\pi} \frac{1}{2\lambda} \ln \frac{d_2}{d_1} \\ t_{w2} - t_{f2} &= \frac{q_l}{\pi} \frac{1}{\alpha_2 d_2} \end{aligned}\right\} \tag{2-48'}$$

Summing up the equations of (2-48'), we obtain the full temperature difference:

$$t_{f1}-t_{f2}=\frac{q_l}{\pi}\left(\frac{1}{\alpha_1 d_1}+\frac{1}{2\lambda}\ln\frac{d_2}{d_1}+\frac{1}{\alpha_2 d_2}\right)$$

Then

$$q_l=\frac{t_{f1}-t_{f2}}{\frac{1}{\alpha_1 d_1}+\frac{1}{2\alpha}\ln\frac{d_2}{d_1}+\frac{1}{\alpha_2 d_2}} \tag{2-49}$$

Let us denote:

$$k_l=\frac{1}{\frac{1}{\alpha_1 d_1}+\frac{1}{2\lambda}\ln\frac{d_2}{d_1}+\frac{1}{\alpha_2 d_2}} \tag{2-50}$$

Taking Eq. (2-50) into consideration, Eq. (2-49) may be written:

$$q_l=k_l\pi\,(t_{f1}-t_{f2}) \tag{2-49'}$$

The value k_l is called the *linear overall heat-transfer coefficient*, it is in W/m·K. It defines the intensity of heat transfer from one fluid to another through a wall separating them. Its value k_l numerically equal to the quantity of heat passing through the wall of a tube one metre long per unit time from one fluid to another at a temperature difference of one degree.

The reciprocal of the linear overall heat-transfer coefficient $R_l=\frac{1}{k_l}$ is referred to as the *linear thermal resistance of heat transfer*:

$$R_l=\frac{1}{k_l}=\frac{1}{\alpha_1 d_1}+\frac{1}{2\lambda}\ln\frac{d_2}{d_1}+\frac{1}{\alpha_2 d_2} \tag{2-51}$$

here R_l is in m·K/W.

The components of the full thermal resistance are:

$\frac{1}{\alpha_1 d_1}$ and $\frac{1}{\alpha_2 d_2}$ = thermal resistance of heat transfer of the wall surfaces, R_{l1} and R_{l2} respectively;

$\frac{1}{2\lambda}\ln\frac{d_2}{d_1}$ = thermal resistance of heat conduction of the wall R_{lw}.

It should be noted that the linear thermal resistances of a tube are determined not only by the heat-transfer coefficients α_1 and α_2, but also by the corresponding diameters.

If the heat flow through a cylindrical wall is reduced to its inner or outer surface, we obtain the rate of heat flow W/m² relative to unit area of a corresponding tube surface:

$$q_1=\frac{Q}{\pi d_1 l}=\frac{k_l}{d_1}(t_{f1}-t_{f2})$$

or

$$q_2=\frac{Q}{\pi d_2 l}=\frac{k_l}{d_2}(t_{f1}-t_{f2})$$

and

$$q_1=k_1(t_{f1}-t_{f2})$$
$$q_2=k_2(t_{f1}-t_{f2})$$

where

$$k_1=\frac{k_l}{d_1}\quad\text{and}\quad k_2=\frac{k_l}{d_2}$$

These last expressions establish the relationship for the heat-transfer coefficients where the heat flow is attributed to unit length and unit area of a cylindrical wall:

$$k_l = d_1 k_1 = d_2 k_2$$

here k_l is in W/m·K.

The formulae for k_1 and k_2, W/m²·K, may be extended as follows:

$$\left.\begin{aligned} k_1 &= \frac{1}{\frac{1}{\alpha_1} + \frac{d_1}{2\lambda} \ln \frac{d_2}{d_1} + \frac{d_1}{\alpha_2 d_2}} \\ k_2 &= \frac{1}{\frac{1}{\alpha_1} \frac{d_2}{d_1} + \frac{d_2}{2\lambda} \ln \frac{d_2}{d_1} + \frac{1}{\alpha_2}} \end{aligned}\right\} \quad (2\text{-}52)$$

In practice the thickness of the walls of cylinders is usually small compared with their diameter. Calculations may therefore be made using simplified formulae which can be obtained in the following way.

Expand the value $\ln \frac{d_2}{d_1}$ in a series:

$$\ln \frac{d_2}{d_1} = \left(\frac{d_2}{d_1} - 1\right) - \frac{1}{2}\left(\frac{d_2}{d_1} - 1\right)^2 + \ldots$$

If the ratio $\frac{d_2}{d_1} \to 1$, the series rapidly becomes converging, and it allows us to obtain sufficiently accurate results using only its first term:

$$\ln \frac{d_2}{d_1} = \left(\frac{d_2}{d_1} - 1\right) = \frac{d_2 - d_1}{d_1} = \frac{2\delta}{d_1}$$

where δ is the thickness of cylindrical wall, m.

Substituting the obtained value $\ln \frac{d_2}{d_1}$ in Eq. (2-52), we get:

$$k_1 = \frac{1}{\frac{1}{\alpha_1} + \frac{\delta}{\lambda} + \frac{1}{\alpha_2}} \quad (2\text{-}53)$$

Hence, for a thin-walled tube practical calculations may be based on the following formula:

$$Q = k\pi d_x l (t_{f1} - t_{f2}) \quad (2\text{-}54)$$

where k, W/m²·K, is to be taken according to formula (2-53), i.e. as for a plane wall. If $\frac{d_2}{d_1} < 2$, the calculation error does not exceed 4 per cent. A maximum error not exceeding 4 per cent is quite admissible for many engineering calculations; and Eq. (2-54) is commonly used when $\frac{d_2}{d_1} \leqslant 1.8$.

The error can be diminished, if for the reference surface in formula (2-54) is taken that with smaller α:

(1) if $\alpha_1 \gg \alpha_2$, then $d_x = d_2$
(2) if $\alpha_2 \gg \alpha_1$, then $d_x = d_1$
(3) if $\alpha_1 \simeq \alpha_2$, then $d_x = \frac{d_1 + d_2}{2}$

For heat transfer through a composite cylindrical wall the system of equations (2-48′) should be replaced by another system to take into account the thermal

resistance to conduction of all the layers, i.e.

$$\left.\begin{aligned} t_{f1}-t_{w1} &= \frac{q_l}{\pi}\frac{1}{\alpha_1 d_1} \\ t_{w1}-t_{w2} &= \frac{q_l}{\pi}\frac{1}{2\lambda_1}\ln\frac{d_2}{d_1} \\ &\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots \\ t_{wn}-t_{w(n+1)} &= \frac{q_l}{\pi}\frac{1}{2\lambda_n}\ln\frac{d_{n+1}}{d_n} \\ t_{w(n+1)}-t_{f2} &= \frac{q_l}{\pi}\frac{1}{\alpha_2 d_{n+1}} \end{aligned}\right\} \qquad (2\text{-}55)$$

Summing up these equalities and solving for q_l, W/m, we get

$$q_l = \frac{\pi\,(t_{f1}-t_{f2})}{\dfrac{1}{\alpha_1 d_1}+\sum\limits_{i=1}^{i=n}\dfrac{1}{2\lambda_i}\ln\dfrac{d_{i+1}}{d_i}+\dfrac{1}{\alpha_2 d_{n+1}}} \qquad (2\text{-}56)$$

or

$$q_l = k_l\pi\,(t_{f1}-t_{f2}) \qquad (2\text{-}56')$$

The value

$$\frac{1}{k_l} = R_l = \frac{1}{\alpha_1 d_1}+\sum_{i=1}^{i=n}\frac{1}{2\lambda_i}\ln\frac{d_{i+1}}{d_i}+\frac{1}{\alpha_2 d_{n+1}} \qquad (2\text{-}56'')$$

is called the *total thermal resistance* of a composite cylindrical wall and is measured in m·K/W.

It follows from Eq. (2-55) that

$$\left.\begin{aligned} t_{w1} &= t_{f1}-\frac{q_l}{\pi}\frac{1}{\alpha_1 d_1} \\ t_{w2} &= t_{f1}-\frac{q_l}{\pi}\left(\frac{1}{\alpha_1 d_1}+\frac{1}{2\lambda_i}\ln\frac{d_2}{d_1}\right) \\ t_{w(i+1)} &= t_{f1}-\frac{q_l}{\pi}\left(\frac{1}{\alpha_1 d_1}+\sum_{i=1}^{i}\frac{1}{2\lambda_i}\ln\frac{d_{i+1}}{d_i}\right) \end{aligned}\right\} \qquad (2\text{-}57)$$

When boundary conditions of the first kind are given they can be considered as the limiting case of boundary conditions of the third kind, when the local heat-transfer coefficients α_1 and α_2 tend to infinity resulting in that fluid temperatures t_{f1} and t_{f2} become equal to wall temperatures t_{w1} and $t_{w(n+1)}$. Then, Eq. (2-56) takes the following appearance:

$$q = \frac{\pi\,(t_{w1}-t_{w(n+1)})}{\sum\limits_{i=1}^{i=n}\dfrac{1}{2\lambda_i}\ln\dfrac{d_{i+1}}{d_i}} \qquad (2\text{-}58)$$

and the calculation formula for interface temperature:

$$t_{w(i+1)} = t_{w1}-\frac{q_l}{\pi}\sum_{i=1}^{i}\frac{1}{2\lambda_i}\ln\frac{d_{i+1}}{d_i} \qquad (2\text{-}59)$$

2-3. Critical Diameter of a Cylindrical Wall

Let us consider how varying the outside diameter affects the thermal resistance of a homogeneous cylindrical wall [2.1]. Equation (2-51) gives:

$$R_l = \frac{1}{\alpha_1 d_1} + \frac{1}{2\lambda} \ln \frac{d_2}{d_1} + \frac{1}{\alpha_2 d_2}$$

The values α_1, d_1, λ and α_2 being constant, the total thermal resistance of heat transfer of the cylindrical wall will depend only on the outside diameter. It follows then from Eq. (2-51) that in this case $\frac{1}{\alpha_1 d_1} \equiv R_{l1} = \text{const}$.

The thermal resistance to conduction, $\frac{1}{2\lambda} \ln \frac{d_2}{d_1} \equiv R_{lw}$, increases with d_2, while the thermal resistance to heat transfer from the wall to the medium, $\frac{1}{\alpha_2 d_2} \equiv R_{l2}$, decreases with increasing d_2. The] total thermal resistance will apparently be determined by variations of the components R_{lw} and R_{l2}. The variations of partial thermal resistances are illustrated in Fig. 2-8.

In order to find the variation of R_l with increasing thickness of a cylindrical wall, let us examine it as a function of d_2, differentiating R_l with respect to d_2 and setting the derivative to zero:

$$\frac{d(R_l)}{d(d_2)} = \frac{1}{2\lambda d_2} - \frac{1}{\alpha_2 d_2^2} = 0$$

Here the value of d_2 corresponds to the extreme point of the curve $R_l = f(d_2)$. Investigating the curve for its maximum and minimum points by any of the known methods, we see that the minimum is located at the extremal point. Thus the thermal resistance is minimum at $d_2 = \frac{2\lambda}{\alpha_2}$.

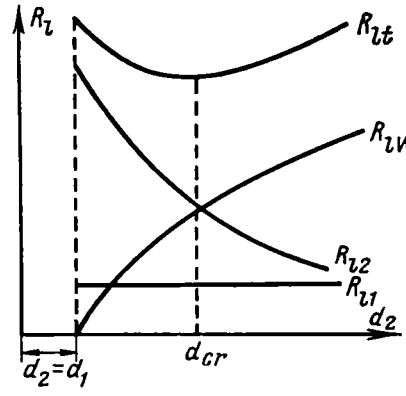

Fig. 2-8. Dependence of the thermal resistance of a cylindrical wall on d_2

The outside diameter of a tube corresponding to the minimum total thermal resistance is called the *critical diameter* and is denoted by d_{cr}. The critical diameter is calculated by the formula

$$d_{cr} = \frac{2\lambda}{\alpha_2} \tag{2-60}$$

With $d_2 < d_{cr}$, the total thermal resistance drops with increasing d_2, because an increase of the outer surface affects the thermal resistance more than an increase in wall thickness.

With $d_2 > d_{cr}$, thermal resistance of heat transfer increases with d_2, which points out to dominating effect of the wall thickness.

These considerations should be taken into account in selecting heat insulation for various cylindrical apparatus and piping.

Consider the critical diameter of the insulation placed on a tube (Fig. 2-9). The thermal resistance of heat transfer of the tube is represented by the formula:

$$R_l = \frac{1}{\alpha_1 d_1} + \frac{1}{2\lambda_w} \ln \frac{d_2}{d_1} + \frac{1}{2\lambda_{ins}} \ln \frac{d_3}{d_2} + \frac{1}{\alpha_2 d_3}$$

As follows from the equation $q_l = \frac{\pi\Delta t}{R_l}$, q_l will first increase with increasing outer diameter d_3 of the insulation, and reach a maximum at $d_3 = d_{cr}$. Further increase in the outer diameter of insulation will then bring about a drop in q_l (Fig. 2-10).

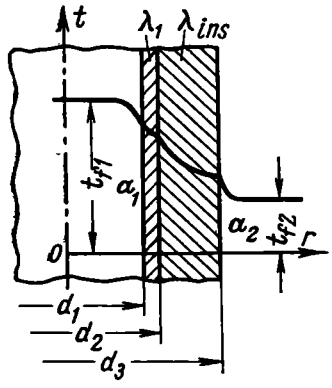

Fig. 2-9. To the notion of critical diameter of insulation

Having chosen some heat-insulating material for a cylindrical surface, it is above all necessary to calculate the critical diameter using the given values of λ_{ins} and α_2 (Eq. 2-60).

If d_{cr} proves larger than the tube outside diameter d_2, it is not worth using the chosen material for heat insulation. An increase in heat loss will be observed with increasing insulation thickness in the zone characterized by $d_2 < d_3 < d_{cr\ ins}$. This will be clearly seen from Fig. 2-10. Only at $d_3 = d_{3ef}$ will the heat loss again equal the amount of heat lost from the initial non-insulated pipe. Consequently any layer of heat insulation will not justify its purpose.

This means that heat insulation, to be effective, must have a critical diameter $d_{cr\ ins}$ equal to, or smaller than, d_2 ($d_{cr\ ins} \leqslant d_2$).

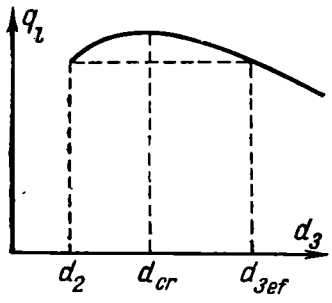

Fig. 2-10. Dependence of heat loss on thickness of insulation laid on a cylindrical wall

Example. A pipe of outside diameter $d = 20$ mm is to be insulated. Asbestos with thermal conductivity $\lambda = 0.1$ W/m·K may be used. The local coefficient of heat transfer to the surroundings $\alpha_2 = 5$ W/m²·K. The problem is whether it is worth using in this case the asbestos as insulating material.

The critical diameter of insulation is:

$$d_{cr\ ins} = \frac{2\lambda_{ins}}{\alpha_2} = \frac{2 \times 0.1}{5} = 0.04 \text{ m} = 40 \text{ mm}$$

Since $d_2 < d_{cr\ ins}$, there is no point in using asbestos.

In this section the problem of critical diameter was treated for a circular cylinder. A similar effect will obviously be observed with bodies of any other geometry having differing inner and outer surfaces [2.2].

2-4. Heat Transfer Through a Spherical Wall

(a) *Boundary Conditions of the First Kind*

Consider a hollow sphere with radii r_1 and r_2, made of a material of constant thermal conductivity λ and given uniformly distributed surface temperatures t_{w1} and t_{w2}.

Since in the case considered temperature varies only in the radial direction of the sphere, the differential equation of conduction acquires in spherical

coordinates the following appearance:

$$\nabla^2 t = \frac{d^2t}{dr^2} + \frac{2}{r}\frac{dt}{dr} = 0 \tag{2-61}$$

The boundary conditions are:

$$\left.\begin{aligned} &\text{at } r = r_1, \quad t = t_{w1} \\ &\text{at } r = r_2, \quad t = t_{w2} \end{aligned}\right\} \tag{2-62}$$

Integrating Eq. (2-61) first, we get:

$$\frac{dt}{dr} = \frac{C_1}{r^2} \tag{a}$$

The second integration gives:

$$t = C_2 - \frac{C_1}{r} \tag{2-63}$$

The integration constants of Eq. (2-63) are determined from the boundary conditions (2-62). In addition to this we get:

$$C_1 = -\frac{t_{w1} - t_{w2}}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \tag{b}$$

$$C_2 = t_{w1} - \frac{t_{w1} - t_{w2}}{\frac{1}{r_1} - \frac{1}{r_2}} \frac{1}{r_1} \tag{c}$$

Substituting the values of C_1 and C_2 in Eq. (2-63), we get the expression of the temperature field for a spherical wall:

$$t = t_{w1} - \frac{t_{w1} - t_{w2}}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} \left(\frac{1}{r_1} - \frac{1}{r}\right) \tag{2-64}$$

In order to find the quantity of heat transferred through an area F of the spherical wall per unit time, we use Fourier's law:

$$Q = -\lambda \frac{dt}{dr} F = -\lambda \cdot 4\pi r^2 \frac{dt}{dr}$$

here Q is in W.

Substituting the value of the temperature gradient dt/dr in this expression, we get:

$$Q = \frac{4\pi\lambda (t_{w1} - t_{w2})}{\left(\frac{1}{r_1} - \frac{1}{r_2}\right)} = \frac{2\pi\lambda\Delta t}{\frac{1}{d_1} - \frac{1}{d_2}} = \pi\lambda \frac{d_1 d_2}{\delta} \Delta t \tag{2-65}$$

These are the calculation formulae for conduction heat transfer through a spherical wall. It follows from Eq. (2-64) that the temperature distribution across a spherical wall is represented by a hyperbola, provided thermal conductivity λ is constant.

(*b*) *Boundary Conditions of the Third Kind* (*Heat Transfer*)

The boundary conditions given include, in addition to r_1 and r_2, the temperatures t_{f1} and t_{f2}, and the local coefficients of heat transfer between the surfaces of the spherical wall and surroundings α_1 and α_2. The temperatures t_{f1}, t_{f2}, and the heat-transfer coefficients α_1 and α_2 are assumed to be constant in time, α_1 and α_2 being constant over the surfaces as well.

Since conduction is steady and the full heat flow Q, W, is constant through all isothermal surfaces, we can write:

$$Q = \alpha_1 \pi d_1^2 (t_{f1} - t_{w1})$$

$$Q = \frac{2\pi\lambda}{\frac{1}{d_1} - \frac{1}{d_2}} (t_{w1} - t_{w2})$$

$$Q = \alpha_2 \pi d_2^2 (t_{w2} - t_{f2})$$

It follows from these equations that

$$Q = \frac{\pi (t_{f1} - t_{f2})}{\frac{1}{\alpha_1 d_1^2} + \frac{1}{2\lambda}\left(\frac{1}{d_1} - \frac{1}{d_2}\right) + \frac{1}{\alpha_2 d_2^2}} = k_{sph} \pi \Delta t \qquad (2\text{-}66)$$

The value

$$k_{sph} = \frac{1}{\frac{1}{\alpha_1 d_1^2} + \frac{1}{2\lambda}\left(\frac{1}{d_1} - \frac{1}{d_2}\right) + \frac{1}{\alpha_2 d_2^2}}$$

is the overall heat-transfer coefficient of a spherical wall measured in W/K, and its reciprocal

$$\frac{1}{k_{sph}} = R_{sph} = \frac{1}{\alpha_1 d_1^2} + \frac{1}{2\lambda}\left(\frac{1}{d_1} - \frac{1}{d_2}\right) + \frac{1}{\alpha_2 d_2^2}$$

is called the thermal resistance to heat flow of a spherical wall and measured in K/W.

2-5. Generalized Solution Method for Problems of Heat Transfer Through a Plain, Cylindrical and Spherical Wall

The offered general solution can be used to calculate heat transfer through a plain, cylindrical and spherical wall, both at a constant thermal conductivity λ and with the latter depending on temperature.

Consider a unidimensional problem for the three different walls at a constant thermal conductivity of the wall material, with the temperature distribution through the wall being represented as $t = f_1(x)$ for a plain wall, $t = f_2(r)$ for a cylindrical wall, and $t = f_3(r)$ for a spherical wall.

If we admit that in the considered solids the isothermal surfaces are closed, temperature becomes a function only of the coordinate n, which is normal to the isothermal surfaces, and the amount of heat flow will be proportional to the temperature gradient $\partial t/\partial n$, and the size of the surface becomes a function of $F = F(n)$.

The isothermal surfaces of a cylinder and sphere are obviously closed, and the plate is to be considered as the extreme case of a closed system for which $n \to \infty$.

Since the isothermal surfaces are closed, the quantity of heat flowing through the wall of any of the solids considered can be expressed through

$$Q = -\lambda \frac{dt}{dn} F(n) \qquad (2\text{-}67)$$

Since Q is constant for any isothermal surface, then separating the variables in Eq. (2-67) and integrating between the limits $n = n_1$ and $n = n_2$, and the temperature interval from t_{w1} to t_{w2}, we get:

$$Q = \frac{\lambda (t_{w1} - t_{w2})}{\int_{n_1}^{n_2} \frac{dn}{F(n)}} \qquad (2\text{-}68)$$

It will be seen that formula (2-68) is similar to the expression previously derived for a plain wall:

$$q = \frac{\lambda (t_{w1} - t_{w2})}{\delta}$$

with Q being similar to the rate of heat flow q and the integral $\int_{n_1}^{n_2} \frac{dn}{F(n)}$, to the wall thickness. From here on $\int_{n_1}^{n_2} \frac{dn}{F(n)} = I_{n_1}^{n_2}$ will be referred to as the reduced wall thickness. Formula (2-68) can be used to describe the heat flow through the walls of all three geometrical forms.

The quantity $\int_{n_1}^{n_2} \frac{dn}{F(n)}$ depends only on the shape of the wall.

(a) For a plain wall (plate) $n = x$, $n_1 = 0$ and $n_2 = \delta$; $F(n) = F = \text{const}$, then

$$I_{n_1}^{n_2} = \int_{n_1}^{n_2} \frac{dn}{F} = \frac{1}{F} \int_0^{\delta} dx = \frac{\delta}{F}$$

Substituting the obtained value of $I_{n_1}^{n_2}$ into Eq. (2-68), we obtain the expression for heat flow Q through a plain wall (plate):

$$Q = \frac{\lambda (t_{w1} - t_{w2})}{\delta} F \tag{2-69}$$

(b) For a cylindrical wall $n = r$, $n_1 = r_1$ and $n_2 = r_2$; $F(n) = F(r) = 2\pi r l$, then

$$I_{n_1}^{n_2} = \int_{n_1}^{n_2} \frac{d(n)}{F(n)} = \int_{r_1}^{r_2} \frac{dr}{2\pi r l} = \frac{1}{2\pi l} \ln \frac{r_2}{r_1}$$

Substituting the obtained value of $I_{n_1}^{n_2}$ into equation (2-68), we get:

$$Q = \frac{2\pi l \lambda (t_{w1} - t_{w2})}{\ln \frac{r_2}{r_1}} \tag{2-70}$$

(c) For a spherical wall $n = r$, $n_1 = r_1$ and $n_2 = r_2$; $F(n) = F(r) = 4\pi r^2$, then

$$I_{n_1}^{n_2} = \int_{n_1}^{n_2} \frac{d(n)}{F(n)} = \int_{r_1}^{r_2} \frac{dr}{4\pi r^2} = \frac{1}{4\pi} \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

and the formula (2-68) acquires the following appearance:

$$Q = \frac{2\pi \lambda (t_{w1} - t_{w2})}{\frac{1}{d_1} - \frac{1}{d_2}} \tag{2-71}$$

Integrating the expression (2-67) between the limits n_1 and any coordinate n, and in the temperature interval from t_{w1} to t, we get the equation for the

temperature field:

$$t = t_{w1} - \frac{Q}{\lambda} \int_{n_1}^{n} \frac{dn}{F(n)}$$

Denoting the integral $\int_{n_1}^{n} \frac{dn}{F(n)} = I_{n_1}^{n}$, the latter expression can be presented as:

$$t = t_{w1} - \frac{Q}{\lambda} I_{n_1}^{n}$$

Substituting into the obtained expression the value of Q from Eq. (2-68), we get:

$$t = t_{w1} - (t_{w1} - t_{w2}) \frac{I_{n_1}^{n}}{I_{n_1}^{n_2}} \tag{2-72}$$

The ratio $I_{n_1}^{n}/I_{n_1}^{n_2}$ of Eq. (2-72) can be considered as some reduced dimensionless coordinate X which depends on the geometrical shape of the wall. Eq. (2-72) can be presented in the dimensionless form:

$$\frac{t - t_{w2}}{t_{w1} - t_{w2}} = 1 - \frac{I_{n_1}^{n}}{I_{n_1}^{n_2}} \tag{2-73}$$

If we denote the dimensionless temperature $\frac{t - t_{w1}}{t_{w1} - t_{w2}}$ by Θ and the ratio $I_{n_1}^{n}/I_{n_1}^{n_2}$ by X, Eq. (2-73) acquires the appearance:

$$\Theta = 1 - X \tag{2-73'}$$

Equation (2-73) is the general expression of the temperature field of the three walls, in dimensionless coordinates.

The reduced dimensionless coordinate present in formula (2-73') is calculated with account taken of the shape of the wall:

for a plain wall

$$X = X_p = x/\delta \tag{2-74}$$

for a cylindrical wall

$$X = X_c = \frac{\ln \frac{r}{r_1}}{\ln \frac{r_2}{r_1}} \tag{2-75}$$

for a spherical wall

$$X = X_{sph} = \frac{\left(\frac{1}{d_1} - \frac{1}{d}\right)}{\left(\frac{1}{d_1} - \frac{1}{d_2}\right)} \tag{2-76}$$

Eqs. (2-68) and (2-73') have been derived assuming the thermal conductivity of the wall to be constant. By analogy, generalized relationships can be derived for the case when the thermal conductivity λ is a function of temperature.

2-6. Intensification of Heat Transfer

(a) Intensification by Increasing Local Heat-transfer Coefficients

From the equation of heat transfer

$$Q = kF\Delta t$$

it follows that, with wall dimensions and fluid temperatures given, heat transfer depends on the overall heat-transfer coefficient k. Since heat transfer is a

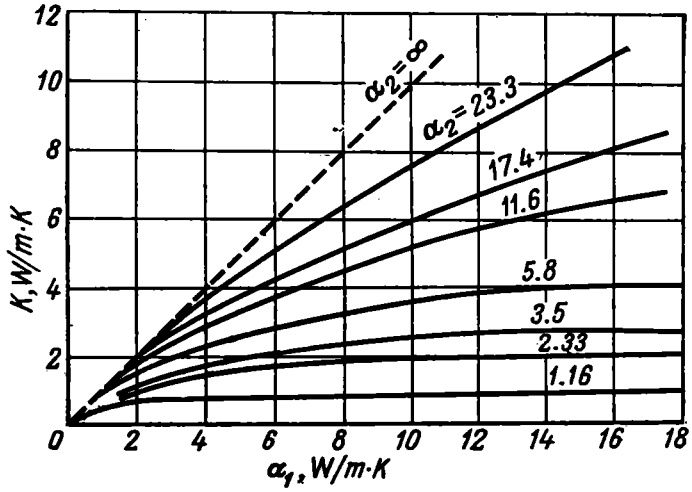

Fig. 2-11. $k = f(\alpha_1, \alpha_2)$ dependence

complicated phenomenon, a correct solution can only be found by analysis of all the particular components characteristic of the process. For instance, given a plane wall for which

$$k = \frac{1}{\frac{1}{\alpha_1} + \frac{\delta}{\lambda} + \frac{1}{\alpha_2}}$$

at $\frac{\delta}{\lambda} \to 0$ (which can be assumed for a thin wall of large thermal conductivity λ), we obtain

$$k' = \frac{1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_2}} = \frac{\alpha_1}{1 + \frac{\alpha_1}{\alpha_2}} = \frac{\alpha_2}{1 + \frac{\alpha_2}{\alpha_1}} \qquad (2\text{-}77)$$

It follows from Eq. (2-77) that the overall heat-transfer coefficient cannot exceed the smallest value of α. With $\alpha_2 \to \infty$, k' tends to its limit value α_1, and with $\alpha_1 \to \infty$ the overall heat-transfer coefficient tends to α_2.

Let us make this clear by numerical examples.

(a) 1. $\alpha_1 = 40$, $\alpha_2 = 5000$ W/m²·K;
2. $\alpha_1 = 40$, $\alpha_2 = 10\,000$ W/m²·K.

From Eq. (2-77) we find that the overall heat-transfer coefficients are:

$$k'_1 = 39.7 \text{ W/m}^2\cdot\text{K} \quad \text{and} \quad k'_2 = 39.8 \text{ W/m}^2\cdot\text{K}$$

(b) 1. $\alpha_1 = 80$ W/m²·K, $\alpha_2 = 5000$ W/m²·K;
2. $\alpha_1 = 200$ W/m²·K, $\alpha_2 = 5000$ W/m²·K.

For the second case the overall heat-transfer coefficients are:

$$k_1' = 78.8 \text{ W/m}^2\cdot\text{K} \quad \text{and} \quad k_2' = 192 \text{ W/m}^2\cdot\text{K}$$

From this example it is clear that with $\alpha_1 \ll \alpha_2$ an increase of the larger of the two heat-transfer coefficients (α_2) causes practically no increase of k_1'. A two- or five-fold increase of the smaller of the two coefficients, (α_1), results in nearly the same increase of k'.

Figure 2-11 shows the dependence of $k' = f(\alpha_1, \alpha_2)$ in accordance with Eq. (2-77). From the graph it follows that with an increasing α_1, k' increases rapidly till α_1 becomes equal to α_2. As soon as α_1 becomes larger than α_2, the rate at which k' is increased diminishes, and further increase in α_1 brings almost no increase in k'. Consequently, when $\alpha_1 \ll \alpha_2$, α_1 should be increased in order to increase k', i.e. the larger of the two thermal resistances, $\frac{1}{\alpha_1}$ should be diminished. In other words, with $\alpha_1 \ll \alpha_2$, it is possible to increase k' only by raising α_1. If $\alpha_1 \cong \alpha_2$, the overall heat-transfer coefficient can be increased by increasing either α.

(b) Intensification of Heat Transfer by Finning Walls

With heat transfer through a cylindrical wall the thermal resistances $\frac{1}{\alpha_1 d_1}$ and $\frac{1}{\alpha_2 d_2}$ depend not only on the values of the heat-transfer coefficients, but also on the size of the surfaces. For a spherical wall the effect of the diameters d_1 and d_2 is still more pronounced as it may be seen from the ratios $1/\alpha_1 d_1^2$ and $1/\alpha_2 d_2^2$. Hence it follows that when α is small thermal resistance can be reduced by increasing the corresponding surface area. The same result may also be ensured for a plane wall, if one of its surfaces is extended with fins. The latter fact is used as the basis for intensifying heat transfer by finning walls. For a finned surface the thermal resistances become proportional to $\frac{1}{\alpha_1 F_1}$ and $\frac{1}{\alpha_2 F_2}$.

It should be mentioned that it is necessary, when using this method to intensify heat transfer, to be guided by the following considerations: if $\alpha_1 \ll \alpha_2$, the surface should be finned on the side of α_1, until $\alpha_1 F_1$ becomes equal to $\alpha_2 F_2$. A further increase of surface F_1 is of little use. Finned surfaces may be made either as solid castings or by attachment of individual fins.

An exact analytical solution of the problem of heat propagation in a fin involves considerable difficulties. An approximate solution can be found using certain assumptions which make it possible to solve the problem in a relatively simple way. A method for solving problems of heat conduction through fins of a simple geometry is considered below.

2-7. Heat Conduction Through a Rod (Fin) of Constant Cross-section

(a) Differential Equation and Its Solution

Fins may be of various cross-sections (rectangular, circular, triangular and so on, including irregular geometric forms). Consider the propagation of heat in a straight rod of constant cross-section. Let the cross-sectional area be denoted by f and its perimeter by u. The rod is placed in a medium of constant tempera-

ture t_f; the coefficient of heat transfer from the surface of the rod to the surroundings is assumed to be constant for the entire surface. Suppose also that the thermal conductivity of the rod material λ is quite large, and the cross-section of the rod extremely small in relation to its length. The latter assumption allows us to neglect temperature variations over the cross-section and to consider that the temperature changes only along the axis of the rod. To simplify calculation, t_f = const will be considered as a reference temperature. The excess in temperature of the rod over this reference value will be denoted by ϑ. It is obvious that

$$\vartheta = t - t_f$$

where t_f = temperature of the medium;
t = rod temperature.

For a given temperature t_1 of the base of the rod, the excess temperature of the base (Fig. 2-12) is:

$$\vartheta_1 = t_1 - t_f$$

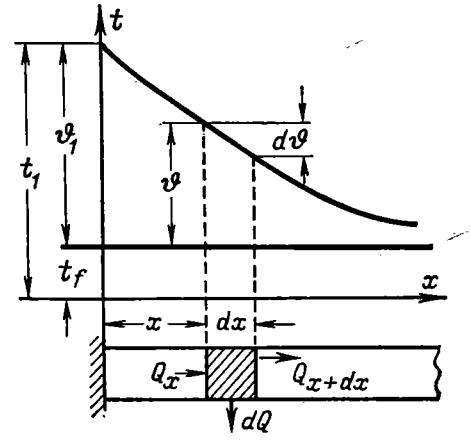

Fig. 2-12. Heat transfer through a rod

Consider a rod element dx long at a distance x from the base. The heat balance equation for this rod element can be written as follows:

$$Q_x - Q_{x+dx} = dQ \qquad \text{(a)}$$

where Q_x = quantity of heat entering the left face of the element per unit time;
Q_{x+dx} = quantity of heat flowing from the opposite face of the element in the same time;
dQ = quantity of heat lost per unit time to the surroundings by the outer surface of the rod.

In accordance with Fourier's law

$$Q_x = -\lambda \frac{d\vartheta}{dx} f$$

and

$$Q_{x+dx} = -\lambda \frac{d}{dx}\left(\vartheta + \frac{d\vartheta}{dx}\,dx\right) f$$

whence

$$Q_{x+dx} = -\lambda f \frac{d\vartheta}{dx} - \lambda f \frac{d^2\vartheta}{dx^2}\,dx$$

Consequently,

$$Q_x - Q_{x+dx} = \lambda f \frac{d^2\vartheta}{dx^2}\,dx \qquad \text{(b)}$$

On the other hand, according to Newton's law

$$dQ = \alpha_f \vartheta u\,dx \qquad \text{(c)}$$

Equating (b) and (c), we get the following differential equation for the temperature distribution in the rod:

$$\frac{d^2\vartheta}{dx^2} = \frac{\alpha_f u}{\lambda f}\,\vartheta = m^2\vartheta \qquad (2\text{-}78)$$

where

$$m = +\sqrt{\frac{\alpha_f u}{\lambda f}} \qquad \text{(d)}$$

here m is in 1/m.

It is clear from expression (d) that for a fin of preset shape and size m is constant, provided that the heat-transfer coefficient α_f is constant over the entire surface, and the thermal conductivity λ is constant within the considered temperature range. Then, the common integral for Eq. (2-78) will be

$$\vartheta = C_1 e^{mx} + C_2 e^{-mx} \qquad (2\text{-}79)$$

The constants C_1 and C_2 are determined from the boundary conditions, which may be given in several ways, depending on the length of the rod and other factors.

(*b*) *Infinite Rod*

A constant temperature is maintained at the base or in the initial cross-section of the rod, i.e. at $x = 0$, $\vartheta = \vartheta_1$. If the rod is of infinite length ($l \to \infty$), all the heat imparted to it will be lost to the surroundings, and at $x \to \infty$, $\vartheta = 0$.

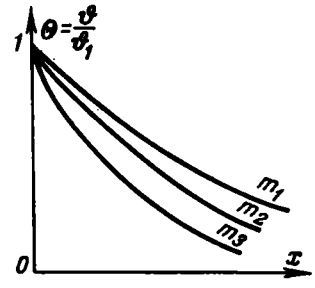

Fig. 2-13. Temperature variation in a rod length

Substituting the boundary conditions in Eq. (2-79) gives

$$\text{at } x = 0, \qquad \vartheta_1 = C_1 + C_2$$
$$\text{at } x \to \infty, \qquad C_1 e^{\infty} = 0$$

The second equality is valid only when $C_1 = 0$. Thus, $C_2 = \vartheta_1$. Substituting the values of C_1 and C_2 in Eq. (2-79), we get:

$$\vartheta = \vartheta_1 e^{-mx} \qquad (2\text{-}80)$$

Equality (2-80) may also be written as follows:

$$\Theta = \frac{\vartheta}{\vartheta_1} = e^{-mx} \qquad (2\text{-}80')$$

where Θ is the dimensionless temperature expressed as a fraction of the temperature of the rod base ϑ_1.

Figure 2-13 shows the dependence of the dimensionless temperature Θ on the length of a rod for different values of m ($m_1 < m_2 < m_3$).

Figure 2-13 shows that the dimensionless temperature falls more with increasing factor m. With $x \to \infty$ all curves approach $\Theta = 0$ asymptotically.

From the equation $m = \sqrt{\frac{\alpha_f u}{\lambda f}}$ it follows that m is proportional to the rate of heat removal from the side surface and inversely proportional to $\sqrt{\lambda f}$, i.e. to the factor determining heat transfer by conduction along the rod. Consequently, fins should be made of a material of high thermal conductivity, which will give a lower m and preserve large excess temperatures along the rod. When $\frac{\alpha_f}{\lambda}$ is constant, m increases with increasing $\frac{u}{f}$. This indicates that the more effective fins are those with a smaller ratio $\frac{u}{f}$ at a given cross-sectional area.

The quantity of heat removed from the rod to the surroundings will seemingly be equal to that passing through its base.

The heat flow is passed through the base of the rod

$$Q = -\lambda f \left(\frac{d\vartheta}{dx}\right)_{x=0}$$

Here Q is in W.

From Eq. (2-80) we find:

$$\left(\frac{d\vartheta}{dx}\right)_{x=0} = -me^{-mx}\vartheta_1|_{x=0} = -m\vartheta_1$$

Substituting the temperature gradient at $x = 0$ in the preceding equation of the heat flow, we obtain the quantity of heat (heat flow) imparted to, or removed by, the rod to its surroundings:

$$Q = \lambda f m \vartheta_1 = \vartheta_1 \sqrt{\alpha_f u \lambda f} \tag{2-81}$$

(c) Finite Rod

The differential equation (2-78) and its solution (2-79) are also valid for a finite rod, but the boundary conditions are different:

$$\left.\begin{aligned} &\text{at } x=0, \quad \vartheta = \vartheta_1 \\ &\text{at } x=l, \quad -\lambda\left(\frac{d\vartheta}{dx}\right)_{x=l} = \alpha_l \vartheta_l \end{aligned}\right\} \tag{2-82}$$

or

$$\left.\left(\frac{d\vartheta}{dx}\right)_{x=l} = -\frac{\alpha_l}{\lambda}\vartheta_l \right\}$$

where ϑ_l = temperature at the end of the rod;

α_l = coefficient of heat transfer from the end of the rod.

When $x = l$, the quantity of heat imparted to the end of the rod by conduction equals the quantity lost from the end of the rod to the surroundings.

The constants C_1 and C_2 in Eq. (2-79) are determined by using boundary conditions (2-82):

$$\left.\begin{aligned} &\text{at } x=0, \quad \vartheta_1 = C_1 + C_2 \\ &\text{at } x=l, \quad \left(\frac{d\vartheta}{dx}\right)_{x=l} = C_1 m e^{ml} - C_2 m e^{-ml} = -\frac{\alpha_l}{\lambda}\vartheta_l \end{aligned}\right\} \tag{2-82'}$$

and

$$\left.\vartheta_l = C_1 e^{ml} + C_2 e^{-ml} \right\}$$

From Eqs. (2-82), C_1 and C_2 are determined as follows:

$$C_1 = \frac{\vartheta_1\left(m - \frac{\alpha_l}{\lambda}\right)}{e^{2ml}\left(m + \frac{\alpha_l}{\lambda}\right) + m - \frac{\alpha_l}{\lambda}}$$

$$C_2 = \vartheta_1 \frac{e^{2ml}\left(m + \frac{\alpha_l}{\lambda}\right)}{e^{2ml}\left(m + \frac{\alpha_l}{\lambda}\right) + m - \frac{\alpha_l}{\lambda}}$$

Substituting C_1 and C_2 in Eq. (2-79), we obtain:

$$\vartheta = \vartheta_1 \left[\frac{e^{mx}\left(m - \frac{\alpha_l}{\lambda}\right)}{e^{2ml}\left(m + \frac{\alpha_l}{\lambda}\right) + m - \frac{\alpha_l}{\lambda}} + \frac{e^{-mx}e^{2ml}\left(m + \frac{\alpha_l}{\lambda}\right)}{e^{2ml}\left(m + \frac{\alpha_l}{\lambda}\right) + m - \frac{\alpha_l}{\lambda}} \right] \tag{2-83}$$

Multiplying and dividing the right side of Eq. (2-83) by e^{-ml} and making simple algebraic transformations, we get:

$$\vartheta = \vartheta_1 \left[\frac{m\left[e^{m(l-x)} + e^{-m(l-x)}\right] + \frac{\alpha_l}{\lambda}\left[e^{m(l-x)} - e^{-m(l-x)}\right]}{m\left(e^{ml} + e^{-ml}\right) + \frac{\alpha_l}{\lambda}\left(e^{ml} - e^{-ml}\right)} \right]$$

Recalling that

$$\frac{e^x + e^{-x}}{2} = \cosh(x) \quad \text{and} \quad \frac{e^x - e^{-x}}{2} = \sinh(x)$$

equation (2-83) may be written in the following form:

$$\vartheta = \vartheta_1 \frac{\cosh\left[m(l-x)\right] + \frac{\alpha_l}{\lambda m}\sinh\left[m(l-x)\right]}{\cosh(ml) + \frac{\alpha_l}{\lambda m}\sinh(ml)} \tag{2-83'}$$

When the loss of heat from the end of the rod can be neglected, the boundary conditions (2-82) can be presented as follows:

at $x = 0$, $\quad \vartheta = \vartheta_1$;

at $x = l$, $\quad \left(\frac{d\vartheta}{dx}\right)_{x=l} = 0$

The latter can be admitted for the case where α_l is small at the end of the rod, and the thermal conductivity λ of the material is large and the ratio $\frac{\alpha_l}{\lambda} \to 0$, i.e. the loss of heat from the end of the rod can be neglected.

With these boundary conditions the second terms of the numerator and denominator of Eq. (2-83') turn into zero and the equation acquires the appearance:

$$\vartheta = \vartheta_1 \frac{\cosh\left[m(l-x)\right]}{\cosh(ml)} \tag{2-84}$$

here ϑ is in °C.

Eqs. (2-83') and (2-84) can be used to calculate temperature in any cross-section of the rod. The fraction of heat lost from the end of the rod is usually small compared to the quantity of heat lost from the surface of the fins, and formula (2-84) is usually used for practical engineering calculations.

In the extreme case, when $x = l$, formula (2-84) acquires the following form:

$$\vartheta_{x=l} = \frac{\vartheta_1}{\cosh(ml)}$$

The amount of heat Q_f lost to the surroundings from the surface of a fin is equal to the amount of heat imparted to its base:

$$Q_f = -\lambda f \left(\frac{d\vartheta}{dx}\right)_{x=0}$$

From Eq. (2-84) we find:

$$\left(\frac{d\vartheta}{dx}\right)_{x=0} = -\vartheta_1 m \frac{\sinh(ml)}{\cosh(ml)} = -\vartheta_1 m \tanh(ml)$$

Then

$$Q_f = \lambda f \vartheta_1 m \tanh (ml) \tag{2-85}$$

Substituting $m = \sqrt{\frac{\alpha_f u}{\lambda f}}$ in Eq. (2-85), we get:

$$Q_f = \vartheta_1 \sqrt{\alpha_f u \lambda f} \tanh (ml) \tag{2-85'}$$

If the rod is very long, then $\cosh (ml) \to \infty$, and $\tanh (ml) \cong 1$. Consequently,

$$\vartheta_{x=l} = 0$$

and Eq. (2-85) is transformed into Eq. (2-81).

2-8. Heat Transfer Through a Finned Plane Wall

The problem is to find the rate of heat flow through a finned wall of infinite area, with fins located on the side of the smaller heat-transfer coefficient (Fig. 2-14).

The constant values of the heat transfer α_1 are given on the finless side of the wall, the flat finned surface α_w and the surface of the fins themselves, α_f. The size of the fins is also given (Fig. 2-14), and the temperatures of the heat-carrying agents, t_{f1} and t_{f2}.

Since the width of a fin is larger than its thickness ($b \gg \delta$), we assume the perimeter of a fin cross-section to be $u = 2b$. The cross-sectional area of a fin is therefore $f = b\delta$.

Consequently,

$$m = \sqrt{\frac{\alpha_f u}{\lambda f}} = \sqrt{\frac{2\alpha_f}{\lambda \delta}}, \ \text{m}^{-1}$$

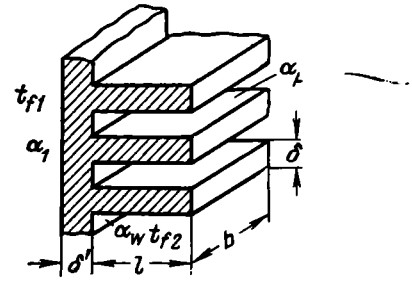

Fig. 2-14. Heat transfer through a finned wall

Substituting this expression in Eq. (2-85), and multiplying and dividing by $2l$, we get:

$$Q_f = \vartheta_1 \sqrt{\alpha_f 2b\lambda b\delta} \frac{2l}{2l} \tanh \left(\frac{l}{\delta}\sqrt{\frac{2\alpha_f \delta}{\lambda}}\right) = \alpha_f \vartheta_1 F_f \frac{\tanh \left(\frac{l}{\delta}\sqrt{\frac{2\alpha_f\delta}{\lambda}}\right)}{\frac{l}{\delta}\sqrt{\frac{2\alpha_f\delta}{\lambda}}}$$

where $\frac{\alpha_f \delta}{\lambda} = \text{Bi}$ is a dimensionless group, known as the *Biot number*, which is an important characteristic of the process of heat conduction. The Biot number represents the ratio between internal thermal resistance to conduction and external thermal resistance to heat transfer:

$$\text{Bi} = \frac{\frac{\delta}{\lambda}}{\frac{1}{\alpha_f}}$$

The equation for heat flow from the surface of a fin can be written in final form as follows:

$$Q_f = \alpha_f \vartheta_1 F_f \frac{\tanh \left(\frac{l}{\delta}\sqrt{2\text{Bi}}\right)}{\frac{l}{\delta}\sqrt{2\text{Bi}}} \tag{2-86}$$

Let us denote $\dfrac{\tanh\left(\frac{l}{\delta}\sqrt{2\mathrm{Bi}}\right)}{\frac{l}{\delta}\sqrt{2\mathrm{Bi}}}$ by E, known as the *fin efficiency*. Then equation (2-86) acquires the form:

$$Q_f = \alpha_f \vartheta_1 F_f E$$

The fin efficiency $E = f\left(\frac{l}{\delta}\sqrt{2\mathrm{Bi}}\right)$ tends to its maximum of unity when $\frac{l}{\delta}\sqrt{2\mathrm{Bi}} \to 0$ (for the given fin dimensions, this is possible when $\lambda \to \infty$, i.e. $\mathrm{Bi} \to 0$).

The amount of heat Q_w, W, removed from the flat surface between fins is:

$$Q_w = \alpha_w \vartheta_1 F_w$$

and the total quantity of heat is

$$Q = Q_f + Q_w = \alpha_f \vartheta_1 F_f E + \alpha_w \vartheta_1 F_w \qquad \text{(a)}$$

or

$$\left.\begin{aligned} Q &= \alpha_{red} \vartheta_1 F_{fw} \\ F_{fw} &= F_f + F_w \end{aligned}\right\} \qquad \text{(b)}$$

Correlation of (a) and (b) shows that

$$\alpha_{red} = \alpha_f E \frac{F_f}{F_{fw}} + \alpha_w \frac{F_w}{F_{fw}} \qquad (2\text{-}87)$$

The value α_{red} in Eq. (2-87) is called the *reduced heat-transfer coefficient*, and is a mean heat-transfer coefficient for a finned wall, accounting for the removal of heat from the fin surfaces and the plain section of the wall between fins, and for fin efficiency.

The process of heat transfer through a finned wall may then be described by the following set of equations:

$$Q = \alpha_1 F_1 (t_{f1} - t_{w1})$$

$$Q = \frac{\lambda}{\delta'} (t_{w1} - t_{w2}) F_1$$

$$Q = \alpha_{red} (t_{w2} - t_{f2}) F_{fw}$$

Here δ'—see Fig. 2-14. From these we obtain:

$$Q = \frac{t_{f1} - t_{f2}}{\frac{1}{\alpha_1 F_1} + \frac{\delta'}{\lambda F_1} + \frac{1}{\alpha_{red} F_{fw}}} \qquad (2\text{-}88)$$

If we relate the heat flow to unit area of the finned wall surface, then

$$\frac{Q}{F_{fw}} = q_{fw} = \frac{t_{f1} - t_{f2}}{\frac{1}{\alpha_1}\frac{F_{fw}}{F_1} + \frac{\delta'}{\lambda}\frac{F_{fw}}{F_1} + \frac{1}{\alpha_{red}}} = k_{fw} (t_{f1} - t_{f2}) \qquad (2\text{-}89)$$

where

$$k_{fw} = \frac{1}{\frac{1}{\alpha_1}\frac{F_{fw}}{F_1} + \frac{\delta'}{\lambda}\frac{F_{fw}}{F_1} + \frac{1}{\alpha_{red}}}$$

is the overall heat-transfer coefficient of a finned wall, with heat flow being reduced to the finned surface of the wall, $\mathrm{W/m^2 \cdot K}$.

If we reduce the heat flow to the area of non-finned surface of the wall, we get:

$$\frac{Q}{F_1} = q_1 = \frac{t_{f1} - t_{f2}}{\frac{1}{\alpha_1} + \frac{\delta'}{\lambda} + \frac{1}{\alpha_{red}} \frac{F_1}{F_{fw}}} = k_1 (t_{f1} - t_{f2}) \quad (2\text{-}90)$$

where

$$k_1 = \frac{1}{\frac{1}{\alpha_1} + \frac{\delta'}{\lambda} + \frac{1}{\alpha_{red}} \frac{F_1}{F_{fw}}}$$

is the overall heat-transfer coefficient for heat flow reduced to the non-finned surface of the wall.

The ratio of the finned surface, F_{fw}, to the plane one, F_1, is called the *finning factor.*

The effect of finning on the heat-transfer coefficient can be illustrated by an example. Let $\alpha_1 = 1000$ and $\alpha_2 = 20$ W/m²·K. Assuming the ratio δ'/λ to be negligible, we have

$$k_1' = \frac{1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_{red}} \frac{F_1}{F_{fw}}}$$

For a plane surface (with the finning factor F_{fw}/F_1 equal to unity) we get:

$$k_1' = \frac{1}{\frac{1}{1000} + \frac{1}{20}} \cong 20 \text{ W/m}^2\text{·K}$$

If a wall has fins on one side, and the finning factor is $F_{fw}/F_1 = 2$, then

$$k_1' = \frac{1}{\frac{1}{1000} + \frac{1}{20} \cdot \frac{1}{2}} \cong 40 \text{ W/m}^2\text{·K}$$

Consequently, with any given values of the heat-transfer coefficients, the rate of heat flow is about doubled by putting fins on the side of a plane wall with the smaller α, and observing a finning factor $\frac{F_{fw}}{F_1} = 2$.

2-9. Heat Transfer Through a Circular Fin of Constant Thickness

The calculation of fins with a variable cross-section is more complicated than that of straight fins of constant thickness. Consider the case of heat transfer through a circular fin of constant thickness (Fig. 2-15), such as those employed for extending cylindrical surfaces (tubes).

The data given for the fin include its inner radius r_1, outer radius r_2, thickness δ, and thermal conductivity λ. The ambient temperature t_f is constant. The excess temperature of the fin is:

$$\vartheta = t - t_f$$

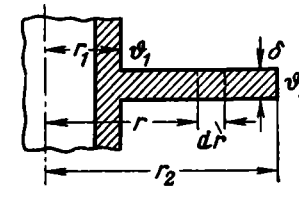

Fig. 2-15. Heat transfer through a circular fin of constant thickness

The constant heat-transfer coefficient α for the entire surface of the fin, and the temperature ϑ_1 at its base are given.

The process is a steady-state one with temperature varying only along the height of the fin. From these conditions, we find the differential equation that

describes conduction of heat through the fin, first working out the energy balance equation for an annular element of the fin dr thick:

$$Q_r - Q_{r+dr} = dQ \tag{2-91}$$

Finding the terms of Eq. (2-91), we obtain a differential equation of the following form:

$$\frac{d^2\vartheta}{dr^2} + \frac{1}{r}\frac{d\vartheta}{dr} - \frac{2\alpha}{\lambda\delta}\vartheta = 0 \tag{2-92}$$

Denoting $\frac{2\alpha}{\lambda\delta} = m^2$, $mr = z$ and $\frac{1}{r} = \frac{m}{z}$, and substituting $\frac{d\vartheta}{dr} = \frac{d\vartheta}{dz}m$ and $\frac{d^2\vartheta}{dr^2} = m^2\frac{d^2\vartheta}{dz^2}$, Eq. (2-92) becomes:

$$\frac{d^2\vartheta}{dz^2} + \frac{1}{z}\frac{d\vartheta}{dz} - \vartheta = 0 \tag{2-93}$$

Equation (2-93) is the Bessel equation whose general solution is:

$$\vartheta = C_1 I_0(z) + C_2 K_0(z) \tag{2-94}$$

where $I_0(z) = I_0(mr)$ = modified Bessel function of the first kind and zero-order imaginary argument

$K_0(z) = K_0(mr)$ = modified Bessel function of the second kind and zero-order imaginary argument.

These functions have the following properties:

at $r = 0$, $I_0(mr) = 1$ and $K_0(mr) \to \infty$

at $r = \infty$, $I_0(mr) \to \infty$ and $K_0(mr) = 0$

The constants C_1 and C_2 are determined from the boundary conditions.

If the loss of heat from the tip of the circular fin is assumed to be negligible, the calculation formulae acquire the following form:

for the current temperature along the fin

$$\vartheta = \vartheta_1 \frac{I_0(mr)\,K_1(mr_2) + I_1(mr_2)\,K_0(mr)}{I_0(mr_1)\,K_1(mr_2) + I_1(mr_2)\,K_0(mr_1)} \tag{2-95}$$

for the temperature at the tip of the fin

$$\vartheta_2 = \vartheta_1 \frac{I_0(mr_2)\,K_1(mr_2) + I_1(mr_2)\,K_0(mr_2)}{I_0(mr_1)\,K_1(mr_2) + I_1(mr_2)\,K_0(mr_1)} \tag{2-96}$$

for the quantity of heat

$$Q = -\lambda 2\pi r_1 \delta \left(\frac{d\vartheta}{dr}\right)_{r=r_1} = 2\pi r_1 \alpha\delta m \vartheta_1 \psi \tag{2-97}$$

where

$$\psi = \frac{I_1(mr_2)\,K_1(mr_1) - I_1(mr_1)\,K_1(mr_2)}{I_0(mr_1)\,K_1(mr_2) + I_1(mr_2)\,K_0(mr_1)}$$

In using these formulae, the heat flow from the tip of the fin can be accounted for by conditionally increasing its height (r_2) by half its thickness.

The formulae obtained, (2-95), (2-96), and (2-97) are cumbersome and inconvenient for engineering calculations. Circular fins of constant thickness and various straight fins of variable cross-section may, therefore, be treated and calculated as straight fins of constant cross-section. Then the quantity of heat lost by the surface of a circular fin of constant thickness is

$$Q' = \varepsilon' F' q \tag{2-98}$$

where Q' = quantity of heat lost by the circular fin, W;
F' = area of the circular fin, m^2;
$q = \frac{Q}{F}$ = quantity of heat lost per unit time from unit area of a straight fin of thickness equal to that of the circular fin and one metre long;
$\varepsilon' = f\left(\frac{\vartheta_2}{\vartheta_1}, \frac{r_2}{r_1}\right)$ = correction factor determined from the graph in Fig. 2-16.

Here $\frac{\vartheta_2}{\vartheta_1}$ is the ratio of temperatures at the base and tip of the fin, calculated by the formulae derived for a straight fin of constant cross-section.

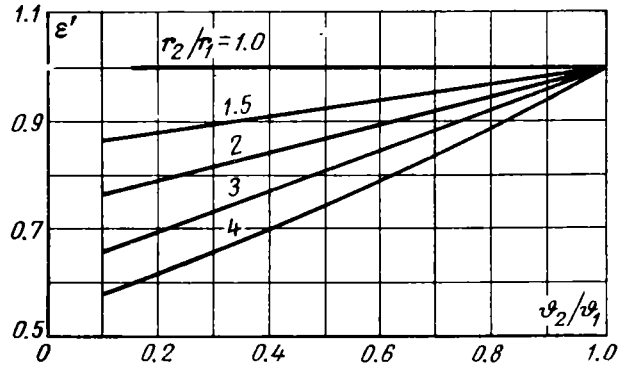

Fig. 2-16. $\varepsilon' = f(\vartheta_2/\vartheta_1;\ r_2/r_1)$—auxiliary calculation graph for circular fins of constant thickness

Thus, by calculating the temperature at the tip of the fin and the rate of heat flow for a straight fin, and substituting q and ε' in Eq. (2-98), we obtain the heat flow for a circular fin.

2-10. Conduction Through a Straight Fin of Variable Thickness

In the design of cooling systems for a variety of machines and equipment, particularly for aircraft, solution of the problem of maximum heat transfer at minimum weight of the heat-exchanger is of special importance. The question arises: what is the optimal profile for a fin with the lowest weight at a given rate of heat transfer.

Fin of minimum weight [2.3]. The essence of the problem resolves itself into the requirement that all sections of the fin should be of equal efficiency, i.e. the specific rate of heat flow must remain constant over the entire cross-section of the fin. This means that the lines of heat flow must be parallel to the axis of the fin. Under such conditions, temperature distribution along the lines of heat flow will be linear (Fig. 2-17).

With the temperature at the base of the fin t_1 given, and with the temperature at the fin tip close to the ambient temperature t_f, the problem becomes one-dimensional; therefore we may write for any cross-section of the fin:

$$t - t_f = \frac{x}{h}(t_1 - t_f) \qquad (2\text{-}99)$$

where x = distance measured along the axis of the fin from its tip;
h = total height of the fin.

Consider an elementary surface of the fin at a distance x. Let this elementary surface form an angle φ with the fin axis. If the rate of heat flow along the fin axis is q, the heat flow through the elementary surface will be $q \sin \varphi$ (Fig. 2-17). The following relationship must therefore hold:

$$q \sin \varphi = \alpha (t - t_f)$$

or

$$q \sin \varphi = \frac{\alpha}{h} x (t_1 - t_f) \qquad (2\text{-}100)$$

It follows from Eq. (2-100) that the angle φ is a function only of x, i.e.

$$\sin \varphi = \frac{\alpha \vartheta_1}{qh} x \qquad (2\text{-}100')$$

The contour of the fin found in this way is a circular arc of radius r, since $\sin \varphi = \frac{x}{r}$. It follows from Eq. (2-100') that $r = \frac{qh}{\alpha \vartheta_1}$. It has been shown that a fin with a profile formed by circular arcs possesses the least weight. Such a fin differs little in weight from a straight triangular fin; technologically, however, the latter is easier to manufacture which is why fins of this kind are more common in practice than circular-profile fins.

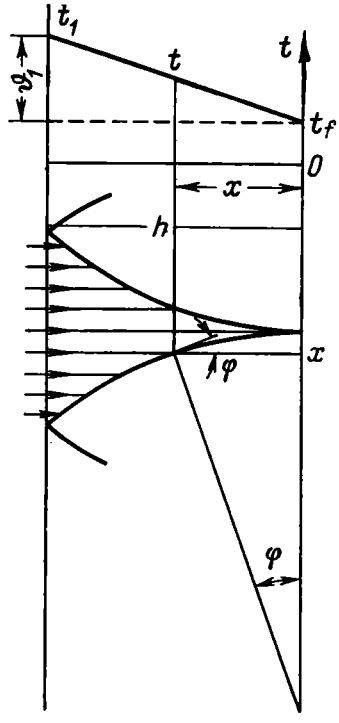

Fig. 2-17. Section through a fin of minimum weight

Triangular and trapezoidal fins. Straight fins of both triangular (pointed) and trapezoidal (truncated) profile are widely used in engineering practice.

Let us assume that the size of a trapezoidal fin (Fig. 2-18) and the excess temperature at its base ϑ_1 are given. It is convenient to take the origin of coordinates at the apex of the triangle, with the x-axis directed along the axis of symmetry of the fin. The direction of heat flow is then opposite to the positive direction of the x-axis [2.4].

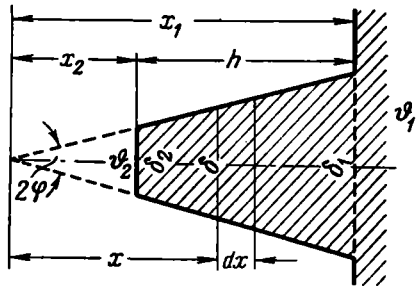

Fig. 2-18. Heat transfer through a straight fin of trapezoidal section

The cross-sectional area f of such a fin is a function only of coordinate x, i.e.

$$f = l\delta = 2lx \tan \varphi \qquad (a)$$

The quantity of heat dissipated to the surroundings from the element dx is equal to:

$$d\left(\lambda f \frac{d\vartheta}{dx}\right) = \alpha u \vartheta \, dx' \qquad (b)$$

where α = heat-transfer coefficient of the fin surface;
u = perimeter of the fin profile at a distance x, which can be expressed as $u = 2l$; $dx' = dx/\cos \varphi$.

Differentiating (b) and accounting for (a), we obtain:

$$\frac{d^2\vartheta}{dx^2}+\frac{1}{x}\frac{d\vartheta}{dx}-\frac{1}{x}\frac{\alpha}{\lambda\sin\varphi}=0 \qquad (c)$$

Upon introduction of a new variable $z=\frac{\alpha}{\lambda\sin\varphi}x$ it becomes:

$$\frac{d^2\vartheta}{dz^2}+\frac{1}{z}\frac{d\vartheta}{dz}-\frac{1}{z}\vartheta=0 \qquad (2\text{-}101)$$

The differential equation (2-101) is a modified Bessel equation whose solution is

$$\vartheta=C_1I_0(2\sqrt{z})+C_2K_0(2\sqrt{z}) \qquad (2\text{-}102)$$

where I_0 and K_0 are modified Bessel functions of the first and second kind of imaginary argument.

The integration constants C_1 and C_2 of Eq. (2-102) are found from the boundary conditions, which are here:

$$\text{at } x=x_1,\ \vartheta=\vartheta_1$$

and, neglecting the dissipation of heat from the tip of the fin, we have:

$$\text{at } x=x_2,\quad \vartheta=\vartheta_2,\quad \text{and}\quad \left(\frac{d\vartheta}{dx}\right)_{x=x_2}=0$$

Having determined the integration constants C_1 and C_2, we obtain:
for the current temperature along the fin

$$\vartheta=\vartheta_1\frac{I_0(2\sqrt{z})K_1(2\sqrt{z_2})+I_1(2\sqrt{z_2})K_0(2\sqrt{z})}{I_0(2\sqrt{z_1})K_1(2\sqrt{z_2})+I_1(2\sqrt{z_2})K_0(2\sqrt{z_1})} \qquad (2\text{-}103)$$

for the temperature at the tip of the fin

$$\vartheta_2=\vartheta_1\frac{I_0(2\sqrt{z_2})K_1(2\sqrt{z_2})+I_1(2\sqrt{z_2})K_0(2\sqrt{z_2})}{I_0(2\sqrt{z_1})K_1(2\sqrt{z_2})+I_1(2\sqrt{z_2})K_0(2\sqrt{z_1})} \qquad (2\text{-}104)$$

The heat flow can be determined by Fourier's law

$$Q=-\lambda f_1\left(\frac{d\vartheta}{dx}\right)_{x=x_1}=\frac{\alpha\vartheta_1\delta_1 l}{\sqrt{z_1}\sin\varphi}\times$$
$$\times\left[\frac{I_1(2\sqrt{z_1})K_1(2\sqrt{z_2})-I_1(2\sqrt{z_2})K_1(2\sqrt{z_1})}{I_0(2\sqrt{z_1})K_1(2\sqrt{z_2})+I_1(2\sqrt{z_2})K_0(2\sqrt{z_1})}\right] \qquad (2\text{-}105)$$

When these formulae are used, the dissipation of heat from the tip can be accounted for by fictitiously increasing the height of the fin h by half of its thickness $\delta_2/2$.

For a fin of triangular profile, $x_2=0$; consequently, $z_2=0$, $I_1(0)=0$, and formulae (2-103), (2-104) and (2-105) become:

$$\vartheta=\vartheta_1\frac{I_0(2\sqrt{z})}{I_0(2\sqrt{z_1})} \qquad (2\text{-}106)$$

$$\vartheta_2=\vartheta_1\frac{1}{I_0(2\sqrt{z_1})} \qquad (2\text{-}107)$$

$$Q=\frac{\alpha\delta_1\vartheta_1 l}{\sqrt{z_1}\sin\varphi}\left[\frac{I_1(2\sqrt{z_1})}{I_0(2\sqrt{z_1})}\right] \qquad (2\text{-}108)$$

Maximum heat flow through a triangular fin of given weight will take place, provided that the equality

$$\frac{h}{\delta_1/2} = 1.309 \sqrt[3]{\frac{2\lambda}{\alpha\delta_1}} \tag{2-109}$$

is fulfilled.

Formulae (2-103), (2-104) and (2-105) are cumbersome and inconvenient for practical use. Therefore variable-section fins are calculated as fins of constant cross-section or profile.

Then

$$Q'' = \varepsilon'' q F'' \tag{2-110}$$

where Q'' = quantity of heat flowing per unit time;

F'' = cooling surface of the fin;

$q = \frac{Q}{F}$ = rate of heat flow for a straight fin whose length, height and thickness are respectively equal to the length, height, and mean thickness of a tapered fin;

$\varepsilon'' = f\left(\frac{\vartheta_2}{\vartheta_1}, \frac{\delta_2}{\delta_1}\right)$ = correction factor on a tapered fin; ε'' is determined from the chart in Fig. 2-19.

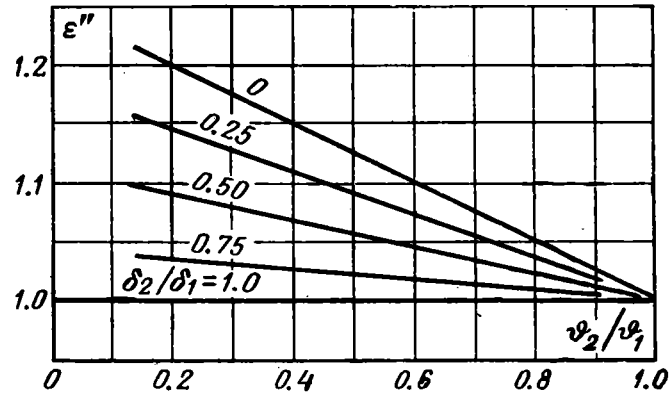

Fig. 2-19. $\varepsilon'' = f(\vartheta_2/\vartheta_1, \delta_2/\delta_1)$—auxiliary calculation graph for fins of trapezoidal and triangular section

The lower curve $\left(\text{at } \frac{\delta_2}{\delta_1} = 1\right)$ corresponds to a straight fin of constant section, and the upper curve $\frac{\delta_2}{\delta_1} = 0$ corresponds to a triangular fin.

The ratio $\frac{\vartheta_2}{\vartheta_1}$ is calculated by formula (2-84), the heat transfer from the tip of the fin being taken into account by increasing the height of the fin h by half of its thickness.

2-11. Conduction Through a Flat Semi-finite Homogeneous Plate

Consider a flat homogeneous plate of thickness δ, of constant thermal conductivity λ, and of infinite size along the Oy-axis (Fig. 2-20) [2.5].

The temperature of the surfaces of the plate, defined by the coordinates $x = 0$, $x = \delta$ and $y \to \infty$, is assumed to remain constant and equal to t_1,

while that along the surface $y = 0$ is a function of coordinate x, i.e. $t = f(x)$. It is assumed that the plate is relatively thin in the direction of the Oz-axis, and the surfaces, parallel to the coordinate plane xOy, are ideally heat insulated. Owing to the last assumption, the temperature gradient $\partial t/\partial z$ can be ignored, so that the temperature field of such a plate will be two-dimensional.

The differential equation for a two-dimensional steady-state conduction problem without inner heat sources is as follows:

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0$$

or

$$\frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2} = 0 \qquad (2\text{-}111)$$

where ϑ is the temperature in excess of the reference temperature t_1, i.e.

$$\vartheta = t - t_1$$

The boundary conditions are

$$\vartheta = \begin{cases} 0 \text{ at } x = 0,\ \delta \\ 0 \text{ at } y \to \infty \end{cases} \qquad (2\text{-}112)$$

$$f(x) - t_1 = F(x) \quad \text{at} \quad y = 0$$

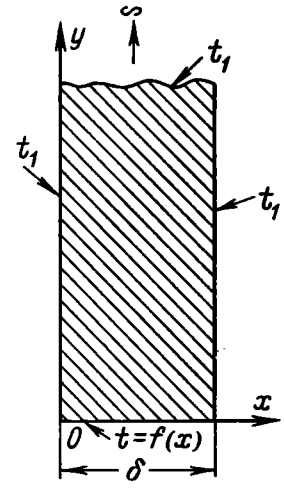

Fig. 2-20. Semi-finite plate

We solve the equations in partial derivatives (2-111) by separating the variables*, and assuming that $\vartheta = f(x, y) = \varphi(x)\,\psi(y)$. Then Eq. (2-111) becomes

$$\frac{\varphi''(x)}{\varphi(x)} = -\frac{\psi''(y)}{\psi(y)} = \text{const} \qquad (2\text{-}113)$$

The left and right sides of this equation are identical and constant. Denoting them by $(-\varepsilon^2)$, we thus obtain two ordinary differential equations:

$$\varphi''(x) + \varepsilon^2 \varphi(x) = 0 \qquad (2\text{-}114)$$

and

$$\psi''(y) - \varepsilon^2 \psi(y) = 0 \qquad (2\text{-}115)$$

The solution of Eq. (2-114) is a function such as

$$\varphi(x) = C_1 \cos(\varepsilon x) + C_2 \sin(\varepsilon x) \qquad (2\text{-}116)$$

According to (2-79), the general solution of Eq. (2-115) has the following form:

$$\psi(y) = C_3 e^{\varepsilon y} + C_4 e^{-\varepsilon y} \qquad (2\text{-}117)$$

The general solution of Eq. (2-111) is obtained by multiplying Eqs. (2-116) and (2-117). Solution (2-116) will satisfy the boundary condition $\vartheta = 0$ at $x = 0$, when $\varphi(x) = 0$ at $x = 0$ as well, and this is possible at $C_1 = 0$. The boundary condition $\vartheta = 0$ at $y \to \infty$ is satisfied when the function $\psi(y)$ is also equal to 0 at $y \to \infty$, and this is possible at $C_3 = 0$. Thus, the solution for (2-111) becomes:

$$\vartheta = Ce^{-\varepsilon y} \sin(\varepsilon x)$$

* This method is treated in greater detail in Chapter 3 as applied to problems in transient conduction.

For the obtained expression to satisfy the boundary conditions $\vartheta = 0$ at $x = \delta$, $\sin(\varepsilon\delta)$ must be equal to zero, or $\varepsilon = \frac{n\pi}{\delta}$ (where $n = 1, 2, 3, \ldots$).

For each value of n there is a partial solution, and for each partial solution there is a corresponding integration constant. The general solution is the sum of the partial solutions for all successive positive values of n:

$$\vartheta = \sum_{n=1}^{n=\infty} C_n e^{\frac{-n\pi}{\delta} y} \sin\left(\frac{n\pi}{\delta} x\right) \tag{2-118}$$

The solution obtained satisfies the third boundary condition as well, i.e. $\vartheta = 0$ at $y \to \infty$.

The remaining integration constant C_n is determined from the boundary conditions $\vartheta = F(x)$ at $y = 0$. Then

$$F(x) = \sum_{n=1}^{n=\infty} C_n \sin\left(\frac{n\pi}{\delta} x\right)$$

This equality is the sine expansion of the function $F(x)$ into Fourier series. The coefficients of this series are determined from the following equation:

$$C_n = \frac{2}{\delta} \int_0^{\delta} F(x) \sin\left(\frac{n\pi}{\delta} x\right) dx$$

Taking last equation into consideration, the final solution of the temperature field may be written as follows:

$$\vartheta = \frac{2}{\delta} \sum_{n=1}^{\infty} e^{-\frac{n\pi}{\delta} y} \sin\left(\frac{n\pi}{\delta} x\right) \int_0^{\delta} F(x) \sin\left(\frac{n\pi}{\delta} x\right) dx \tag{2-119}$$

Thus, having determined the integration constants, the final solution of the two-dimensional problem appears as the sum of an infinite series.

By analogy we may obtain the solution for a solid cylinder with the temperature field varying in two dimensions, which will appear as the sum of an infinite series, as in the case of a flat plate.

In solving particular problems, the integral of Eq. (2-119) is taken on the assumption that the temperature is given. The next step consists in calculating the terms of the series, depending on the conditions of divergence and the accuracy required.

For instance, if $t = t_2 = \text{const}$ at $y = 0$, then $f(x) = t_2$, and $F(x) = t_2 - t_1$. The integral is

$$\int_0^{\delta} F(x) \sin\left(\frac{n\pi}{\delta} x\right) dx = -\frac{\delta}{n\pi} (t_2 - t_1) \left(-\cos\frac{\pi n}{\delta} x\right) \Bigg|_0^{\delta} = \frac{2\delta}{n\pi} (t_2 - t_1)$$

where $n = 1, 3, 5, 7 \ldots$.

Substituting this in Eq. (2-119), we find that

$$\vartheta = (t_2 - t_1) \frac{4}{\pi} \left[e^{-\frac{\pi}{\delta} y} \sin\left(\frac{\pi}{\delta} x\right) + \frac{1}{3} e^{-\frac{3\pi}{\delta} y} \sin\left(\frac{3\pi}{\delta} x\right) + \right.$$

$$\left. + \frac{1}{5} e^{-\frac{5\pi}{\delta} y} \sin\left(\frac{5\pi}{\delta} x\right) + \ldots \right] \tag{2-120}$$

It can be shown that the series obtained is converging. The isotherms can be calculated by various methods; the most precise is that in which y is assumed to be constant. In plotting the isotherms use is made of the family of curves conforming to the constant y.

2-12. Porous Cooling of a Plane Plate

Porous materials are widely used in high-temperature heat exchangers, turbine blades, jet nozzles, etc. In practice, cooling of porous structures is achieved by forcing liquid or gas through the capillaries of the solid. The process of heat transfer in these systems is extremely complex. In solving the problem, it is assumed that heat transfer inside a plane plate is due to conduction through the solid phase and that the temperatures of the solid and liquid differ only slightly at any point of the system. These assumptions considerably simplify solution [2.6].

Consider the plane plate of constant thermal conductivity λ_w illustrated in Fig. 2-21. The dimensions of the plate in the y- and z-directions are large, and the temperature field inside it is assumed to be one-dimensional. This assumption also holds for the cooling fluid, i.e. $t=t\,(x)$ at $0 \leqslant x \leqslant \delta$, and $t_f = t_f\,(x)$ at $-\infty \leqslant x \leqslant 0$.

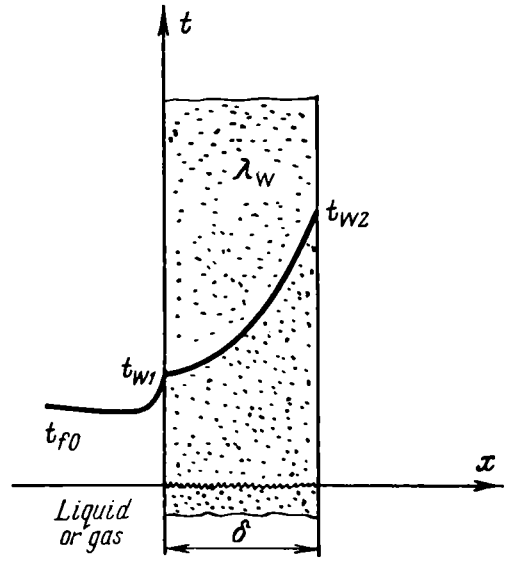

Fig. 2-21. Porous cooling of a plane plate

At $x = \delta$ the surface temperature of the plate is t_{w2}. The temperature of the fluid forced through the plate along Ox-axis is t_{f0}, at a distance $x \to -\infty$. Both temperatures t_{w2} and t_{f0} are known. The given values also include the mass rate of fluid flow G in kg/m^2·s; the specific heat c_{pf} and thermal conductivity λ_f of the fluid are assumed to be constant. It is necessary to find the temperature distribution in the porous wall.

We consider the porosity of the plate p as the ratio of the volume of the pores to the total volume of the material of the plate. With uniform porosity it may be assumed that the area for fluid flow in a unit area of the plate normal to the direction of fluid flow is $f_f = p$, and that the solid area for heat conduction is $f_w = 1 - f_f = 1 - p$. We also take note that if the mass rate of flow of the fluid upstream of the plate is G, then the mass rate of flow within the plate is G/p.

The process of heat transport in such a porous plate can be visualized as consisting of heat transfer by conduction through the plate proper and heat transfer between the solid and liquid flowing through the pores of the plate.

The heat flow due to conduction in the plate sections x and $x+dx$ is written as:

$$q_x = -\lambda_w \frac{dt}{dx}(1-p)$$

and

$$q_{x+dx} = -\lambda_w \frac{d}{dx}\left(t+\frac{dt}{dx}dx\right)(1-p)$$

Owing to heat transfer between the solid and the fluid, the heat imparted to element dx will be consumed for heating the fluid:

$$dq = q_x - q_{x+dx} = Gc_{pf}\,dt$$

or

$$-\lambda_w(1-p)\frac{dt}{dx} + \lambda_w(1-p)\frac{dt}{dx} + \lambda_w(1-p)\frac{d^2t}{dx^2}\,dx = Gc_{pf}\,dt$$

Hence the differential equation for the region $0 \leqslant x \leqslant \delta$ is

$$\frac{d^2t}{dx^2} - \frac{Gc_{pf}}{\lambda_w(1-p)}\frac{dt}{dx} = 0 \qquad \text{(a)}$$

or, if we denote

$$\frac{Gc_{pf}}{\lambda_w(1-p)} = \xi_w$$

then Eq. (a) is written as

$$\frac{d^2t}{dx^2} - \xi_w\frac{dt}{dx} = 0 \qquad (2\text{-}121)$$

The differential equation for the region $-\infty \leqslant x \leqslant 0$ may be obtained in a similar manner:

$$\frac{d^2t}{dx^2} - \xi_f\frac{dt}{dx} = 0 \qquad (2\text{-}122)$$

where

$$\xi_f = \frac{Gc_{pf}}{\lambda_f}$$

The general solution of Eq. (2-121) is:

$$t = C_1e^{\xi_w x} + C_2$$

The constants C_1 and C_2 are determined from the boundary conditions:

$$\text{at} \quad x = 0, \quad t = t_{w1} \quad \text{and at} \quad x = \delta, \quad t = t_{w2}$$

Having determined C_1 and C_2, we obtain t for the region $0 \leqslant x \leqslant \delta$:

$$t = t_{w1} + \frac{t_{w2} - t_{w1}}{e^{\xi_w\delta} - 1}(e^{\xi_w x} - 1) \qquad (2\text{-}123)$$

The general solution for Eq. (2-122) is:

$$t_f = C_3e^{\xi_f x} + C_4$$

which must conform to the boundary conditions for fluid flow:

$$\text{at} \quad x = -\infty, \quad t_f = t_{f0}$$

$$\text{at} \quad x = 0, \quad \lambda_f\frac{dt_f}{dx} = \lambda_w(1-p)\frac{dt}{dx}$$

From these conditions we find the integration constants; $C_4 = t_{f0}$ and $C_3 = \frac{t_{w2} - t_{w1}}{e^{\xi_w\delta} - 1}$; then the solution of Eq. (2-122) is written:

$$t_f = t_{f0} + \frac{t_{w2} - t_{w1}}{e^{\xi_w\delta} - 1}e^{\xi_f x} \quad (-\infty \leqslant x \leqslant 0) \qquad (2\text{-}124)$$

Based on Eq. (2-124), the unknown temperature t_{w1} in Eq. (2-123) can be eliminated. At $x = 0$,

$$t_f = t_{w1} = t_{f0} + (t_{w2} - t_{f0})\, e^{-\xi_w \delta}$$

Substituting the value of t_{w1} in Eq. (2-123), we obtain the final expression for the temperature distribution in a porous plate ($0 \leqslant x \leqslant \delta$):

$$\frac{t - t_{f0}}{t_{w2} - t_{f0}} = e^{-\xi_w \delta (1 - x/\delta)} \tag{2-125}$$

Denoting the dimensionless temperature of the plate $\frac{t - t_{f0}}{t_{w2} - t_{f0}}$ by Θ, Eq. (2-125) may be written as follows:

$$\Theta = e^{-\xi_w \delta (1 - x/\delta)} \tag{2-125'}$$

The mean temperature inside the porous plate for the given value of $\xi_w \delta$, calculated from the integral $\overline{\Theta} = \frac{1}{\delta} \int_0^{\delta} \Theta \, dx$, is equal to:

$$\overline{\Theta} = \frac{\bar{t} - t_{f0}}{t_{w2} - t_{f0}} = \frac{1}{\xi_w \delta} (1 - e^{-\xi_w \delta}) \tag{2-126}$$

If the value $\xi_w \delta$ is taken as a parameter, Eq. (2-125) may be presented as shown in Fig. 2-22, where the mean temperature calculated by Eq. (2-126) is plotted against corresponding values of $\xi_w \delta$.

A solution for the problem of temperature distribution in a sweat-cooled porous plate at different boundary conditions has been proposed by Isachenko

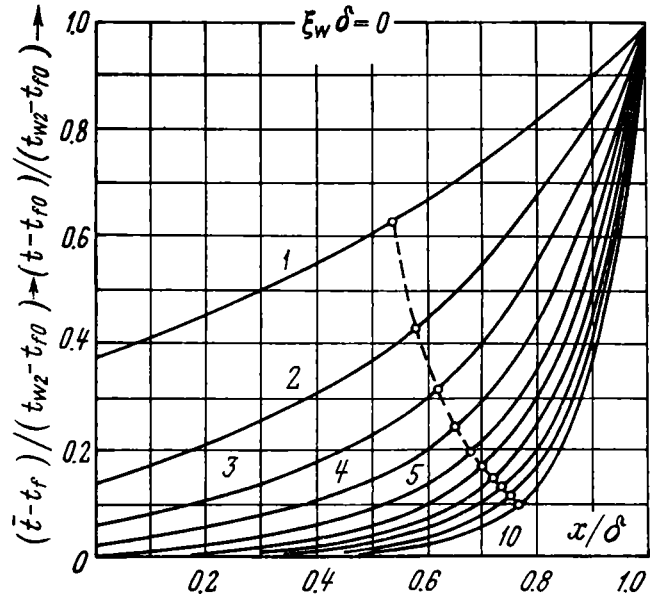

Fig. 2-22. Temperature distribution and mean temperature in a porous plate

[2.7]. For this solution, it is assumed that pores of small diameter are uniformly distributed over the volume of a plane wall and penetrate it in the transverse direction, as shown in Fig. 2-23. The mass rate of fluid flow through the pores is G_f kg/m^2; the temperatures of the fluid and wall are equal in any given section; the physical parameters are independent of temperature.

The equations of conduction and the boundary conditions here are as follows:

$$\frac{d^2t}{dx^2}+\frac{G_f c_{pf}}{\lambda}\frac{dt}{dx}=0 \tag{2-127}$$

$$\alpha_g(t_g-t_{w1})-G_f r=-\lambda\left(\frac{dt}{dx}\right)_{x=0} \tag{2-128}$$

$$\alpha_f(t_{w2}-t_f)=-\lambda\left(\frac{dt}{dx}\right)_{x=\delta} \tag{2-129}$$

where (apart from the notations in Fig. 2-23)

r = latent heat of evaporation;

c_{pf} = specific heat of the liquid;

α_g and α_f = heat-transfer coefficients for the wall surfaces respectively exposed to gas and liquid.

In the general case the thermal conductivity λ in Eq. (2-127) must account for the conduction of the solid proper and of the cooling liquid. With porous metal walls of high thermal conductivity and small total volume of pores, the thermal conductivity of the liquid can be ignored. As in the preceding problem, here also the assumption $\lambda=\lambda_w(1-p)$ is justified.

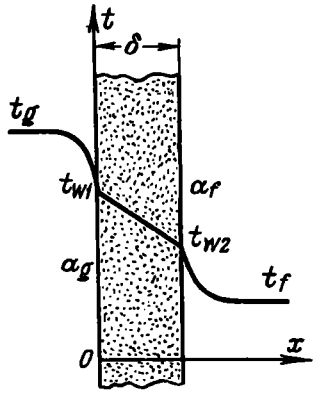

Fig. 2-23. Porous cooling of a plate (boundary conditions of the third kind)

Omitting the intermediate calculations, let us present the final solution of Eq. (2-127) with boundary conditions (2-128) and (2-129):

$$t=t_g-\frac{G_f r}{\alpha_g}-\frac{\left[(t_g-t_f)-\frac{G_f r}{\alpha_g}\right]}{(1+k_g)-(1-k_f)e^{-\xi\delta}}[1+k_g-e^{-\xi x}] \tag{2-130}$$

where

$$k_g=\frac{G_f c_{pf}}{\alpha_g};\qquad k_f=\frac{G_f c_{pf}}{\alpha_f};\qquad \xi=\frac{G_f c_{pf}}{\lambda}$$

If cooling of the porous wall does not involve evaporation of the cooling liquid, i.e. if $r=0$, then Eq. (2-130) takes the following form:

$$t=t_g-\frac{t_g-t_f(1+k_g-e^{-\xi x})}{(1+k_g)-(1-k_f)e^{-\xi\delta}} \tag{2-131}$$

2-13. Conduction with Inner Heat Sources

In the problems so far considered there have been no inner heat sources. Processes, however, may often develop inside the objects studied, which result in liberation or absorption of heat. Processes of this kind are exemplified by the following: liberation of Joule heat by an electric current flowing through a conductor; volumetric liberation of heat from the heat-liberating elements of atomic reactors as a consequence of deceleration of fission products of the atomic fuel, and of moderation of neutron flux; liberation or absorption of heat during many chemical reactions, etc.

When investigating heat transport in such cases, it is essential to know the intensity of volumetric heat liberation (absorption) which is represented quantitatively by the capacity of inner heat sources q_v, W/m³. If q_v is positive, the body is said to have positive inner heat sources. A negative q_v signifies the presence of negative inner heat sources (or sinks).

Depending on the peculiarities of the variation of q_v in space, we may speak of point, linear, surface and volume heat sources.

For steady-state processes $\frac{\partial t}{\partial \tau} = 0$ the differential equation of conduction (1-24) in the presence of inner heat sources takes the following form:

$$\nabla^2 t + \frac{q_v}{\lambda} = 0 \qquad (2\text{-}132)$$

(a) Conduction Through a Homogeneous Plate

Consider a long plate whose thickness 2δ is small in comparison with its other dimensions.

Heat sources are uniformly distributed over the entire volume and amount to q_v = const. The given conditions of heat transfer include the heat-transfer coefficient α and the temperature t_f of the fluid at a distance from the plate, both values being constant. The temperatures of both surfaces of the plate are equal, owing to uniform cooling. Under the given conditions the temperature of the plate will vary only in the direction of the x-axis which is normal to its surface. The temperature at the axis of the plate and its surface temperature are respectively denoted by t_0 and t_w and are unknown (Fig. 2-24). It is necessary to find, in addition, temperature distribution within the plate and the quantity of heat lost to the surroundings.

The differential equation (2-132) is simplified here and becomes:

$$\frac{d^2 t}{dx^2} + \frac{q_v}{\lambda} = 0 \qquad (2\text{-}133)$$

while the boundary conditions are

$$\text{at } x = \pm\delta,$$

$$\pm\lambda \left(\frac{\partial t}{\partial x}\right)_{x=\pm\delta} = \alpha (t_w - t_f)$$

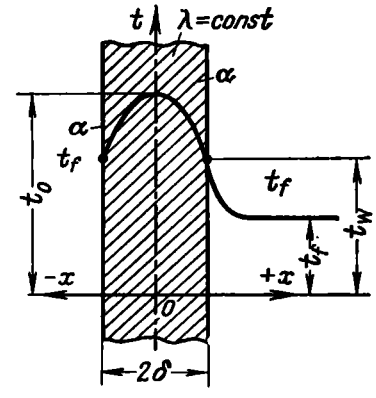

Fig. 2-24. Heat transfer through a plane plate with inner heat sources

The boundary conditions being similar for both sides of the plate, the temperature field within it must be symmetrical with respect to the surface $x=0$. Heat is dissipated through the left and right surface of the body with equal intensity, and the amount of heat liberated from both halves of the plate is the same. This means that only one half of the plate need be considered further, for example the right half (see Fig. 2-24) and the boundary conditions for this half can be written as follows:

$$\left.\begin{aligned} &\text{at } x = 0, \quad \left(\frac{\partial t}{\partial x}\right)_{x=0} = 0 \\ &\text{at } x = \delta, \quad -\lambda \left(\frac{\partial t}{\partial x}\right)_{x=\delta} = \alpha (t_w - t_f) \end{aligned}\right\} \qquad (2\text{-}134)$$

Integration of Eq. (2-133) gives:

$$\frac{dt}{dx} = -\frac{q_v x}{\lambda} + C_1 \tag{2-135}$$

$$t = -\frac{q_v}{2\lambda} x^2 + C_1 x + C_2 \tag{2-136}$$

The integration constants C_1 and C_2 are determined from the boundary conditions (2-134).

At $x = 0$ we find from Eq. (2-135) that $C_1 = 0$;

at $x = \delta$ we obtain:

$$-\lambda \left(\frac{dt}{dx}\right)_{x=\delta} = -\alpha (t_w - t_f)$$

From Eq. (2-135)

$$\left(\frac{dt}{dx}\right)_{x=\delta} = -\frac{q_v \delta}{\lambda}$$

Then $t_w = \frac{q_v \delta}{\alpha} + t_f$; substituting this expression in Eq. (2-136), at $x = \delta$ we obtain:

$$C_2 = t_f + \frac{q_v \delta}{\alpha} + \frac{q_v \delta^2}{2\lambda}$$

Substituting C_1 and C_2 in Eq. (2-136), we find the equation of the temperature field:

$$t_x = t_f + \frac{q_v \delta}{\alpha} + \frac{q_v \delta^2}{2\lambda}\left[1 - \left(\frac{x}{\delta}\right)^2\right] \tag{2-137}$$

In the problem considered heat flow varies along the x-axis:

$$q = q_v x$$

At $x = 0$, $q = 0$; the same follows from the condition that at $x = 0$, $\left(\frac{dt}{dx}\right)_{x=0} = 0$. The heat flow per unit area of the plate surface at $x = \delta$ is

$$q = \alpha (t_w - t_f) = q_v \delta \tag{2-138}$$

and the total quantity of heat dissipated from the whole surface per unit time (the whole surface F is the sum of the two side surfaces F_1) is

$$Q = qF = q_v \delta 2F_1 \tag{2-139}$$

It follows from Eq. (2-137) that temperature distribution in a plane wall in a symmetric problem is parabolic.

If we set $\alpha \to \infty$ in Eq. (2-137), the resulting expression will represent the temperature field for the boundary conditions of the first kind, since $t_f \equiv t_w$ at $\alpha \to \infty$.

Taking this into consideration, Eq. (2-137) thus acquires the following form:

$$t = t_w + \frac{q_v}{2\lambda}(\delta^2 - x^2) \tag{2-140}$$

The temperature at the symmetry axis of the plate ($x = 0$) is

$$t_0 = t_w + \frac{q_v}{2\lambda}\delta^2$$

and the temperature difference between the symmetry axis and the surface of the plate is:

$$t_0 - t_w = \frac{q_v}{2\lambda}\delta^2 = \frac{q}{2\lambda}\delta \tag{2-141}$$

Thus far the thermal conductivity of the material of the wall was assumed to be constant. With large temperature differences, however, it may become necessary to take into account the dependence of thermal conductivity on temperature. This dependence is often linear, i.e.

$$\lambda = \lambda_0 (1 + bt)$$

Then

$$q_v x = -\lambda_0 (1 + bt) \frac{dt}{dx} \tag{a}$$

Separating the variables and integrating Eq. (a), we obtain:

$$t + b\frac{t^2}{2} = -\frac{1}{\lambda_0}\frac{q_v x^2}{2} + C \tag{b}$$

At $x = 0$, $t = t_0$, and it then follows from Eq. (b) that

$$C = t_0 + \frac{b}{2} t_0^2$$

Substituting the determined value of C in Eq. (b) and solving the quadratic equation with respect to t, we find the following equation for the temperature curve:

$$t = -\frac{1}{b} + \sqrt{\left(t_0 + \frac{1}{b}\right)^2 - \frac{q_0 x^2}{\lambda_0 b}} \tag{2-142}$$

(b) Conduction Through a Homogeneous Cylindrical Rod

Consider a circular cylinder (Fig. 2-25) whose radius is small in comparison with its length. Under these conditions temperature will vary only along the radius.

The inner heat sources are uniformly distributed over the volume of the solid. The constant ambient temperature t_f, and the heat-transfer coefficient, constant over the entire surface, are given. Under these conditions all points of the inner surface of the cylinder will be at an equal temperature.

For such a cylinder the problem is one-dimensional and symmetrical, as for the previously considered plate, and Eq. (2-132) acquires the following form:

$$\frac{d^2t}{dr^2} + \frac{1}{r}\frac{dt}{dr} + \frac{q_v}{\lambda} = 0 \tag{2-143}$$

Boundary conditions are as follows:

$$\left.\begin{aligned} &\text{at } r = 0, \quad \left(\frac{dt}{dr}\right)_{r=0} = 0 \\ &\text{at } r = r_0, \quad \left(\frac{dt}{dr}\right)_{r=r_0} = -\frac{\alpha}{\lambda}(t_w - t_f) \end{aligned}\right\} \tag{2-144}$$

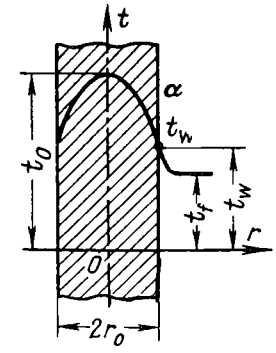

Fig. 2-25. Heat transfer through a cylindrical homogeneous rod with inner heat sources

It is necessary to find the equation of the temperature field and the heat flow, and the temperatures at the axis t_0 and the surface t_w.

Let us integrate Eq. (2-143), substituting u for $\frac{dt}{dr}$. Then Eq. (2-143) is written as

$$\frac{du}{dr} + \frac{u}{r} + \frac{q_v}{\lambda} = 0$$

or

$$r\,du + u\,dr + \frac{q_v}{\lambda} r\,dr = 0$$

After integration we obtain:

$$u + \frac{q_v r}{2\lambda} = \frac{C_1}{r}$$

or

$$\frac{dt}{dr} + \frac{q_v r}{2\lambda} = \frac{C_1}{r} \tag{2-145}$$

After a second integration, we get:

$$t = -\frac{q_v r^2}{4\lambda} + C_1 \ln r + C_2 \tag{2-146}$$

where the constants C_1 and C_2 are determined from the boundary conditions (2-144).

At $r = 0$, we find from Eq. (2-145) that $C_1 = 0$; at $r = r_0$

$$\left(\frac{dt}{dr}\right)_{r=r_0} = -\frac{q_v r_0}{2\lambda}$$

Substituting the latter expression in Eq. (2-144), we obtain:

$$\frac{q_v r_0}{2} = \alpha (t_w - t_f)$$

and

$$t_w = \frac{q_v r_0}{2\alpha} + t_f$$

From Eq. (2-146) we find the integration constant C_2:

$$C_2 = t_f + \frac{q_v r_0}{2\alpha} + \frac{q_v r_0^2}{4\lambda}$$

Substituting C_1 and C_2 in Eq. (2-146), we have:

$$t = t_f + \frac{q_v r_0}{2\alpha} + \frac{q_v}{4\lambda}(r_0^2 - r^2) \tag{2-147}$$

The expression obtained makes it possible to calculate the temperature at any point of the rod, and shows that the temperature distribution in a circular rod is parabolic.

At $r = 0$ the temperature at the cylinder axis can be found from Eq. (2-147)

$$t_0 = t_f + \frac{q_v r_0}{2\alpha} + \frac{q_v r_0^2}{4\lambda} \tag{2-147'}$$

The rate of heat flow per unit area of the cylinder is

$$q = \alpha (t_w - t_f) = \frac{q_v r_0}{2} \tag{2-148}$$

The total heat flow from the surface of the cylinder is:

$$Q = qF = \frac{q_v r_0}{2} 2\pi r_0 l = q_v \pi r_0^2 l \tag{2-148'}$$

It follows from Eq. (2-148) that the rate of heat flow depends only on the capacity of the inner heat sources and on the size of the outer surface (r_0) through which it passes.

Now assume that boundary conditions of the first kind are given, i.e. a surface temperature t_w. This is a special case of the preceding problem, when the heat-transfer coefficient is equal to infinity, $\alpha \to \infty$. Then $t_f \equiv t_w$ obviously, and Eq. (2-147) becomes:

$$t = t_w + \frac{q_v r_0^2}{4\lambda}\left[1 - \left(\frac{r}{r_0}\right)^2\right] \tag{2-149}$$

The temperature at the cylinder axis ($r = 0$) is

$$t_0 = t_w + \frac{q_v r_0^2}{4\lambda} \tag{2-150}$$

If the dependence of thermal conductivity on temperature [given as $\lambda(t) = \lambda_0(1 + bt)$] must be taken into account then, integrating the expression

$$q_v \pi r^2 = -2\pi r \lambda_0 (1 + bt)\frac{dt}{dr}$$

we obtain:

$$t + \frac{b}{2}t^2 = -\frac{1}{4\lambda_0} q_v r_0^2 + C \tag{2-151}$$

The integration constant C is determined from the boundary conditions. At $r=0, t=t_0$ and $C = t_0 + \frac{b}{2}t_0^2$. Substituting this value in Eq. (2-151) and solving for t, we obtain the following formula for the temperature curve:

$$t = -\frac{1}{b}\sqrt{\left(\frac{1}{b} + t_0\right)^2 - \frac{q_v r^2}{2\lambda_0 b}} \tag{2-152}$$

(c) Conduction Through a Cylindrical Wall

Consider an infinitely long cylindrical wall (tube) of inside radius r_1, outside radius r_2, and constant thermal conductivity λ. The heat sources of a heat capacity q_v are uniformly distributed within the wall.

In such a wall temperature will vary only in a radial direction, and the process of conduction is described by Eq. (2-143)

$$\frac{d^2t}{dr^2} + \frac{1}{r}\frac{dt}{dr} + \frac{q_v}{\lambda} = 0$$

The integral of this equation is represented by Eq. (2-146):

$$t = -\frac{q_v r^2}{4\lambda} + C_1 \ln r + C_2$$

The integration constants C_1 and C_2 are determined from the boundary conditions. Let us consider cases where heat is removed either from the inner or the outer surface only, or from both simultaneously.

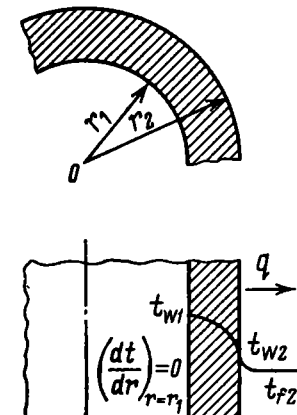

Fig. 2-26. Removal of heat through the outer surface of a cylindrical wall with inner heat sources

1. Heat removed only through the outer surface. Here we shall consider the case where boundary conditions of the third kind are given, i. e. with an ambient temperature t_{f2} at the outer surface and a constant heat-transfer coefficient for the outer tube surface (Fig. 2-26). The boundary conditions will then be

written as follows:

$$\text{at } r=r_1, \quad q=0 \text{ or } \left(\frac{dt}{dr}\right)_{r=r_1}=0$$

$$\text{at } r=r_2, \quad \left(\frac{dt}{dr}\right)_{r=r_1}=-\frac{\alpha}{\lambda}(t_{w2}-t_{f2})$$

From Eq. (2-146) we obtain:

$$\frac{dt}{dr}=-\frac{q_v}{2\lambda}r+\frac{C_1}{r}$$

At $r=r_1$, $\left(\frac{dt}{dr}\right)_{r=r_1}=-\frac{q_v}{2\lambda}r_1+\frac{C_1}{r_1}=0$ whence

$$C_1=\frac{q_v r_1^2}{2\lambda}$$

Taking this value of C_1 into account at $r=r_2$, we obtain from Eq. (2-146):

$$t_{w2}=-\frac{q_v r_2^2}{4\lambda}+\frac{q_v r_1^2}{2\lambda}\ln r_2+C_2 \qquad \text{(a)}$$

Noting that $\left(\frac{dt}{dr}\right)_{r=r_2}=-\frac{\alpha}{\lambda}(t_{w2}-t_{f2})$, we get:

$$t_{w2}=t_{f2}+\frac{q_v r_2}{2\lambda}-\frac{q_v r_1^2}{2\alpha r_2} \qquad \text{(b)}$$

Equating (a) and (b), we find:

$$C_2=t_{f2}+\frac{q_v r_2}{2\alpha}+\frac{q_v r_2^2}{4\lambda}-\frac{q_v}{2\alpha}\frac{r_1^2}{r_2}-\frac{q_v r_1^2}{2\lambda}\ln r_2$$

Substituting the found constants C_1 and C_2 in Eq. (2-146), we obtain the following expression for the temperature field:

$$t=t_{f2}+\frac{q_v r_2}{2\alpha}\left[1-\left(\frac{r_1}{r_2}\right)^2\right]+\frac{q_v r_2^2}{4\lambda}\left[1+\left(\frac{r_1}{r_2}\right)^2 2\ln\frac{r}{r_2}-\left(\frac{r}{r_2}\right)^2\right] \qquad (2\text{-}153)$$

For the outer surface losing heat (at $r=r_2$)

$$t_{w2}=t_{f2}+\frac{q_v r_2}{2\alpha}\left[1-\left(\frac{r_1}{r_2}\right)^2\right] \qquad (2\text{-}154)$$

The rate of heat flow from unit area of this surface will be determined as

$$q=\alpha(t_{w2}-t_{f2})=\frac{q_v r_2}{2}\left[1-\left(\frac{r_1}{r_2}\right)^2\right] \qquad (2\text{-}155)$$

The temperature of the inner surface of the wall (tube) is found from Eq. (2-153) by introducing the value of $r=r_1$:

$$t_{w1}=t_{f2}+\frac{q_v r_2}{2\alpha}\left[1-\left(\frac{r_1}{r_2}\right)^2\right]+\frac{q_v r_2^2}{4\lambda}\left[1+\left(\frac{r_1}{r_2}\right)^2 2\ln\frac{r_1}{r_2}-\left(\frac{r_1}{r_2}\right)^2\right] \qquad (2\text{-}156)$$

Consider now that boundary conditions of the first kind are given, i.e. a temperature t_{w2} on the surface losing heat. These conditions may be treated as a special case of the problem, when the heat-transfer coefficient for the surface is sufficiently large ($\alpha\to\infty$). Then, fluid temperature will equal the temperature of the tube surface. Taking this into account, Eq. (2-153) becomes:

$$t=t_{w2}+\frac{q_v r_2^2}{4\lambda}\left[1+\left(\frac{r_1}{r_2}\right)^2\ln\frac{r}{r^2}-\left(\frac{r}{r_2}\right)^2\right] \qquad (2\text{-}157)$$

Assuming $r = r_1$ and $t = t_{w1}$, we find the temperature difference across the cylindrical wall:

$$t_{w1} - t_{w2} = \frac{q_v r_1^2}{4\lambda}\left[\left(\frac{r_2}{r_1}\right)^2 - 2\ln\frac{r^2}{r_1} - 1\right] \qquad (2\text{-}158)$$

2. Heat removed only through the inner surface of the tube (Fig. 2-27). With the heat-transfer coefficient α for the inner surface and the ambient temperature t_{f1} given, the boundary conditions are:

$$\text{at } r = r_1,\ \left(\frac{dt}{dr}\right)_{r=r_1} = -\frac{\alpha}{\lambda}(t_{w1} - t_{f1})$$

$$\text{at } r = r_2,\ \left(\frac{dt}{dr}\right)_{r=r_0} = 0$$

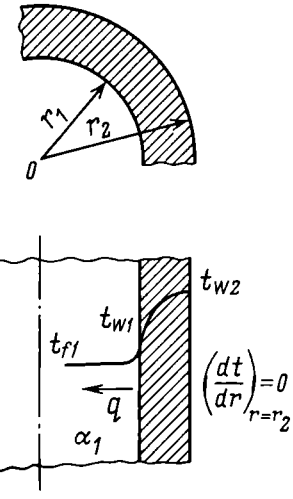

Fig. 2-27. Removal of heat through the inner surface of a cylindrical wall with inner heat sources

As with the previous case, the integration constants C_1 and C_2 of Eq. (2-146) are determined from these boundary conditions.

Having determined the integration constants and substituted them in Eq. (2-146), we get:

$$t = t_{f1} + \frac{q_v r_1}{2\alpha}\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right] + \frac{q_v r_2^2}{4\lambda}\left[2\ln\frac{r}{r_1} + \left(\frac{r_1}{r_2}\right)^2 - \left(\frac{r}{r_2}\right)^2\right] \qquad (2\text{-}159)$$

The temperature difference between the surroundings and the exothermic surface is found by substituting r_1 in Eq. (2-159). Then,

$$t_{w1} - t_{f1} = \frac{q_v r_1}{2\alpha}\left[\left(\frac{r_2}{r_1}\right)^2 - 1\right] \qquad (2\text{-}160)$$

When the temperature of the surface losing heat is given (corresponding to $\alpha \to \infty$), Eq. (2-159) acquires the following form:

$$t = t_{w1} + \frac{q_v r_2^2}{4\lambda}\left[2\ln\frac{r}{r_1} + \left(\frac{r_1}{r_2}\right)^2 - \left(\frac{r}{r_2}\right)^2\right] \qquad (2\text{-}161)$$

Assuming $r = r_2$ and, correspondingly, $t = t_{w2}$, we obtain the total temperature difference across the wall:

$$t_{w2} - t_{w1} = \frac{q_v r_2^2}{4\lambda}\left[2\ln\frac{r_2}{r_1} + \left(\frac{r_1}{r_2}\right)^2 - 1\right] \qquad (2\text{-}162)$$

3. Heat removed through both inner and outer surfaces of the tube. A maximum temperature must exist inside the wall when heat is dissipated to the surroundings from both the inner and outer surfaces of the tube. The isothermal surface of the maximum temperature t_0 separates the wall into two layers. Heat flows toward the tube axis in the inner layer and toward the outside in the outer layer. The maximum temperature corresponds to the condition $dt/dr = 0$; consequently, $q = 0$.

Thus, the problem can be solved by using the relationships already obtained. For this purpose it is necessary to know the radius r_0 (Fig. 2-28) at which maximum temperature t_0 occurs.

In accordance with Eqs. (2-156) and (2-162), the maximum temperature differences across the outer and inner layers are determined from the following equations:

$$t_0 - t_{w2} = \frac{q_v r_0^2}{4\lambda}\left[\left(\frac{r_2}{r_0}\right) - 2\ln\frac{r_2}{r_0} - 1\right] \quad \text{(a)}$$

$$t_0 - t_{w1} = \frac{q_v r_0^2}{4\lambda}\left[\left(\frac{r_1}{r_0}\right)^2 + 2\ln\frac{r_0}{r_1} - 1\right] \quad \text{(b)}$$

Subtracting the left and right sides respectively of equations (a) and (b) we obtain:

$$t_{w1} - t_{w2} = \frac{q_v r_0^2}{4\lambda}\left[\left(\frac{r_2}{r_0}\right)^2 - \left(\frac{r_1}{r_0}\right)^2 + 2\ln\frac{r_0}{r_2} - 2\ln\frac{r_0}{r_1}\right]$$

This equation must be solved for r_0; we then obtain:

$$r_0^2 = \frac{4\lambda\,(t_{w1} - t_{w2})}{q_v 2\ln\frac{r_1}{r_2}} - \frac{r_2^2 - r_1^2}{2\ln\frac{r_1}{r_2}}$$

or

$$r_0^2 = \frac{q_v\,(r_2^2 - r_1^2) - 4\lambda\,(t_{w1} - t_{w2})}{q_v 2\ln\frac{r_2}{r_1}} \quad \text{(2-163)}$$

Substituting the value of r_0 calculated from Eq. (2-163) in expression (a) or (b) we obtain the maximum temperature of the wall.

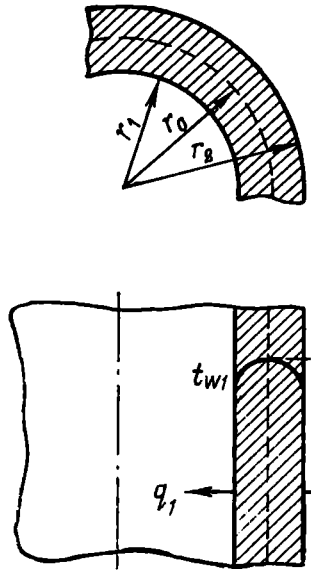

Fig. 2-28. Heat of inner sources is removed through both surfaces of a cylindrical wall

Temperature distribution in the inner layer is found by substituting the actual values of the coordinate $r_1 < r < r_0$ in Eq. (2-161), and the temperature distribution in the outer layer by substituting $r_0 < r < r_2$ in Eq. (2-157).

If the outer surfaces of the cylindrical wall are at equal temperatures, i.e. t_{w1} is equal to t_{w2}, Eq. (2-163) simplifies, and then

$$r_0^2 = \frac{r_2^2 - r_1^2}{2\ln\frac{r_2}{r_1}} \quad \text{(2-163')}$$

i.e. r_0 depends only on the dimensions of the wall and is independent of the thermal conditions. For instance, with $r_2 = 2$ and $r_1 = 1$, $r_0 = 1.46$.

If the temperatures of the wall surfaces, t_{w1} and t_{w2} are not known, bu twe know the temperatures of the fluids t_{f1} and t_{f2}, to which the inner and outer surfaces are respectively exposed, and also the heat-transfer coefficients α_1 and α_2, Eq. (2-163) must be supplemented with the following equations in order to determine r_0:

$$\left.\begin{aligned} q_{l1} &= \alpha_1\,(t_{w1} - t_{f1})\,2\pi r_1 \\ q_{l2} &= \alpha_2\,(t_{w2} - t_{f2})\,2\pi r_2 \end{aligned}\right\} \quad \text{(c)}$$

where

$$q_{l1} = q_v \pi (r_0^2 - r_1^2)$$

and

$$q_{l2} = q_v \pi (r_2^2 - r_0^2)$$

To determine r_0, equation (c) is solved together with Eq. (2-163).

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CHAPTER

3

TRANSIENT CONDUCTION

3-1. General

This chapter is devoted to the heat transfer by conduction in the absence of inner heat sources with temperature of the system considered varying not only from point to point, but also with time. The processes of conduction where temperature varies in time as well as in space, are called unsteady, non-stationary, or transient. Transient conduction takes place in the heating (or cooling) of various blanks and articles, glass manufacture, brick burning, vulcanization of rubber, and during starting and stopping of various heat-exchangers, power installations, etc.

Among the practical problems of transient conduction, two groups of processes are of particular importance: (a) when the body tends to thermal equilibrium, and (b) when the temperature of the body is subject to periodic variation.

The first group includes processes involving the heating or cooling of bodies placed in a medium of given thermal state, for example, the heating of an ingot in a furnace, the cooling of bars and pigs in steelworks, the cooling of hardened parts, etc.

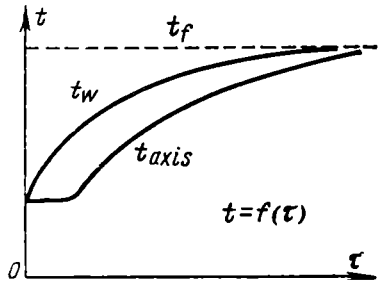

Fig. 3-1. Time variation of temperature within a body

The second group embraces processes developing in periodically functioning heaters, for instance, the heat process of regenerators whose packings are periodically and alternately heated by flue gases and cooled by air.

The nature of the curves obtained in heating a homogeneous solid in a medium of constant temperature t_f is shown in Fig. 3-1. As the body heats, the temperature at each point asymptotically approaches the temperature of the medium. The temperatures of points near the surface of the body change most rapidly. The differences in the rate of heating of individual points diminish with time and theoretically become zero with a sufficiently large interval of time.

When heat is transferred through a wall and the temperature of one of the heat-carrying agents suddenly changes, all the heat involved will not be transferred through it, and a fraction will be consumed for the change in the internal energy of the wall itself (i.e. the change in its temperature); all the heat will be transferred through the separating wall from one fluid to the other only after steady-state conduction is established.

The examples given indicate that transient thermal processes are always linked with a change in the inner energy, or enthalpy, of the substance concerned.

This chapter will treat only some of the most important problems associated with processes in which the body tends to thermal equilibrium, with the aim of showing the general physical peculiarities of the process of this kind, giving

familiarity with the method of solving problems in transient conduction and obtaining mathematical relationships for practical calculations. A more detailed treatment of the methods of solving a great number of problems in transient conduction, including cases when the temperature of the body tends to equilibrium or changes periodically, will be found in A.V. Lykov's work [3.1] and in other literature on the subject [3.2-3.4].

3-2. Analytical Description of the Process

The analytical description of heat conduction includes the differential equation and the conditions of unambiguity.

In the absence of inner heat sources (or sinks) the differential equation of conduction is as follows:

$$\frac{\partial t}{\partial \tau} = a\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right) \tag{3-1}$$

The conditions of unambiguity are given as follows:

physical parameters λ, c, ρ;

geometric form and dimensions of the object l_0, l_1, l_2, . . ., l_n;

temperature of the body at the initial moment of time

$$\tau = 0,\ t = t_0 = f(x, y, z) \tag{3-2}$$

Use may be made of the boundary conditions of the third kind:

$$\left(\frac{\partial t}{\partial n}\right)_{n=0} = -\frac{\alpha}{\lambda}(t_{n=0} - t_f)$$

Together with the conditions of unambiguity (3-2) the differential equation (3-1) gives a full mathematical description of the problem under consideration. Solution consists in finding the function

$$t = f(x, y, z, \tau, \alpha, a, t_0, t_f, l_0, l_1, \ldots, l_n) \tag{3-3}$$

which will satisfy Eq. (3-1) and conditions (3-2).

3-3. Cooling (Heating) of an Infinite Plate

Statement of the problem. Let there be given a plate of thickness 2δ. If the thickness of the plate is small compared with its width and length, such a plate is usually assumed to be infinite.

Under the boundary conditions given the local heat-transfer coefficient α is the same for all points on the surface of the plate. Temperature varies only in the direction x, remaining constant in the two other directions ($\partial t/\partial y = \partial t/\partial z = 0$), hence, in respect to space the problem is one-dimensional. The initial temperature distribution is expressed by some function $t\,(x, 0) = f\,(x)$. Cooling takes place in surroundings of a constant temperature $t_f = \text{const}$. Heat is transferred from both surfaces at a time-constant local heat-transfer coefficient. The reference temperature for the plate at any moment of time is that of the surroundings, i.e. $t - t_f = \vartheta$.

Since the problem is one-dimensional, the differential equation (3-1) takes the form:

$$\frac{\partial \vartheta}{\partial \tau} = a\frac{\partial^2 \vartheta}{\partial x^2} \tag{3-4}$$

The initial conditions for the problem are:

$$\text{at } \tau = 0,\ \vartheta = \vartheta_0 = f(x) - t_f = F(x) \tag{3-5}$$

The given conditions of cooling make the problem symmetrical; and it is convenient to place the origin of coordinates on the axis of the plate, as shown in Fig. 3-2. Then the boundary conditions on the axis and surface of the plate are:

$$\left.\begin{array}{l}\text{(a) on the axis, at } x=0, \left(\dfrac{\partial\vartheta}{\partial x}\right)_{x=0}=0\\ \text{(b) on the surface, at } x=\delta, \left(\dfrac{\partial\vartheta}{\partial x}\right)_{x=\delta}=-\dfrac{\alpha}{\lambda}\vartheta_{x=\delta}\end{array}\right\} \quad (3\text{-}6)$$

The considered problems are uniquely formulated by the differential equation (3-4), the initial conditions (3-5) and the boundary conditions (3-6). The unknown distribution of temperature in a plane plate is obtained by solving Eq. (3-4), taking the initial and boundary conditions into account.

The solution of Eq. (3-4) is envisaged as the product of two functions, one of which is a function only of time τ, and the other only of distance x (the method of separation of variables)

$$\vartheta=\vartheta(\tau, x)=\varphi(\tau)\psi(x) \quad (3\text{-}7)$$

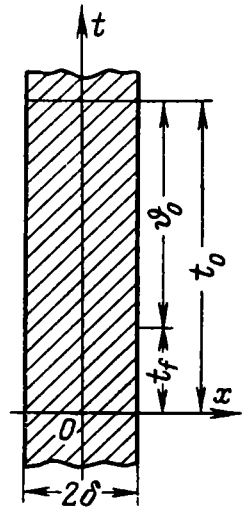

Fig. 3-2. Cooling of an infinite plane plate with $\tau = 0$, $t_0 = \text{const}$, and $\vartheta_0 = \text{const}$

Substituting expression (3-7) in Eq. (3-4), we obtain:

$$\frac{\partial\varphi(\tau)}{\partial\tau}\psi(x)=a\frac{\partial^2\psi(x)}{\partial x^2}\varphi(\tau)$$

or

$$\varphi'(\tau)\psi(x)=a\psi''(x)\varphi(\tau)$$

The variables of this equation are readily separated, so that it can be written in the following form:

$$\frac{\varphi'(\tau)}{\varphi(\tau)}=a\frac{\psi''(x)}{\psi(x)} \quad (3\text{-}8)$$

The left side of Eq. (3-8) is a function only of τ, and the right side a function only of x.

If the argument x is assumed constant and only the time τ is varied, then at any value of it the left-hand side of Eq. (3-8) is equal to the constant quantity present in the right-hand side of the equation, i.e. $\varphi'(\tau)/\varphi(\tau) = \text{const}$. By analogy, if the time τ is held constant and x is varied, for any magnitude of x the right-hand side of Eq. (3-8) must be equal to the constant found in the left-hand side which depends only on τ, i.e. $\psi''(x)/\psi(x) = \text{const}$.

Since Eq. (3-8) must hold for any values of x and τ, its both sides must be equal to the same constant value. Denoting this constant by ε, we rewrite Eq. (3-8):

$$\frac{1}{a}\frac{\varphi'(\tau)}{\varphi(\tau)}=\frac{\psi''(x)}{\psi(x)}=\text{const}=\varepsilon$$

It will be noted that a non-trivial solution of the function $\psi(x)$ is obtained not at all values of ε, but only at $\varepsilon < 0$. Since numerically ε is as yet an arbitrary constant, we assume that $\varepsilon = -k^2$. Substituting this value for ε, we obtain

$$\frac{1}{a}\frac{\varphi'(\tau)}{\varphi(\tau)}=\frac{\psi''(x)}{\psi(x)}=-k^2$$

whence

$$\varphi'(\tau) + ak^2\varphi(\tau) = 0 \tag{3-9}$$

$$\psi''(x) + k^2\psi(x) = 0 \tag{3-10}$$

The constant k is determined from boundary conditions, and the minus sign is selected on the basis of physical reasonings. A minus sign must be taken for processes of heat transfer tending to thermal equilibrium.

We have thus obtained a system of ordinary, readily integrated, equations (3-9) and (3-10).

Equation (3-9) is satisfied by the function $\varphi(\tau) = C_1 e^{-ak^2\tau}$.

Equation (3-10) is satisfied by functions of the kind:

$$\varphi(x) = C_2 \sin(kx) + C_3 \cos(kx)$$

Substituting the expressions for $\varphi(\tau)$ and $\psi(x)$ in Eq. (3-7), we obtain:

$$\vartheta = [C_2 \sin(kx) + C_3 \cos(kx)]\, C_1 e^{-ak^2\tau} \tag{3-11}$$

The expression (3-11) satisfies the original equation (3-4) at any values of the constants C_1, C_2, C_3, and k.

For Eq. (3-11) to be the solution of the problem considered, it must be subordinated to the initial and boundary conditions. Subordinating it to the boundary conditions at $x = 0$

$$\left(\frac{\partial\vartheta}{\partial x}\right)_{x=0} = 0$$

we find:

$$\left(\frac{\partial\vartheta}{\partial x}\right)_{x=0} = C_1 e^{-ak^2\tau} k\, [C_2 \cos(kx) - C_3 \sin(kx)]\,|_{x=0} = 0$$

or

$$C_2 \cos(0) = C_3 \sin(0)$$

whence

$$C_2 = 0$$

This means that the particular solution

$$\psi(x) = C_2 \sin(kx)$$

must be rejected, since it does not satisfy the given boundary conditions.

Taking into account that $C_2 = 0$, and denoting $C_1 C_3 = A$, Eq. (3-11) may be presented in the form

$$\vartheta = Ae^{-ak^2\tau} \cos(kx) \tag{3-12}$$

Subordinating this particular solution to the boundary condition

$$\left(\frac{\partial\vartheta}{\partial x}\right)_{x=\delta} = -\frac{\alpha}{\lambda}\vartheta_{x=\delta}$$

we obtain:

$$-kAe^{-ak^2\tau} \sin(k\delta) = -\frac{\alpha}{\lambda} Ae^{-ak^2\tau} \cos(k\delta) \tag{3-13}$$

hence after simple rearrangement we obtain:

$$\cot(k\delta) = \frac{k\delta}{\frac{\alpha\delta}{\lambda}}$$

where $\frac{\alpha\delta}{\lambda} = \mathrm{Bi}$ is the Biot number. If we denote $k\delta = \mu$, this last expression becomes

$$\cot\mu = \frac{\mu}{\mathrm{Bi}} \tag{3-14}$$

From analysis of this trigonometric equation it follows that for each value of the Biot number there exists an infinite number of solutions. Equation (3-14) is more readily solved graphically.

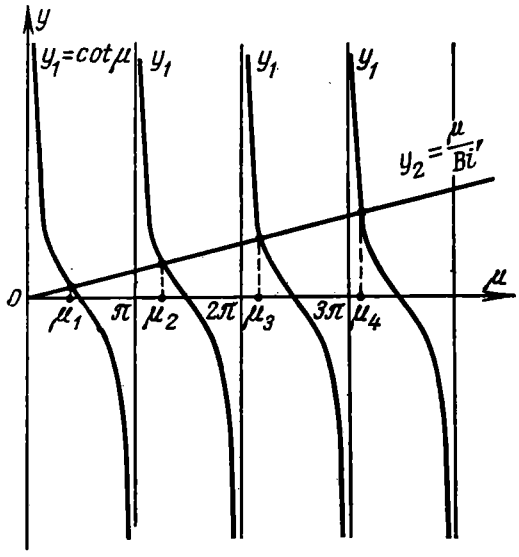

Fig. 3-3. Graphical solution of Eq. (3-14)

Let us denote the left-hand side of Eq. (3-14) by $y_1 = \cot\mu$, and the right by $y_2 = \mu/\mathrm{Bi}$. The intersection of the cotangent curve y_1 with the straight line y_2 gives the roots μ of the characteristic equation (Fig. 3-3).

It follows from Fig. 3-3 that there is an infinite number of solutions for μ_n, with each subsequent value larger than the preceding one:

$$\mu_1 < \mu_2 < \mu_3 < \ldots < \mu_n < \ldots$$

It is important to note that each value of the Biot number gives its own roots for Eq. (3-14).

The first four roots of Eq. (3-14) are given in Table 3-1 for different values of the Bi (from 0 to ∞).

With $\mathrm{Bi} \to \infty$ the straight line $y_2 = \mu/\mathrm{Bi}$ coincides with the x-axis, and the roots of the equation are:

$$\mu_1 = \frac{\pi}{2}; \quad \mu_2 = \frac{3}{2}\pi; \quad \mu_3 = \frac{5}{2}\pi; \ldots; \quad \mu_n = (2n-1)\frac{\pi}{2}$$

With $\mathrm{Bi} \to 0$ the straight line $y_2 = \frac{\mu}{\mathrm{Bi}}$ coincides with the y-axis and its slope tends to infinity, with the roots of Eq. (3-14) equal to:

$$\mu_1 = 0; \quad \mu_2 = \pi; \quad \mu_3 = 2\pi; \ldots; \quad \mu_n = (n-1)\pi$$

where $n = 1, 2, 3, \ldots$.

For other final meanings of the Biot number the values of μ_n are intermediate (see Table 3-1).

Hence, each found μ has its special temperature distribution:

$$\left.\begin{aligned}
\vartheta_1 &= A_1 \cos\left(\mu_1 \frac{x}{\delta}\right) e^{-\mu_1^2 \frac{a\tau}{\delta^2}} \\
\vartheta_2 &= A_2 \cos\left(\mu_2 \frac{x}{\delta}\right) e^{-\mu_2^2 \frac{a\tau}{\delta^2}} \\
&\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots \\
\vartheta_n &= A_n \cos\left(\mu_n \frac{x}{\delta}\right) e^{-\mu_n^2 \frac{a\tau}{\delta^2}}
\end{aligned}\right\} \tag{3-15}$$

TABLE 3-1

Values of μ_n for a Plate

Bi	μ_1	μ_2	μ_3	μ_4	Bi	μ_1	μ_2	μ_3	μ_4
0	0.0000	3.1416	6.2832	9.4248	1.0	0.8603	3.4256	6.4373	9.5293
0.001	0.0316	3.1419	6.2833	9.4249	1.5	0.9882	3.5422	6.5097	9.5801
0.002	0.0447	3.1422	6.2835	9.4250	2.0	1.0769	3.6436	6.5783	9.6296
0.004	0.0632	3.1429	6.2838	9.4252	3.0	1.1925	3.8088	6.7040	9.7240
0.006	0.0774	3.1435	6.2841	9.4254	4.0	1.2646	3.9352	6.8140	9.8119
0.008	0.0893	3.1441	6.2845	9.4256	5.0	1.3138	4.0336	6.9096	9.8928
0.01	0.0998	3.1448	6.2848	9.4258	6.0	1.3496	4.1116	6.9924	9.9667
0.02	0.1410	3.1479	6.2864	9.4269	7.0	1.3766	4.1746	7.0640	10.0339
0.04	0.1987	3.1543	6.2895	9.4290	8.0	1.3978	4.2264	7.1263	10.0949
0.06	0.2425	3.1606	6.2927	9.4311	9.0	1.4149	4.2694	7.1806	10.1502
0.08	0.2791	3.1668	6.2959	9.4333	10.0	1.4289	4.3058	7.2281	10.2003
0.1	0.3111	3.1731	6.2991	9.4354	15.0	1.4729	4.4255	7.3959	10.3898
0.2	0.4328	3.2039	6.3148	9.4459	20.0	1.4961	4.4915	7.4954	10.5117
0.3	0.5218	3.2341	6.3305	9.4565	30.0	1.5202	4.5615	7.6057	10.6543
0.4	0.5932	3.2636	6.3461	9.4670	40.0	1.5325	4.5979	7.6647	10.7334
0.5	0.6533	3.2923	6.3616	9.4775	50.0	1.5400	4.6202	7.7012	10.7832
0.6	0.7051	3.3204	6.3770	9.4879	60.0	1.5451	4.6353	7.7259	10.8172
0.7	0.7506	3.3477	6.3923	9.4983	80.0	1.5514	4.6543	7.7573	10.8606
0.8	0.7910	3.3744	6.4074	9.5087	100.0	1.5552	4.6658	7.7764	10.8871
0.9	0.8274	3.4003	6.4224	9.5190	∞	1.5708	4.7124	7.8540	10.9956

The partial solutions obtained in (3-15) will satisfy the differential equation of conduction at any value of the constants A_1, A_2, . . ., A_n, but none will agree with the actual temperature distribution at the initial moment of time. But any real temperature curve representing the initial temperature distribution can be plotted, however, by superimposing an infinite number of such distribution curves on a corresponding choice of values of A_n.

From what has been said here, the general solution can be represented by the sum of an infinite series:

$$\vartheta = \sum_{n=1}^{n\to\infty} A_n \cos\left(\mu_n \frac{x}{\delta}\right) e^{-\mu_n^2 \frac{a\tau}{\delta^2}} \tag{3-16}$$

If the individual distributions (3-15) satisfy the differential equations and boundary conditions, their sum will also satisfy them.

The constant A_n of Eq. (3-16) is found from the initial conditions. Subordinating Eq. (3-16) to the initial condition, we obtain:

$$\vartheta_0 = F(x) = \sum_{n=1}^{n\to\infty} A_n \cos\left(\mu_n \frac{x}{\delta}\right) \tag{3-17}$$

Equation (3-17) is the Fourier expansion of an even function with given values of μ_n determined by the characteristic equation (3-14). This order of μ_n

numbers agrees with the formula

$$\int_{-\delta}^{+\delta} \cos\left(\mu_n \frac{x}{\delta}\right) \cos\left(\mu_m \frac{x}{\delta}\right) dx \begin{cases} = 0 & \text{at} \quad n \neq m \\ \neq 0 & \text{at} \quad n = m \end{cases}$$

which may be used to determine all the coefficients A_n of Eq. (3-17). For this purpose we multiply both sides of Eq. (3-17) by $\cos\left(\mu_n \frac{x}{\delta}\right) dx$, then integrate the relation obtained for plate thickness. Then

$$\int_{-\delta}^{+\delta} F(x) \cos\left(\mu_n \frac{x}{\delta}\right) dx = A_n \int_{-\delta}^{+\delta} \cos^2\left(\mu_n \frac{x}{\delta}\right) dx, \tag{3-18}$$

because all other terms of the right side, for which $n \neq m$, become zero. The integral of the right-hand side of Eq. (3-18) is equal to $\delta\left(1 + \frac{1}{2\mu_n} \sin 2\mu_n\right)$.

Hence

$$A_n = \frac{\mu_n}{\delta(\mu_n + \sin\mu_n \cos\mu_n)} \int_{-\delta}^{+\delta} F(x) \cos\left(\mu_n \frac{x}{\delta}\right) dx \tag{3-19}$$

It follows from Eq. (3-19) that A_n is a function only of the root of the characteristic equation and of initial temperature distribution.

Substituting the expression for the constant A_n into Eq. (3-16), we obtain the final expression for the temperature field of a cooled homogeneous plate:

$$\vartheta = \sum_{n=1}^{n\to\infty} \frac{\mu_n}{\delta(\mu_n + \cos\mu_n \sin\mu_n)} \left[\int_{-\delta}^{+\delta} F(x) \cos\left(\mu_n \frac{x}{\delta}\right) dx\right] \cos\left(\mu_n \frac{x}{\delta}\right) e^{-\mu_n^2 \frac{a\tau}{\delta^2}} \tag{3-20}$$

Equation (3-20) makes it possible to determine the temperature at any point of the plate and for any moment of time τ for any initial temperature distribution ϑ_0.

If at the initial moment of time ($\tau = 0$) temperature distribution in the plate is uniform (Fig. 3-2), i.e. $t_0 - t_f = \vartheta_0 = \text{const}$, the integral in Eq. (3-19) is equal to $(\vartheta_0 2\delta/\mu_n) \sin\mu_n$. Allowing for the foregoing, the expression for the constant A_n becomes:

$$A_n = \vartheta_0 \frac{2\sin\mu_n}{\mu_n + \sin\mu_n \cos\mu_n} \tag{3-21}$$

Substituting the value of A_n obtained for the case of uniform temperature distribution in the plate at the initial moment of time into Eq. (3-20), we get:

$$\vartheta = \sum_{n=1}^{n\to\infty} \frac{\vartheta_0 2\sin\mu_n}{\mu_n + \sin\mu_n \cos\mu_n} \cos\left(\mu_n \frac{x}{\delta}\right) e^{-\mu_n^2 \frac{a\tau}{\delta^2}} \tag{3-22}$$

It is convenient to write this equation for the temperature field (3-22) in a dimensionless form. For this purpose we divide the right and left sides of the equation by ϑ_0, denoting

$$\frac{A_n}{\vartheta_0} = D_n$$

With such a rearrangement, we obtain:

$$\frac{\vartheta}{\vartheta_0} = \sum_{n=1}^{n\to\infty} \frac{2\sin\mu_n}{\mu_n + \sin\mu_n \cos\mu_n} \cos\left(\mu_n \frac{x}{\delta}\right) e^{-\mu_n^2 \frac{a\tau}{\delta^2}} \tag{3-23}$$

The components of Eq. (3-23), such as $\frac{\vartheta}{\vartheta_0}$; D_n; μ_n; $\frac{a\tau}{\delta^2}$ and $\frac{x}{\delta}$ are dimensionless, and have the following meaning:

$\frac{\vartheta}{\vartheta_0} = \Theta$ = dimensionless temperature;

$\frac{x}{\delta} = X$ = dimensionless coordinate;

$\frac{a\tau}{\delta^2} = \mathrm{Fo}$ = *Fourier number*, or dimensionless time;

D_n = dimensionless coefficient.

Taking this notation into account, Eq. (3-23) becomes:

$$\Theta = \sum_{n=1}^{n\to\infty} \frac{2\sin\mu_n}{\mu_n + \sin\mu_n\cos\mu_n} \cos(\mu_n X) \exp(-\mu_n^2 \mathrm{Fo}) \tag{3-24}$$

Analysis of the solution. Since $\mu_1, \mu_2, \ldots, \mu_n$ represent a series of increasing numbers, the larger μ the smaller the significance of the successive numbers of the series compared with the preceding one. In addition, the larger the Fourier number, the greater the rate at which the terms of the series decrease with increasing μ_n.

Numerous studies have shown that the series (3-24) becomes already rapidly converging with $\mathrm{Fo} \geqslant 0.3$ so that temperature distribution is described with sufficient accuracy by its first term:

$$\Theta = \frac{2\sin\mu_1}{\mu_1 + \sin\mu_1\cos\mu_1} \cos(\mu_1 X) \exp(-\mu_1^2 \mathrm{Fo}) \tag{3-25}$$

The term $\frac{2\sin\mu_1}{\mu_1 + \sin\mu_1\cos\mu_1}$ was previously denoted by D_1. This makes it possible to write Eq. (3-25) in the form:

$$\Theta = D_1 \cos(\mu_1 X) \exp(-\mu_1^2 \mathrm{Fo}) \tag{3-25'}$$

The value D_1 is a function only of the Biot number, and can be calculated and tabulated beforehand. Besides, if we consider the temperature for a certain value $X = x/\delta$, the term $\cos(\mu_1 X)$ is also a function of Bi. For the axis of the plate, $X = x/\delta = 0$ and $\cos(\mu_1 0) = 1$, and for its surface $X = x/\delta = 1$ and $\cos(\mu_1 1) = \cos\mu_1$.

For the axis of the plate let us denote the product $D_1 \cos(0)$ as a certain function N (Bi). This allows Eq. (3-25) to be presented in the following form:

$$\Theta_{X=0} = N(\mathrm{Bi}) \exp(-\mu_1^2 \mathrm{Fo}) \tag{3-26}$$

If we denote the product $D_1 \cos\mu_1$ for the surface as a function P (Bi), then Eq. (3-25′) becomes:

$$\Theta_{X=1} = P(\mathrm{Bi}) \exp(-\mu_1^2 \mathrm{Fo}) \tag{3-27}$$

Tables of the functions N (Bi) and P (Bi) have been compiled and published [3.5, 3.6, 3.7]. It further follows from Eqs. (3-26) and (3-27) that with a given coordinate the dimensionless temperature is a function of two dimensionless terms, the Biot and Fourier numbers, i.e.

$$\Theta_{X=0} = f_1(\mathrm{Bi}, \mathrm{Fo}), \quad \text{and} \quad \Theta_{X=1} = f_2(\mathrm{Bi}, \mathrm{Fo})$$

Taking the logarithm of Eq. (3-26), we obtain:

$$\ln\Theta_{X=0} = \mathrm{n}\, N(\mathrm{Bi}) - \mu_1^2 \mathrm{Fo} \tag{3-28}$$

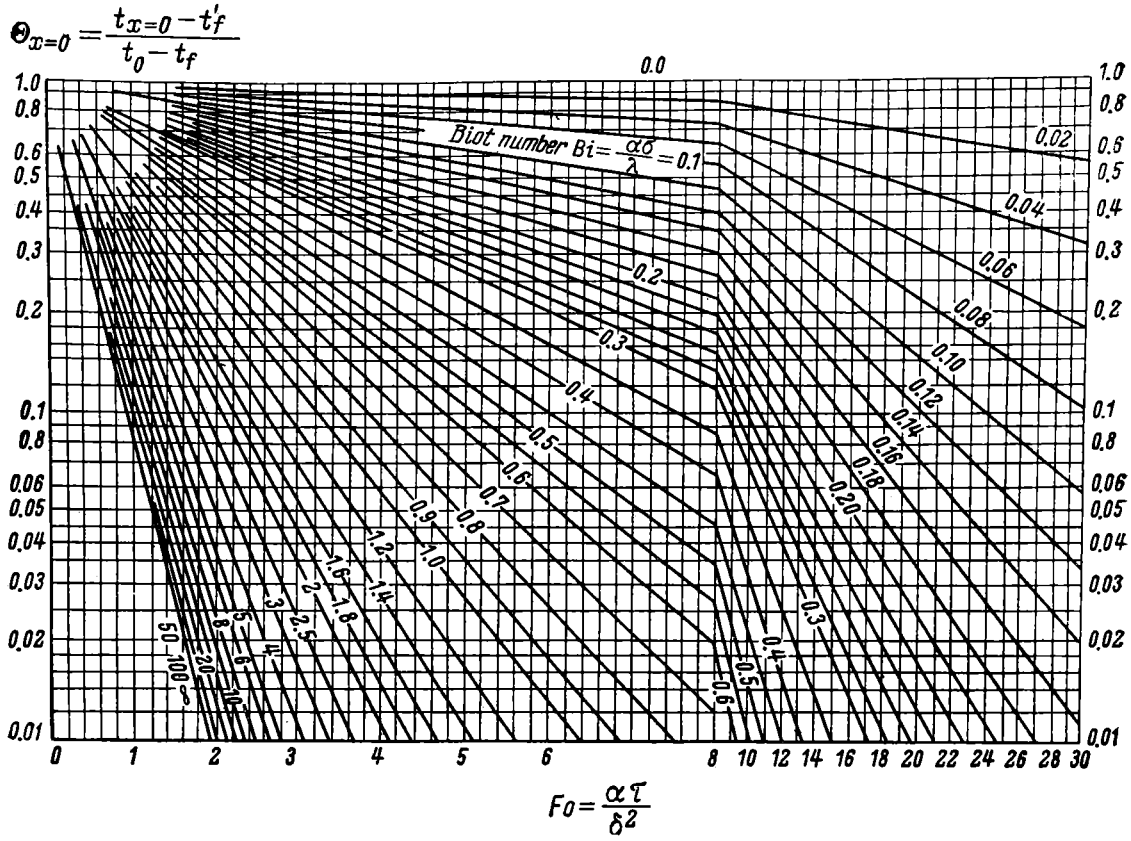

Fig. 3-4. $\Theta = f_1$ (Fo, Bi) dependence for the middle of plate

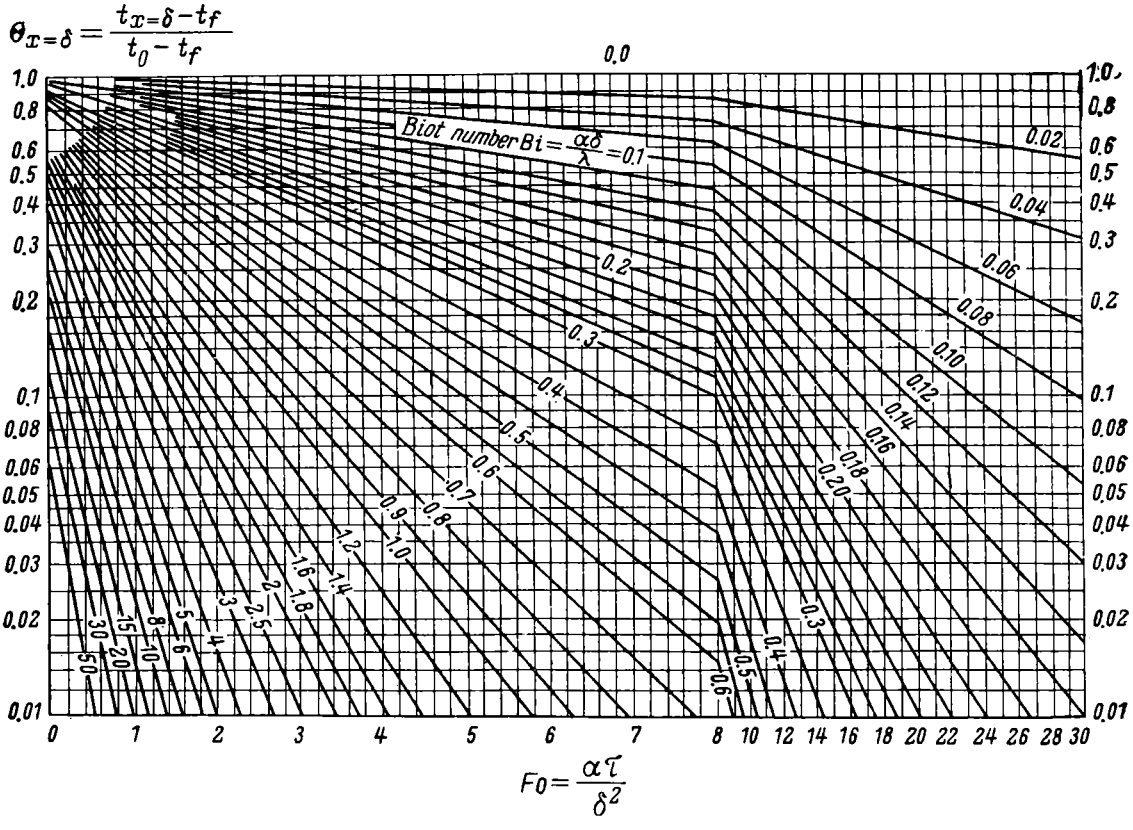

Fig. 3-5. $\Theta = f_2$ (Fo, Bi) dependence for the surface of a plate

A similar equation can be obtained by taking the logarithm of Eq. (3-27).

It follows from Eq. (3-28) that with a given coordinate and given Biot number the natural logarithm of the dimensionless temperature depends on time in a linear way. This circumstance makes a graphical solution of Eqs. (3-26) and (3-27) possible (Figs. 3-4 and 3-5).

It follows from Eq. (3-24) that at any moment of time and given boundary conditions the field of temperature distribution in a cooled (heated) plate is represented by a symmetrical curve with its maximum point at the axis of the plate ($X = 0$). Each subsequent moment of time will have its own curve, decreasing monotonically in the direction of the surfaces. Furthermore, at any moment of time the extensions of the tangents to the curves at points ($X = \pm 1$) pass through two guide points $+A$ and $-A$ located at a distance $\pm X_0$ from the surface of the plate, equal to $X_0 = 1/\mathrm{Bi}$ (Fig. 3-6).

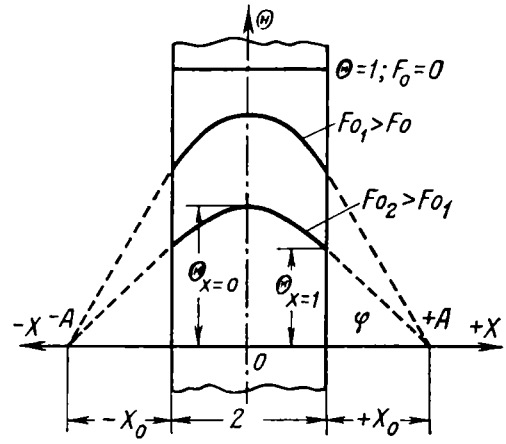

Fig. 3-6. Variation of temperature field of an infinite plane plate during cooling

In order to prove this important property, let us consider the temperature field for an arbitrary moment of time $\mathrm{Fo} > 0$.

Multiplying the boundary condition (3-6) at $x = \pm\delta$ by δ/ϑ_0 we obtain:

$$-\left[\frac{\partial\left(\frac{\vartheta}{\vartheta_0}\right)}{\partial\left(\frac{x}{\delta}\right)}\right]_{x=\delta} = \frac{\alpha\delta}{\lambda}\left(\frac{\vartheta}{\vartheta_0}\right)_{x=\delta}$$

Writing this expression in dimensionless terms, we have:

$$\left(\frac{\partial\Theta}{\partial X}\right)_{X=1} = -\mathrm{Bi}\Theta_{X=1} \qquad \text{(a)}$$

From Fig. 3-6 it follows that

$$-\left(\frac{\partial\Theta}{\partial X}\right)_{X=1} = \frac{\Theta_{X=1}}{X_0} = \tan\varphi \qquad \text{(b)}$$

Comparing (a) and (b), we obtain:

$$X_0 = \frac{1}{\mathrm{Bi}} \qquad (3\text{-}29)$$

It follows from Eq. (3-29) that the distance from point A to the surface is determined by the given conditions of unambiguity which are true for any given moment of time. Hence the tangents to all temperature curves at the point of intersection with the plate surface will always pass through point A, provided the boundary conditions are invariable. This statement is true not only for a plate, but also for a cylinder, sphere, and bodies of other geometric forms.

The property of the temperature curves here demonstrated makes it possible to determine the nature of temperature variation in a body at a given value of the Biot number. Let us consider three cases.

(1) $Bi \to \infty$ (*practically* $Bi > 100$).

If the Biot number tends to infinity, the temperature of the plate surface at once becomes equal to the temperature of the surroundings in which it is placed. This is evident from Eq. (3-29): at $\mathrm{Bi} \to \infty$, $X_0 = \frac{1}{\mathrm{Bi}} = 0$. This means that the intersection point of the tangents to the temperature curves is located on the surface of the plate. From $\mathrm{Bi} = \frac{\delta/\lambda}{1/\alpha}$ it follows that $\mathrm{Bi} \to \infty$ at the given physical parameters and thickness of the plate when the heat-transfer coefficient $\alpha \to \infty$, i.e. when heat is intensively removed from the surface. Under these conditions the process of cooling is governed only by the physical properties and dimensions of the body. The value $\mu_n = (2n-1)\frac{\pi}{2}$ and the coefficient of series (3-24) is then equal to:

$$D_n = \frac{2\sin\mu_n}{\mu_n + \sin\mu_n\cos\mu_n} = \frac{2\sin\left[(2n-1)\frac{\pi}{2}\right]}{(2n-1)\frac{\pi}{2} + \sin\left[(2n-1)\frac{\pi}{2}\right]\cos\left[(2n-1)\frac{\pi}{2}\right]} = \frac{4(-1)^{n+1}}{\pi(2n-1)}$$

The general solution for the considered case becomes:

$$\Theta = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\left[\frac{(2n-1)}{2}\pi X\right]\exp\left[-\left(\frac{2n-1}{2}\right)^2\pi^2\mathrm{Fo}\right] \qquad (3\text{-}30)$$

Then the temperature at the axis of the plate ($X = 0$) is

$$\Theta_{X=0} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\exp\left[-\left(\frac{2n-1}{2}\right)^2\pi^2\mathrm{Fo}\right] \qquad (3\text{-}31)$$

At $X = 1$, $\cos[2(n-1)\frac{\pi}{2}X] = 0$ and, hence, $\Theta_{X=1} = 0$.

The distribution of temperature in the plate at $\mathrm{Fo} > 0$ is shown in Fig. 3-7 where $\mathrm{Fo}_1 < \mathrm{Fo}_2 < \mathrm{Fo}_3 < \mathrm{Fo}_4 < \ldots < \mathrm{Fo}_n$. As has already been noted, the series (3-24) rapidly converges at $\mathrm{Fo} \geqslant 0.3$ and the error does not exceed 1 per cent, if all the terms of the series, except the first, are disregarded. Under these conditions Eq. (3-31) becomes:

$$\Theta_{X=0} = \frac{4}{\pi}\exp\left[-\left(\frac{\pi}{2}\right)^2\mathrm{Fo}\right] \qquad (3\text{-}32)$$

Taking the logarithm of Eq. (3-32) and solving it for the Fourier number, we obtain:

$$\mathrm{Fo} = \frac{4}{\pi^2}\ln\left(\frac{4}{\pi}\frac{1}{\Theta_{X=0}}\right) \qquad (3\text{-}33)$$

Considering that $\mathrm{Fo} = \frac{a\tau}{\delta^2}$, Eq. (3-33) becomes:

$$\tau = \left(\frac{2\delta}{\pi}\right)^2\frac{1}{a}\ln\left(\frac{1}{\Theta_{X=0}}\frac{4}{\pi}\right) \qquad (3\text{-}33')$$

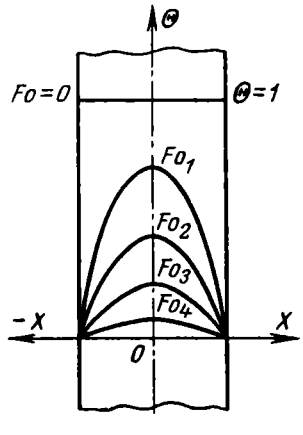

Fig. 3-7. Temperature distribution in a plane wall cooled at $\mathrm{Bi} \to \infty$; $\mathrm{Fo}_1 < \mathrm{Fo}_2 < \mathrm{Fo}_3 < \mathrm{Fo}_4$ conditions

This formula may be used to determine the time required for heating the middle of the plate to the given temperature.

(2) *Very small Biot numbers (practically Bi < 0.1).*

If the Biot number is small, all the coefficients of the series tend to zero, i.e. $D_n \to 0$, inasmuch as $\mu_n = (n-1)\pi$ now, except D_1 which is equal to:

$$D_1 = \left| \frac{2 \sin \mu_1}{\mu_1 + \sin \mu_1 \cos \mu_1} \right|_{\mu_1 \to 0} = 1$$

The expression $\mathrm{Bi} = \frac{\delta/\lambda}{1/\alpha}$ shows that small Biot numbers can hold with thin plates, and with large thermal conductivities λ and small heat-transfer coefficients α. It should be noted that at small values of μ_1 the functions $\tan \mu_1$ and $\sin \mu_1$ may be replaced by their arguments, and then the characteristic equation (3-14) becomes:

$$\frac{1}{\mu_1} = \frac{\mu_1}{\mathrm{Bi}}$$

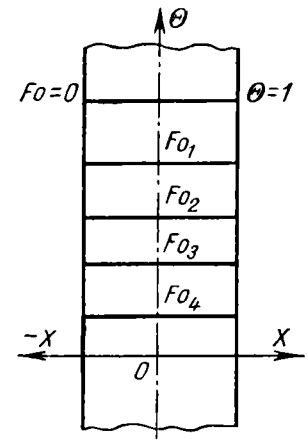

Fig. 3-8. Temperature distribution in a plane wall cooled at $\mathrm{Bi} \to 0$; $\mathrm{Fo}_1 < \mathrm{Fo}_2 < \mathrm{Fo}_3 < \mathrm{Fo}_4$ conditions

In the light of the foregoing Eq. (3-24) acquires the following form:

$$\Theta = \cos(\mu_1 X) \exp(-\mu_1^2 \mathrm{Fo}) = \cos\left(\sqrt{\mathrm{Bi}}\, X\right) \exp(-\mathrm{BiFo}) \qquad (3\text{-}34)$$

Let us determine the temperatures at the axis and surface of the plate:

$$\text{at } X = 0, \quad \Theta_{X=0} = \exp(-\mathrm{BiFo}) \qquad (3\text{-}35)$$

$$\text{at } X = 1, \quad \Theta_{X=1} = \cos\left(\sqrt{\mathrm{Bi}}\right) \exp(-\mathrm{BiFo}) \qquad (3\text{-}36)$$

The ratio of temperatures at the axis and the surface of the plate is

$$\frac{\Theta_{X=0}}{\Theta_{X=1}} = \frac{\exp(-\mathrm{BiFo})}{\cos\left(\sqrt{\mathrm{Bi}}\right) \exp(-\mathrm{BiFo})} \to 1$$

At small Biot numbers the surface temperature differs little from the temperature at the axis. This shows that temperature distribution across the plate is uniform, and that its distribution curve remains almost parallel with the Ox-axis for any moment of time (Fig. 3-8).

The tangents to the temperature curves at the points where the latter cross the surface must intersect the x-axis at infinity: with $\mathrm{Bi} \to 0$, $X_0 = \frac{1}{\mathrm{Bi}} \to \infty$.

In the present case, the rate of heating and cooling of the body depends only on the intensity of heat transfer. In other words, the process of temperature equalization in the plate develops at a rate essentially exceeding that at which heat is removed from its surface. The problem becomes outward.

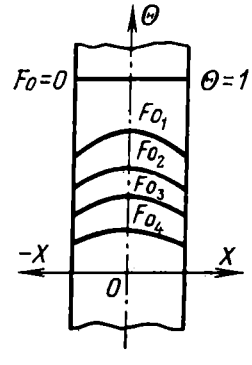

Fig. 3-9. Temperature distribution in a plane wall cooled at the conditions of a finite Biot number; $\mathrm{Fo}_1 < \mathrm{Fo}_2 < \mathrm{Fo}_3 < \mathrm{Fo}_4$

(3) *The Biot number falls within* $0.1 \leqslant Bi < 100$.

Here μ_n is a function of Bi, i.e. it depends on the thickness of the plate. The temperature distribution curves for any moment of time will appear as

shown in Fig. 3-9. The intensity of the process of cooling (heating) here is determined both by the internal and external thermal resistances.

3-4. Determining Quantity of Heat Q Removed from a Plate During Cooling

The quantity of heat removed from, or imparted to, a plate from both sides per time between $\tau = 0$ and $\tau = \infty$ must be equal to the change in internal energy, or enthalpy, of the plate during its complete cooling or heating:

$$Q_p = 2\delta f \rho c\,(t_0 - t_f) \tag{3-37}$$

Then the change in enthalpy during any interval of time between $\tau = 0$ and τ_1 or, what is the same thing, between $Fo = 0$ and Fo_1 will change by

$$Q = Q_p - Q_1 = 2\delta f \rho c\,(t_0 - t_f)\left(1 - \frac{\bar{t}_1 - t_f}{t_0 - t_f}\right)$$

or

$$Q = Q_p\,(1 - \bar{\Theta}_1) \tag{3-38}$$

where $\bar{\Theta}_1 = \frac{\bar{t}_1 - t_f}{t_0 - t_f}$ is the dimensionless mean temperature across the plate at time τ_1.

From Eqs. (3-37) and (3.38) it follows that calculation of the heat flux into or from the plate consists in finding the dimensionless mean temperature at the desired moment of time. The dimensionless mean temperature of a layer of the plate confined between the axis of symmetry and plane X is found as

$$\bar{\Theta} = \frac{1}{X}\int_0^X \Theta\, dX$$

in accordance with the mean value theorem.

Substituting the value of Θ from Eq. (3-24) in the integral expression and taking the integral between the limits of zero and unity, we obtain:

$$\bar{\Theta} = \sum_{n=1}^{\infty} \frac{2\sin^2 \mu_n}{\mu_n^2 + \mu_n \sin \mu_n \cos \mu_n} \exp\left(-\mu_n^2 Fo\right) \tag{3-39}$$

Substituting the mean temperature of the plate calculated from formula (3-39) in Eq. (3-38), we obtain the quantity of heat lost by the plate to the surroundings during the period in question.

At $Bi \to \infty$ (practically $Bi > 100$), Eq. (3-39) becomes:

$$\bar{\Theta} = \sum_{n=1}^{\infty} \frac{8}{\pi^2 (2n-1)^2} \exp\left[-\left(\frac{(2n-1)}{2}\right)^2 \pi^2 Fo\right] \tag{3-40}$$

If $Bi \to 0$ (in practice $Bi < 0.1$), Eq. (3-39) acquires the following form:

$$\bar{\Theta} = \exp(-BiFo) \tag{3-41}$$

With Fourier numbers $Fo \geqslant 0.3$ for a plate we can limit ourselves to the first term of series (3-39); then

$$\bar{\Theta} = \frac{2\sin^2 \mu_1}{\mu_1^2 + \mu_1 \sin \mu_1 \cos \mu_1} \exp\left(-\mu_1^2 Fo\right) \tag{3-42}$$

The factor $\frac{2\sin^2\mu_1}{\mu_1^2+\mu_1\cos\mu_1\sin\mu_1}$ depends only on the Bi number and can be presented as a function M (Bi); then Eq. (3-42) becomes:

$$\overline{\Theta} = M\,(\mathrm{Bi})\exp(-\mu_1^2\,\mathrm{Fo}) \tag{3-42'}$$

The function M (Bi) can be calculated and presented in tables. Then calculation of the mean temperature amounts to calculation of the exponent.

3-5. Cooling (Heating) of an Infinite Cylinder

Let a cylinder with radius r_0 lose heat to the surroundings from its surface; the heat-transfer coefficient α is equal at all points of the surface and remains constant throughout the entire period of cooling. The temperature of the surroundings, t_f, is constant. At the initial moment of time ($\tau = 0$) the temperature is some function of the $t\,(r,\ 0) = f\,(r)$. The reference temperature for the cylinder is assumed equal to that of the surroundings, i.e. $t - t_f = \vartheta$. In this case the equation of heat transfer takes the form:

$$\frac{\partial\vartheta}{\partial\tau} = a\left(\frac{\partial^2\vartheta}{\partial r^2}+\frac{1}{r}\frac{\partial\vartheta}{\partial r}\right) \tag{3-43}$$

Boundary and initial conditions are:

at $\tau = 0$ and $0 \leqslant r \leqslant r_0$,

$$\vartheta = \vartheta_0 = f\,(r) - t_f = F\,(r)$$

at $r = 0$ and $\tau > 0$,

$$\left(\frac{\partial\vartheta}{\partial r}\right)_{r=0} = 0$$

at $r = r_0$ and $\tau > 0$,

$$\left(\frac{\partial\vartheta}{\partial r}\right)_{r=r_0} = -\frac{\alpha}{\lambda}\vartheta_{r=r_0}$$

Let us solve the formulated problem by separating the variables, i.e. $\vartheta\,(r,\ \tau) = \varphi\,(\tau)\,\psi\,(r)$. Substituting this expression into Eq. (3-43), we obtain two ordinary differential equations of the kind

$$\varphi'\,(\tau) + ak^2\varphi\,(\tau) = 0 \tag{3-44}$$

$$\psi''\,(r) + \frac{1}{r}\psi'(r) + k^2\psi\,(r) = 0 \tag{3-45}$$

From the preceding paragraph it is known that the solution of Eq. (3-44) is as follows:

$$\varphi\,(\tau) = C_1 e^{-\alpha k^2\tau} \tag{a}$$

Equation (3-45) is a Bessel equation whose integral has the appearance:

$$\psi\,(r) = C_2 J_0\,(kr) + C_3 Y_0\,(kr) \tag{b}$$

where C_1 and C_2 = integration constants;
J_0 and Y_0 = zero-order Bessel functions of the first and second kind.

Since the temperature at the cylinder axis ($r = 0$) must be a finite quantity, and $Y_0\,(0)$ tends to infinity, then from the physical considerations a special solution of Eq. (3-45) should not contain a Bessel function of the second kind and the integration constant C_3 must be equal to zero.

Allowing for the foregoing, Eq. (b) takes the form:

$$\psi(r) = C_2 J_0(kr) \qquad (c)$$

If we denote $kr_0 = \mu$, then the special solution of Eq. (3-43) will have the following appearance:

$$\vartheta(r, \tau) = Ce^{-\mu^2 \frac{a\tau}{r_0^2}} J_0\left(\mu \frac{r}{r_0}\right) \qquad (3\text{-}46)$$

The constant μ in Eq. (3-46) is determined from the boundary conditions ($r = r_0$) the solution of which leads to the characteristic equation

$$\frac{J_0\mu}{J_1\mu} = \frac{\mu}{\mathrm{Bi}} \qquad (3\text{-}47)$$

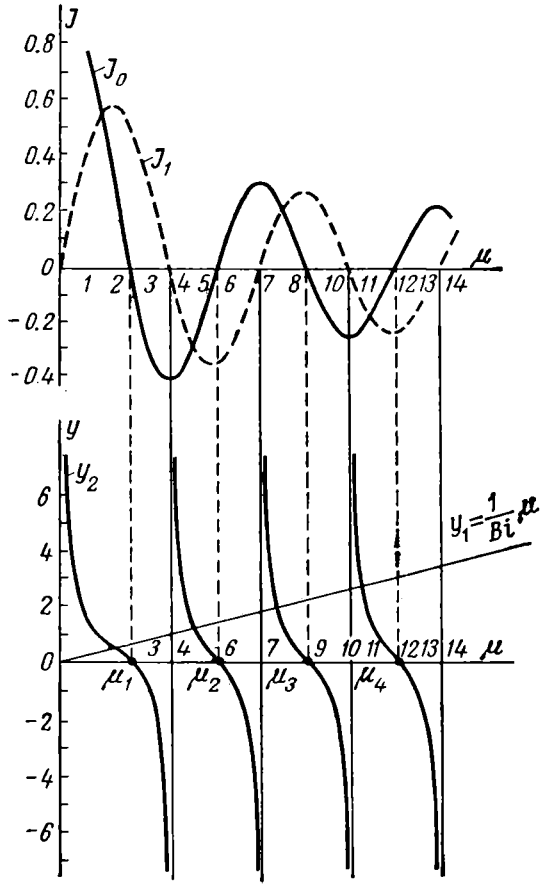

Fig. 3-10. To the solution of Eq. (3-47)

where $J_1(\mu)$ is first-order Bessel function of the first kind.

Equation (3-47) is a transcendental one and graphically it is solved conveniently, denoting

$$\mu/\mathrm{Bi} = y_1;\ J_0(\mu)/J_1(\mu). = y_2$$

It will be noted that y_2 turns into zero at points for which $J_0(\mu) = 0$.

In points at which the function $J_1(\mu)_1$ becomes zero, the function y_2 becomes discontinuous and gets equal to $\pm\infty$. The functions $J_0(\mu)$ and $J_1(\mu)$ are periodical transient functions and the curve $y_2 = J_0(\mu)/J_1(\mu)$ bears resemblance to a cotangent curve, but with a diminishing period. The graph of the function $y_1 = \mu/\mathrm{Bi}$ is a straight line passing through the origin of coordinates. Having carried out the plotting, as shown in Fig. 3-10, the points at which the function y_2 intersects the straight line y_1 yield the roots of the characteristic equation (3-47). It follows from Fig. 3-10 that Eq. (3-47) has an infinite number of solutions, and the roots proper, just as for a plate, represent a series of increasing numbers, i.e. $\mu_1 < \mu_2 < \mu_3 < \ldots < \mu_n$, where $n = 1, 2, 3, \ldots, \infty$.

The first four roots of Eq. (3-47), μ_1, μ_2, μ_3 and μ_4, are given in Table 3-2 for different values of the Biot number (Bi), varying from 0 to ∞.

The general solution will be the sum of all special solutions of Eq. (3-46):

$$\vartheta = \sum_{n=1}^{n=\infty} C_n J_0\left(\mu_n \frac{r}{r_0}\right) \exp\left(-\mu_n^2 \frac{a\tau}{r_0^2}\right) \qquad (3\text{-}48)$$

The constant C_n in Eq. (3-48) is found from the initial conditions. At $\tau = 0$ $\vartheta = \vartheta_0 = F(r)$ and Eq. (3-48) reduces into:

$$\vartheta_0 = F(r) = \sum_{n=1}^{n\to\infty} C_n J_0\left(\mu_n \frac{r}{r_0}\right) \qquad (3\text{-}49)$$

TABLE 3-2

Values of μ_n for a Cylinder

Bi	μ_1	μ_2	μ_3	μ_4	Bi	μ_1	μ_2	μ_3	μ_4
0.0	0.0000	3.8317	7.0156	10.1735	2.0	1.5994	4.2910	7.2884	10.3658
0.01	0.1412	3.8343	7.0170	10.1745	3.0	1.7887	4.4634	7.4103	10.4566
0.02	0.1995	3.8369	7.0184	10.1754	4.0	1.9081	4.6018	7.5201	10.5423
0.04	0.2814	3.8421	7.0213	10.1774	5.0	1.9898	4.7131	7.6177	10.6223
0.06	0.3438	3.8473	7.0241	10.1794	6.0	2.0490	4.8033	7.7039	10.6964
0.08	0.3960	3.8525	7.0270	10.1813	7.0	2.0937	4.8772	7.7797	10.7646
0.10	0.4417	3.8577	7.0298	10.1833	8.0	2.1286	4.9384	7.8464	10.8271
0.15	0.5376	3.8706	7.0369	10.1882	9.0	2.1566	4.9897	7.9051	10.8842
0.20	0.6170	3.8835	7.0440	10.1931	10.0	2.1795	5.0332	7.9569	10.9363
0.30	0.7465	3.9091	7.0582	10.2029	15.0	2.2509	5.1773	8.1422	11.1367
0.40	0.8516	3.9344	7.0723	10.2127	20.0	2.2880	5.2568	8.2534	11.2677
0.50	0.9408	3.9594	7.0864	10.2225	30.0	2.3261	5.3410	8.3771	11.4221
0.60	1.0184	3.9841	7.1004	10.2322	40.0	2.3455	5.3846	8.4432	11.5081
0.70	1.0873	4.0085	7.1143	10.2419	50.0	2.3572	5.4112	8.4840	11.5621
0.80	1.1490	4.0325	7.1282	10.2519	60.0	2.3651	5.4291	8.5116	11.5990
0.90	1.2048	4.0562	7.1421	10.2613	80.0	2.3750	5.4516	8.5466	11.6461
1.0	1.2558	4.0795	7.1558	10.2710	100.0	2.3809	5.4652	8.5678	11.6747
1.5	1.4569	4.1902	7.2233	10.3188	∞	2.4048	5.5201	8.6537	11.9309

It can be seen that Eq. (3-49) represents an expansion of the continuous function $F(r)$ in a Fourier-Bessel series and for such a sequence the number C_n is determined by the formula

$$C_n = \frac{\int_0^{r_0} rF(r)\, J_0\left(\mu_n \frac{r}{r_0}\right) dr}{\int_0^{r_0} rJ_0^2\left(\mu_n \frac{r}{r_0}\right) dr} \tag{3-50}$$

Integration of the denominator gives:

$$C_n = \frac{2}{r_0^2 [J_0^2(\mu_n) + J_1^2(\mu_n)]} \int_0^{r_0} rF(r)\, J_0\left(\mu_n \frac{r}{r_0}\right) dr \tag{3-51}$$

Substituting the obtained expression for C_n into Eq. (3-48), we obtain:

$$\vartheta = \sum_{n=1}^{n\to\infty} \frac{r}{r_0^2 [J_0^2(\mu_n) + J_1^2(\mu_n)]} \left[\int_0^{r_0} rF(r)\, J_0\left(\mu_n \frac{r}{r_0}\right) dr\right] \times$$

$$\times J_0\left(\mu_n \frac{r}{r_0}\right) \exp\left[-\mu_n^2 \frac{a\tau}{r_0^2}\right] \tag{3-52}$$

Equation (3-52) is valid for any initial temperature distribution in the cylinder.

If at the initial moment of time ($\tau = 0$) temperature distribution is uniform, i.e. $\vartheta_0 = F(r) = \text{const}$, the integral in Eq. (3-52)

$$\int_0^{r_0} r\vartheta_0 J_0\left(\mu_n \frac{r}{r_0}\right) dr = \frac{r_0^2}{\mu_n} \vartheta_0 J_1(\mu_n)$$

For these conditions the equation of the temperature field reduces into:

$$\frac{\vartheta}{\vartheta_0} = \sum_{n=1}^{n\to\infty} \frac{2J_1(\mu_n)}{\mu_n [J_0^2(\mu_n) + J^2(\mu_n)]} J_0\left(\mu_n \frac{r}{r_0}\right) \exp\left(-\mu_n^2 \frac{a\tau}{r_0^2}\right) \qquad (3\text{-}53)$$

Let us denote: $\frac{\vartheta}{\vartheta_0} = \Theta$ = dimensionless temperature;

$\frac{r}{r_0} = R$ = dimensionless coordinate varying within the limits $0 \leqslant R \leqslant 1$;

$a\tau/r_0^2 = \text{Fo}$ = Fourier number for a cylinder.

Taking these into account, the last equation will be:

$$\Theta = \sum_{n=1}^{n\to\infty} = \frac{2J_1(\mu_n)}{\mu_n [J_0^2(\mu_n) + J_1^2(\mu_n)]} J_0(\mu_n R) \exp(-\mu_n^2 \text{Fo}) \qquad (3\text{-}53')$$

It will be noted that all the main conclusions made in analysing the solution for a plate are also true for a cylinder.

The characteristic equation (3-47) shows the roots μ_n to depend only on Bi. The equation of the temperature field can therefore be presented in a dimensionless form:

$$\Theta = F\left(\frac{r}{r_0}, \frac{\alpha r_0}{\lambda}, \frac{a\tau}{r_0^2}\right) = F(R, \text{Bi}, \text{Fo}) \qquad (3\text{-}54)$$

For the temperature at the axis of the cylinder ($R = 0$) Eq. (3-53) becomes:

$$\Theta = \sum_{n=1}^{\infty} \frac{2J_1(\mu_n)}{\mu_n [J_0^2(\mu_n) + J_1^2(\mu_n)]} \exp[-\mu_n^2 \text{Fo}] \qquad (3\text{-}55)$$

and on the surface of the cylinder

$$\Theta_{R=1} = \sum_{n=1}^{\infty} \frac{2J_1(\mu_n) J_0(\mu_n)}{\mu_n [J_0^2(\mu_n) + J_1^2(\mu_n)]} \exp[-\mu_n^2 \text{Fo}] \qquad (3\text{-}56)$$

With $\text{Bi} \to \infty$ (practically $\text{Bi} > 100$) the straight line coincides with the x-axis, and the roots of the characteristic equation are independent of the Biot number, but are determined from the condition $J_0(\mu) = 0$.

The process of cooling is determined solely by the physical properties of the solid and its geometrical dimensions. Thus Eq. (3-53) acquires the form:

$$\Theta = \sum_{n=1}^{\infty} \frac{2}{\mu_n J_1(\mu_n)} J_0(\mu_n R) \exp[-\mu_n^2 \text{Fo}] \qquad (3\text{-}57)$$

If cooling occurs under the condition that $\text{Bi} \to 0$ (practically $\text{Bi} < 0.1$), the power series of the $J_0(\mu)$ and $J_1(\mu)$ functions become rapidly converging so that it becomes possible to limit ourselves to the first terms of the series, then $\mu^2 = 2\text{Bi}$.

In fact,

$$\frac{J_0(\mu)}{J_1(\mu)} = \frac{\mu}{\mathrm{Bi}} = \frac{1 - \frac{1}{2^2}\mu^2 + \ldots}{\frac{1}{2}\mu - \frac{1}{2^2 \times 4}\mu^3 + \ldots}$$

Hence

$$\mu = \sqrt{2\mathrm{Bi}}$$

Moreover, the coefficients of all terms of the infinite sum (3-53) are all equal to zero, except D_1:

$$D_1 = \frac{2J_1(\mu_1)}{\mu_1 [J_0^2(\mu_1) + J_1^2(\mu_1)]}$$

which is equal to unity. For conditions $\mathrm{Bi} \to 0$, Eq. (3-53) becomes:

$$\Theta = J_0(\mu_1 R) \exp[-\mu_1^2 \mathrm{Fo}] \tag{3-58}$$

At the axis of the cylinder ($R = 0$):

$$\Theta_{R=0} = \exp[-\mu_1^2 \mathrm{Fo}] \tag{3-59}$$

at the surface ($R = 1$):

$$\Theta_{R=1} = J_0(\mu_1) \exp[-\mu_1^2 \mathrm{Fo}] \tag{3-60}$$

Since $\mu = \sqrt{2\mathrm{Bi}}$, both the function $J_0(\mu_1)$ and the ratio of the temperature at the axis to the temperature on the surface tend to unity, i.e.

$$\frac{\Theta_{R=0}}{\Theta_{R=1}} = \frac{\exp[-\mu_1^2 \mathrm{Fo}]}{J_0(\mu_1) \exp(-\mu_1^2 \mathrm{Fo})} \to 1$$

The last equation points to the fact that temperature distribution across the wall is uniform and practically independent of the radius of the cylinder. The problem becomes an external one, and the development of the process is controlled by the conditions of cooling at the surface of the cylinder.

When $\mathrm{Fo} \geqslant 0.25$, we can confine ourselves to the first term of the series in calculating the dimensionless temperature Θ, the admissible error not exceeding 1 per cent. Thus the dimensionless temperatures at the axis and surface of the cylinder can be calculated by the following formulae:
at the axis

$$\Theta_{R=0} = N_0(\mathrm{Bi}) \exp[-\mu_1^2 \mathrm{Fo}] \tag{3-61}$$

at the surface

$$\Theta = P_0(\mathrm{Bi}) \exp[-\mu_1^2 \mathrm{Fo}] \tag{3-62}$$

The numerical values of the functions $N_0(\mathrm{Bi})$ and $P_0(\mathrm{Bi})$ can be calculated independently and tabulated [3.5-3.7]. Since the dimensionless temperature Θ in Eqs. (3-61) and (3-62) is a function of only two dimensionless terms

$$\Theta_{R=0} = \Phi_1(\mathrm{Bi}, \mathrm{Fo})$$

and

$$\Theta_{R=1} = \Phi_2(\mathrm{Bi}, \mathrm{Fo})$$

the temperatures at the axis and surface of the cylinder can be found from the charts shown in Figs. 3-11 and 3-12.

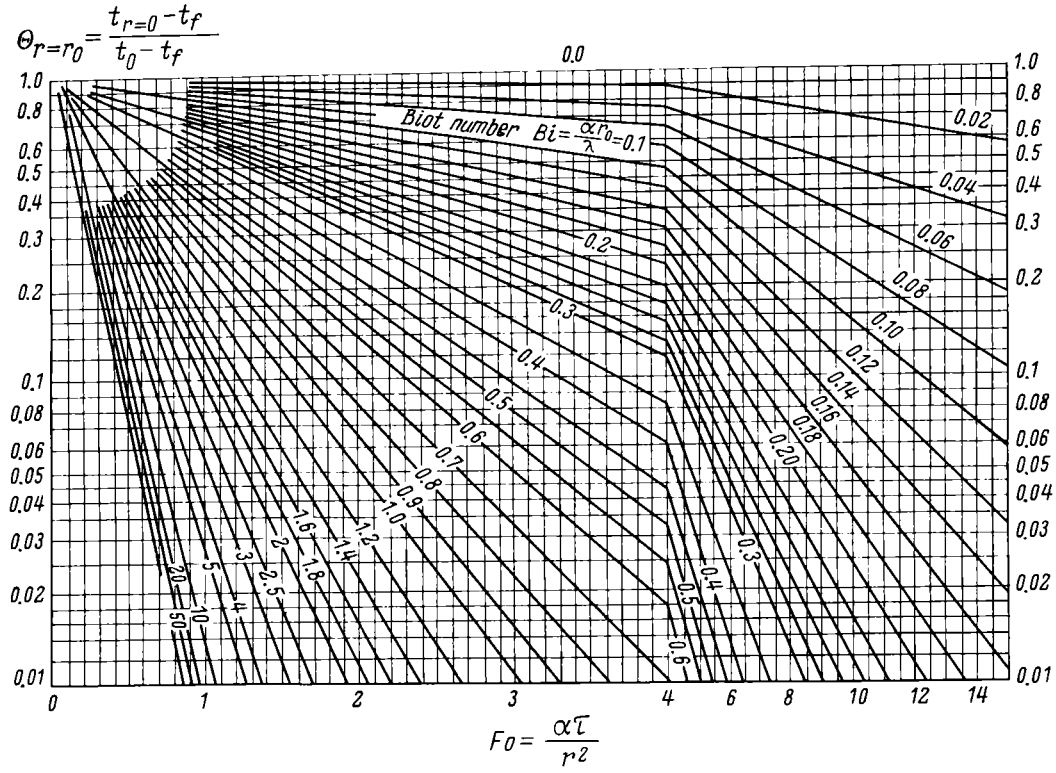

Fig. 3-11. $\Theta = \Phi_1$ (Fo, Bi) dependence for a cylinder axis

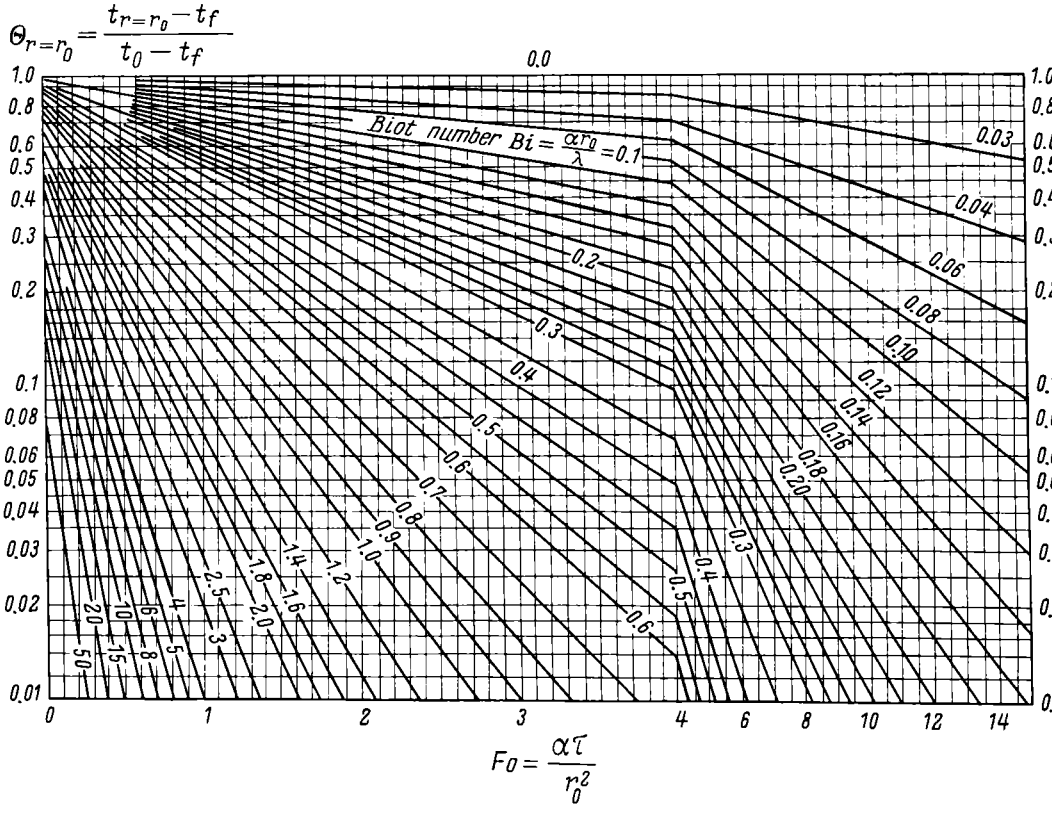

Fig. 3-12. $\Theta = \Phi_2$ (Fo, Bi) dependence for a cylinder surface

3-6. Determination of the Quantity of Heat Removed from a Cylinder During Cooling

Just as with a plate, the heat removed from, or imparted to, the surface of a cylinder in time between $\tau = 0$ and $\tau = \infty$ must be equal to the change of its internal energy, or enthalpy, during the period of complete cooling:

$$Q_P = \pi r_0^2 l \rho c (t_0 - t_f) \tag{3-63}$$

The enthalpy of the cylinder must change during any period from $\tau = 0$ to τ_1 by

$$Q = Q_P (1 - \bar{\Theta}_1) \tag{3-64}$$

where, as for a plate,

$$\bar{\Theta}_1 = \frac{\bar{t}_1 - t_f}{t_0 - t_f}$$

The dimensionless mean temperature of the cylinder is found from the equation

$$\bar{\Theta} = \frac{1}{\pi R^2} \int_0^R \Theta 2\pi R \, dR = 2 \int_0^1 \Theta R \, dR$$

(R varies between 0 and 1).

Substituting the value of Θ from Eq. (3-53) into this equation and integrating it between the previously indicated limits, we obtain:

$$\bar{\Theta} = \sum_{n=1}^{\infty} \frac{4J_1^2(\mu_n)}{\mu_n^2 [J_0^2(\mu_n) + J_1^2(\mu_n)]} \exp(-\mu_n^2 \, \mathrm{Fo})$$

or, taking into account that $\frac{J_0(\mu)}{J_1(\mu)} = \frac{\mu}{\mathrm{Bi}}$,

$$\bar{\Theta} = \sum_{n=1}^{\infty} \frac{4\mathrm{Bi}^2}{\mu_n^2 (\mu_n^2 + \mathrm{Bi}^2)} \exp(-\mu_n^2 \, \mathrm{Fo}) \tag{3-65}$$

In calculating the mean temperature Θ it is also permissible to limit ourselves to the first term of the series (3-65), provided $\mathrm{Fo} \geqslant 0.25$; then

$$\bar{\Theta} = \frac{4\mathrm{Bi}}{\mu_1^2 (\mu_1^2 + \mathrm{Bi}^2)} \exp(-\mu_1^2 \, \mathrm{Fo}) \tag{3-65'}$$

The function $\frac{4\mathrm{Bi}^2}{\mu_1^2 (\mu_1^2 + \mathrm{Bi}^2)} = M(\mathrm{Bi})$ may be calculated independently for corresponding values of Bi and tabulated.

3-7. Cooling of a Sphere

Consider the cooling of a sphere placed in a medium of constant temperature, with a constant surface heat-transfer coefficient α. At the initial moment $\tau = 0$ all points of the sphere with radius r_0 are at an equal temperature t_0. Under the conditions given the temperature at any point will be a function only of time and the radius. The problem is to find the distribution of temperature, or temperature profile, inside the sphere.

Denoting the excess temperature for any point by $\vartheta = t - t_f$, the differential equation of conduction becomes, in spherical coordinates:

$$\frac{\partial \vartheta}{\partial \tau} = a \left(\frac{\partial^2 \vartheta}{\partial r^2} + \frac{2}{r} \frac{\partial \vartheta}{\partial r} \right) \tag{3-66}$$

(a) Boundary conditions are:
at the surface of the sphere at $r = r_0$

$$\left(\frac{\partial \vartheta}{\partial r}\right)_{r=r_0} = -\frac{\alpha}{\lambda}\vartheta_{r=r_0}$$

Since the problem is symmetrical, at the centre of the sphere ($r = 0$):

$$\left(\frac{\partial \vartheta}{\partial r}\right)_{r=0} = 0 \qquad (3\text{-}67)$$

(b) Initial conditions at $\tau = 0$ are:

$$\vartheta = \vartheta_0 = t_0 - t_f \text{ for } 0 \leqslant r \leqslant r_0$$

Solving Eq. (3-66) by separating the variables and subordinating the obtained result to conditions (3-67) we obtain*:

$$\Theta = \sum_{n=1}^{\infty} \frac{2(\sin\mu_n - \mu_n \cos\mu_n)\sin(\mu_n R)}{(\mu_n - \sin\mu_n \cos\mu_n)\mu_n R} \exp(-\mu_n^2 \text{Fo}) \qquad (3\text{-}68)$$

where $\Theta = \vartheta/\vartheta_0$, and $R = r/r_0$.

The constant μ in Eq. (3-68) is the root of the characteristic equation, which has the following form for a sphere:

$$\tan\mu = -\frac{\mu}{\text{Bi} - 1} \qquad (3\text{-}69)$$

This trigonometric equation has an infinite number of solutions at a given value of Bi, and is solved in the same way as Eq. (3-14). The numerical values of six roots of Eq. (3-69) for different Bi numbers are tabulated [3.1], Table 6-5.

In accordance with characteristic equation (3-69), at $\text{Bi} \to \infty$

$$\mu_n = n\pi$$

and the "initial thermal amplitude" of Eq. (3-68) is

$$D_n = \frac{2(\sin\mu_n - \mu_n \cos\mu_n)}{\mu_n - \sin\mu_n \cos\mu_n} = 2(-1)^{n+1}$$

Taking this equation into consideration, Eq. (3-68) becomes:

$$\Theta = \sum_{n=1}^{\infty} 2(-1)^{n+1} \frac{1}{n\pi R} \sin(n\pi R) \exp[-(n\pi)^2 \text{Fo}] \qquad (3\text{-}70)$$

At Bi = 1 and in accordance with Eq. (3-69):

$$\mu_n = (2n-1)\frac{\pi}{2} \quad \text{and} \quad D_n = 2(-1)^{n+1}\frac{1}{\mu_n}$$

and Eq. (3-68) becomes:

$$\Theta = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\mu_n} \frac{\sin(\mu_n R)}{\mu_n R} \exp(-\mu_n^2 \text{Fo}) \qquad (3\text{-}71)$$

The initial amplitudes (D_n) of all terms of series (3-68), except the first term, tend to zero at small values of Bi ($\text{Bi} < 0.1$). The initial amplitude of the first series $D_1 = 1$, and $\mu_1^2 = 3\text{Bi}$. Under these conditions Eq. (3-68) becomes:

$$\Theta = \frac{\sin(\sqrt{3\text{Bi}}\, R)}{\sqrt{3\text{Bi}}\, R} \exp(-3\text{BiFo}) \qquad (3\text{-}72)$$

* A detailed solution is given in [3.1].

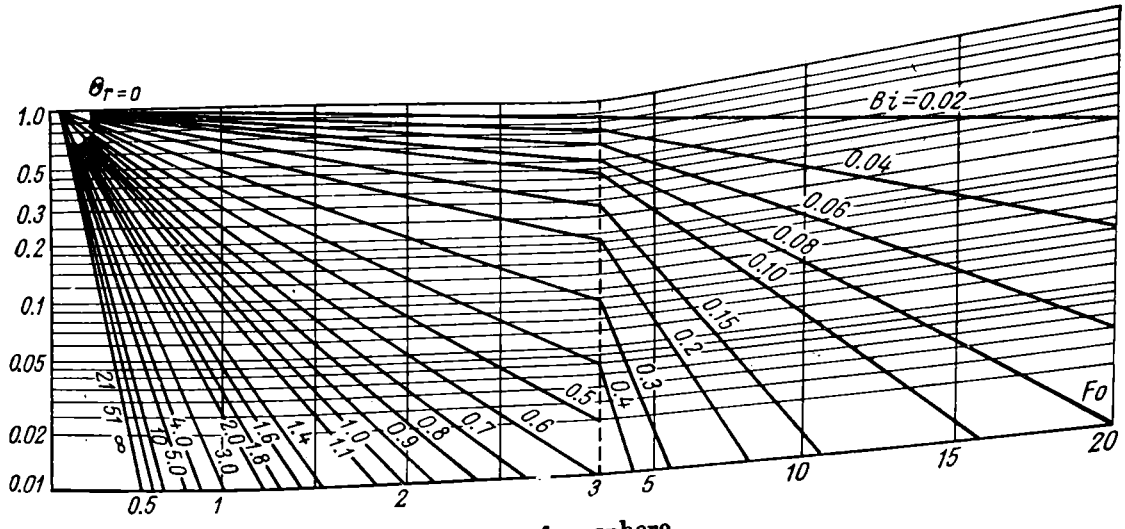

Fig. 3-13. $\Theta = F_1$ (Fo, Bi) for the centre of a sphere

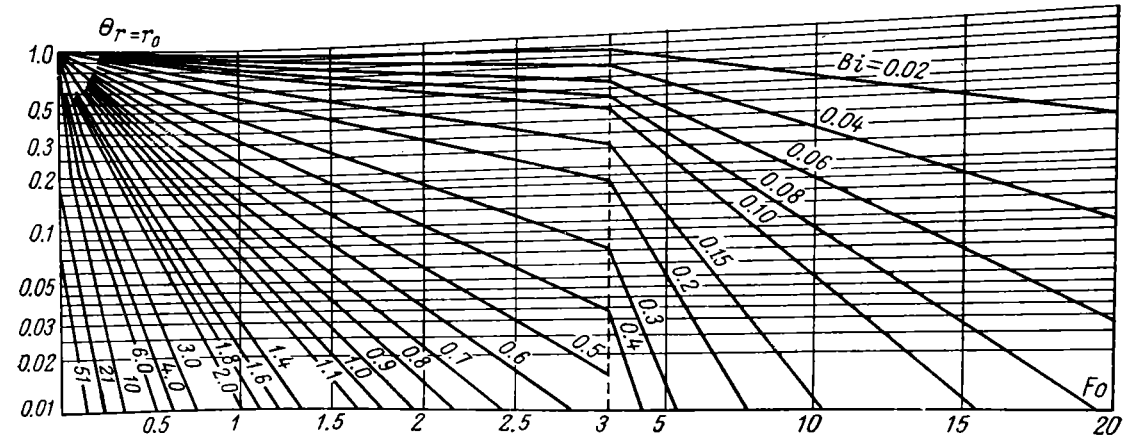

Fig. 3-14. $\Theta = F_2$ (Fo, Bi) for a spherical surface

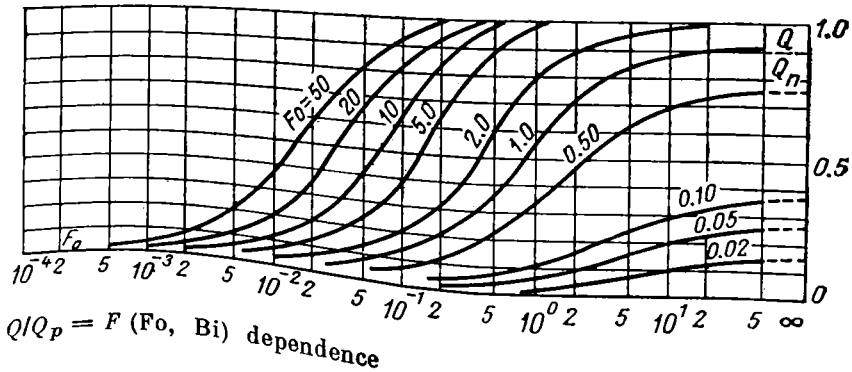

Fig. 3-15. $Q/Q_p = F$ (Fo, Bi) dependence

Analysis of Eq. (3-68) shows that at Fo $\geqslant 0.25$ the series becomes so rapidly converging that it is sufficient to use only the first term of the series to describe the temperature field:

$$\Theta = \frac{2(\sin\mu_1 - \mu_1\cos\mu_1)}{\mu_1 - \sin\mu_1\cos\mu_1}\,\frac{\sin(\mu_1 R)}{\mu_1 R}\exp(-\mu_1^2\,\mathrm{Fo}) \tag{3-73}$$

Since μ_n in Eq. (3-68) depends solely on the Biot number, the equation of the temperature field may be written as

$$\Theta = F(R,\ \mathrm{Bi},\ \mathrm{Fo}) \tag{3-74}$$

For the centre of the sphere

$$\Theta_{R=0} = F_1(\mathrm{Bi},\ \mathrm{Fo}) \tag{3-74'}$$

For its surface

$$\Theta_{R=1} = F_2(\mathrm{Bi},\ \mathrm{Fo}) \tag{3-74''}$$

The functions (3-74') and (3-74'') are plotted against various Fourier and Biot numbers in Figs. 3-13 and 3-14.

As with a plate or a cylinder, the quantity of heat dissipated from, or imparted to, the sphere in time interval between $\tau = 0$ and τ is found by the formula

$$\frac{Q}{Q_p} = \sum_{n=1}^{\infty}\frac{6}{\mu_n^3}\,\frac{(\sin\mu_n - \mu_n\cos\mu_n)^2}{\mu_n - \sin\mu_n\cos\mu_n}\,(1 - e^{-\mu_n^2\,\mathrm{Fo}}) \tag{3-75}$$

The term $Q_p = \frac{4}{3}\pi r_0^3 \rho c\,(t_0 - t_f)$ in Eq. (3-75) is the initial excess enthalpy of the sphere. An examination of Eq. (3-75) shows that

$$\frac{Q}{Q_p} = F(\mathrm{Bi},\ \mathrm{Fo}) \tag{3-76}$$

The numerical values of this function for various Fo and Bi numbers are shown in Fig. 3-15.

3-8. Cooling (Heating) of Bodies of Finite Size

(a) *Cooling of a Parallelepiped*

Consider the cooling of a parallelepiped placed in a medium of constant temperature, and possessing a heat-transfer coefficient α constant for all its faces. All points are at one and the same temperature t_0 at the initial moment of time ($\tau = 0$). The parallelepiped, measuring $2\delta_x \times 2\delta_y \times 2\delta_z$, is homogeneous and isotropic. It is necessary to find temperature distribution inside it at any moment of time, and the mean temperature needed to determine the quantity of heat imparted (or removed).

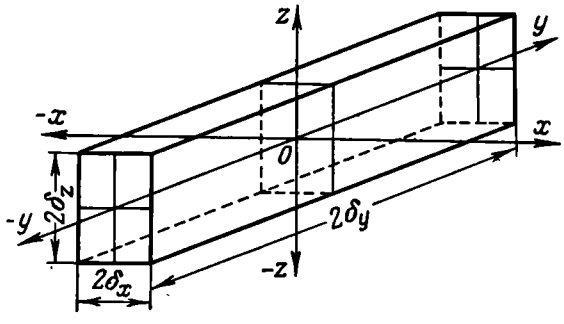

Fig. 3-16. Cooling of a parallelepiped

Let us place the origin of coordinates at the centre of the parallelepiped, as shown in Fig. 3-16.

The differential equation for this case is:

$$\frac{\partial t(x, y, z, \tau)}{\partial\tau} = a\nabla^2 t(x, y, z, \tau) \tag{3-77}$$

Initial conditions ($\tau = 0$) are:

$$t(x, y, z) = t_0 = \text{const} \tag{3-78}$$

The conditions given make the problem symmetrical with respect to the centre of the parallelepiped. Introducing the notation $\vartheta = t - t_f$, the boundary conditions become:

(a) for the surface at $\tau > 0$,

$$\left.\begin{aligned} \mp\lambda\left(\frac{\partial\vartheta}{\partial x}\right)_{x=\pm\delta_x} &= \alpha\vartheta_{x=\pm\delta_x} \\ \mp\lambda\left(\frac{\partial\vartheta}{\partial y}\right)_{y=\pm\delta_y} &= \alpha\vartheta_{y=\pm\delta_y} \\ \mp\lambda\left(\frac{\partial\vartheta}{\partial z}\right)_{z=\pm\delta_z} &= \alpha\vartheta_{z=\pm\delta_z} \end{aligned}\right\} \tag{3-79}$$

(b) at the centre of the parallelepiped at $\tau > 0$,

$$\left.\begin{aligned} \left(\frac{\partial\vartheta}{\partial x}\right)_{x=0,\ 0\leqslant y\leqslant\delta_y,\ 0\leqslant z\leqslant\delta_z} &= 0 \\ \left(\frac{\partial\vartheta}{\partial y}\right)_{y=0,\ 0\leqslant x\leqslant\delta_x,\ 0\leqslant z\leqslant\delta_z} &= 0 \\ \left(\frac{\partial\vartheta}{\partial z}\right)_{z=0,\ 0\leqslant x\leqslant\delta_x,\ 0\leqslant y\leqslant\delta_y} &= 0 \end{aligned}\right\} \tag{3-80}$$

Bodies, such as parallelepipeds, finite cylinders, and rectangular rods, can be considered as being formed respectively by the intersection of three mutually perpendicular plates, a cylinder and plate and two plates of infinite size but finite thickness.

The solutions can be shown to be rightfully represented as the product of the dimensionless temperatures of the infinite bodies, whose intersections form the body concerned.

As noted above, the parallelepiped is formed by intersection of three mutually perpendicular infinite plates of finite thickness. Hence, the solution of the problem can be considered as the product of the dimensionless temperatures of the three infinite plates:

$$\Theta = \Theta_x\Theta_y\Theta_z \tag{3-81}$$

where

$$\Theta_x = \frac{t(x, \tau) - t_f}{t_0 - t_f}$$

$$\Theta_y = \frac{t(y, \tau) - t_f}{t_0 - t_f}$$

$$\Theta_z = \frac{t(z, \tau) - t_f}{t_0 - t_f}$$

The general solution of Eq. (3-81) in explicit form is:

$$\Theta = \frac{t(x, y, z, \tau) - t_f}{t_0 - t_f} = \frac{t(x, \tau) - t_f}{t_0 - t_f}\,\frac{t(y, \tau) - t_f}{t_0 - t_f}\,\frac{t(z, \tau) - t_f}{t_0 - t_f} \tag{3-81'}$$

This general solution satisfies both the differential equation and the boundary conditions describing the process of heat conduction in the parallelepiped.

Thus, the solution of a problem involving a body of finite dimensions comes down to the solution of one dealing with an infinite plane of finite thickness. Equation (3-81) may be presented in the form:

$$\Theta = F_x\left(\frac{x}{\delta_x}, \frac{\alpha\delta_x}{\lambda}, \frac{a\tau}{\delta_x^2}\right) F_y\left(\frac{y}{\delta_y}, \frac{\alpha\delta_y}{\lambda}, \frac{a\tau}{\delta_y^2}\right) F_z\left(\frac{z}{\delta_z}, \frac{\alpha\delta_z}{\lambda}, \frac{a\tau}{\delta_z^2}\right)$$

or

$$\Theta = F_x(X, \mathrm{Bi}_x, \mathrm{Fo}_x)\, F_y(Y, \mathrm{Bi}_y, \mathrm{Fo}_y)\, F_z(Z, \mathrm{Bi}_z, \mathrm{Fo}_z) \qquad (3\text{-}81'')$$

The multipliers in the latter are calculated from (3-24).

The method we have considered is known in the theory of conduction as the *theorem of multiplied solutions*. The solution obtained is also suitable for finding the mean temperature.

The dimensionless mean temperature of a parallelepiped is:

$$\overline{\Theta} = \frac{\bar{t}(\tau) - t_f}{t_0 - t_f} = \frac{\bar{t}(\tau)_x - t_f}{t_0 - t_f} \frac{\bar{t}(\tau)_y - t_f}{t_0 - t_f} \frac{\bar{t}(\tau)_z - t_f}{t_0 - t_f}$$

or

$$\overline{\Theta} = \overline{\Theta}_x \overline{\Theta}_y \overline{\Theta}_z = \overline{F}_x(\mathrm{Bi}_x, \mathrm{Fo}_x)\, \overline{F}_y(\mathrm{Bi}_y, \mathrm{Fo}_y)\, \overline{F}_z(\mathrm{Bi}_z, \mathrm{Fo}_z) \qquad (3\text{-}82)$$

The multipliers of Eq. (3-82) are calculated from Eq. (3-39). It may be noted here that the theorem of multiplied solutions is also true for the more general case where the thermal conductivities are different in different directions, and the face heat-transfer coefficients are different as well.

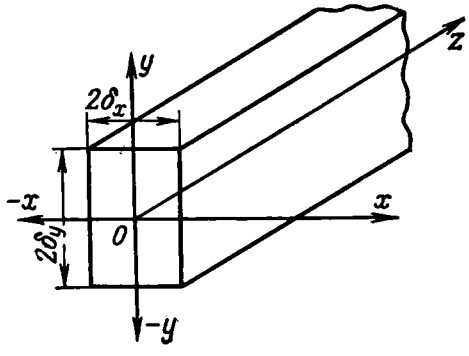

Fig. 3-17. Cooling of a semi-infinite rectangular rod

(b) *Cooling of a Long Rectangular Rod*

A homogeneous rod is cooled in surroundings of constant temperature t_f, at a constant surface heat-transfer coefficient. At the initial moment of time ($\tau = 0$) all points of the rod are at a uniform temperature.

The cross-section of the rod is a rectangle $2\delta_x \times 2\delta_y$ (Fig. 3-17). Such a body may be considered as being formed by the intersection of two plates $2\delta_x$ and $2\delta_y$ thick, whose conditions of unambiguity are similar to those of the rod so formed. The dimensionless temperature field for the problem can be represented by

$$\Theta = \Theta_x \Theta_y \qquad (3\text{-}82')$$

where

$$\Theta_x = F_x(X, \mathrm{Bi}_x, \mathrm{Fo}_x) \text{ and } \Theta_y = F_y(Y, \mathrm{Bi}_y, \mathrm{Fo}_y)$$

The multipliers of Eq. (3-82) are calculated by formula (3-24).

(c) *Cooling of a Finite Cylinder*

A homogeneous cylinder is cooled in a medium of constant temperature t_f. The heat-transfer coefficient α at the bases and surface of the cylinder is the same. At the initial moment of time ($\tau = 0$) all points are at an equal tempe-

rature t_0. The diameter of the cylinder is $2r_0$ and its length $l = 2\delta_z$ (Fig. 3-18) It is required to find the temperature distribution inside the cylinder for any moment of time and the mean temperature as a function of time for the given conditions of unambiguity. A finite cylinder may be considered as being formed by the intersection of an infinite cylinder of a diameter $2r_0$ and an infinite plate of thickness $2\delta_z$. The dimensionless temperature of such a body may, consequently, be presented as

$$\Theta = \Theta_z \Theta_r = \frac{t(z, \tau) - t_f}{t_0 - t_f} \frac{t(r, \tau) - t_f}{t_0 - t_f} \quad (3\text{-}83)$$

or

$$\Theta = F_z\left(\frac{z}{\delta_z}, \frac{\alpha\delta_z}{\lambda}, \frac{a\tau}{\delta_z^2}\right) \times \times F_r\left(\frac{r}{r_0}, \frac{\alpha r_0}{\lambda}, \frac{a\tau}{r_0^2}\right) \quad (3\text{-}83')$$

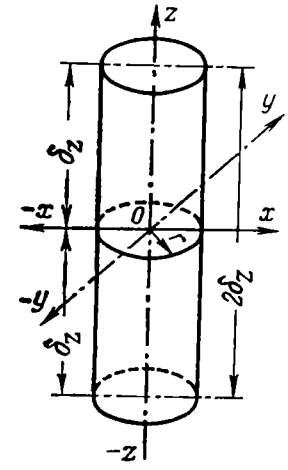

Fig. 3-18. Cooling of a finite cylinder

The multipliers of the right side of Eq. (3-83) are calculated by formulae (3-24) and (3-53), the reference linear dimensions of Eq. (3-24) being taken as one half of the cylinder height δ_z, and of Eq. (3-53) as the radius r_0 of the cylinder. The dimensionless mean temperature within the cylinder for any moment of time is:

$$\bar{\Theta} = \frac{\bar{t}(\tau) - t_f}{t_0 - t_f} = \frac{\bar{t}(\tau)_z - t_f}{t_0 - t_f} \frac{\bar{t}(\tau)_r - t_f}{t_0 - t_f}$$

or

$$\bar{\Theta} = \bar{\Theta}_z \bar{\Theta}_r = \bar{F}_z(\mathrm{Bi}_z, \mathrm{Fo}_z)\, \bar{F}_r(\mathrm{Bi}_r, \mathrm{Fo}_r) \quad (3\text{-}84)$$

The multipliers of Eq. (3-84) are calculated by formulae (3-39) and (3-65).

3-9. Dependence of Cooling (Heating) on the Form and Size of Bodies

The rate of heat propagation in a body depends on the ratio of its surface to its volume. Research in cooling processes indicates that the larger the ratio of surface to volume, the faster the rate at which the process proceeds. This is true at any Biot number, and can be visualized by considering as examples the cooling of a plate, a long cylinder and a sphere. With Bi = 0 the equations of their temperature fields are respectively:

$$\Theta_p = \exp(-\mathrm{BiFo})$$

$$\Theta_c = \exp(-2\mathrm{BiFo})$$

$$\Theta_s = \exp(-3\mathrm{BiFo})$$

From these equations it follows that given an identical reference dimension and other identical conditions the greatest rate of temperature variation in time will be observed in the sphere. Comparison of the surface/volume ratios of the plate, cylinder and sphere shows them to be in a 1 : 2 : 3 relationship.

The curves shown in Fig. 3-19 represent the variation of temperature in time at the axis and centre of bodies of various geometrical forms at an equal Biot number. From Fig. 3-19 it follows that the rate of cooling is faster for a sphere

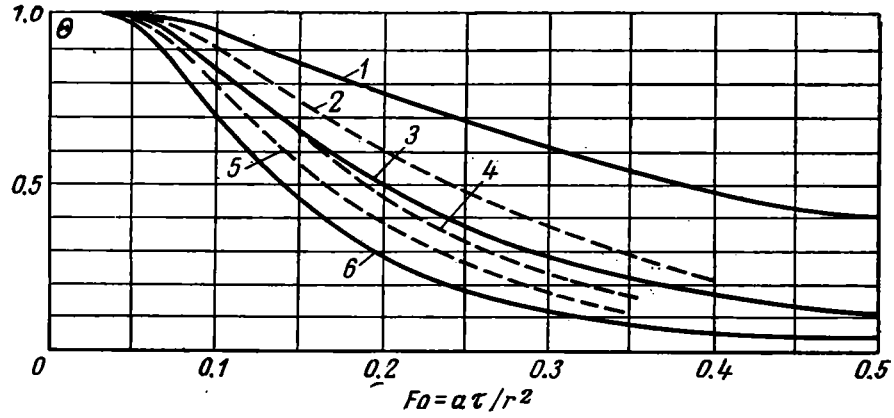

Fig. 3-19. Rate of cooling in the centre for various bodies with equal reference dimension l_0: *1*—infinite plate; *2*—square beam of infinite length; *3*—cylinder of infinite length; *4*—cube; *5*—cylinder of length equal to diameter; *6*—sphere

than for any other body. It should be remembered that everything that has been said here also holds for bodies with an idential linear dimension l_0.

3-10. Regular Cooling (Heating) of Bodies

The analysis of the solutions obtained for bodies of various geometric forms shows them to be identical in structure, i.e. to be the sum of an infinite series whose terms are rapidly decreasing exponential functions. The solution involving, for instance, an infinite plate cooled in a medium of constant temperature t_f and at a constant surface heat-transfer coefficient α is:

$$\vartheta = \sum_{n=1}^{\infty} A_n \cos\left(\mu_n \frac{x}{\delta}\right) e^{-\mu_n^2 \frac{a\tau}{\delta^2}}$$

where A_n is a constant coefficient, with a value proper for each term of the series (depending neither on the coordinates nor on time), and is found from the initial conditions.

The factor $\cos\left(\mu_n \frac{x}{\delta}\right)$ is a function only of the coordinate x, and can be denoted by U_n. The exponent will decrease in proportion to time τ. The term $\mu_n^2 a/\delta^2$ is a constant real positive number which can be denoted by m_n, with m varying in dependence on the index number in the same way as μ, i.e.

$$m_1 < m_2 < m_3 < \ldots < m_n \tag{3-85}$$

where $n = 1, 2, 3, \ldots$.

Taking this into account, the solution for a plate can be conceived as

$$\vartheta = \sum_{n=1}^{\infty} A_n U_n e^{-m_n \tau} \tag{3-86}$$

The temperature field of bodies of other form may also be described by an equation of the type of (3-86). The influence of the geometry is accounted for

by the different forms of the factors A_n and U_n. For every initial temperature distribution curve of bodies of one and the same form there is a different set of A_n numbers.

With small values of τ between $\tau = 0$ and $\tau = \tau_1$, the temperature distribution within the body and the rate of temperature variation at individual points of the body in time depend on the peculiarities of the initial temperature distribution. In these conditions the temperature field of the body will be determined not only by the first, but also by the subsequent terms of series (3-86).

This initial period of cooling, during which the rate of temperature variation depends on the kind of initial temperature distribution, is referred to as the *disordered stage* of cooling (or heating). By virtue of the inequality (3-85), an increase in time τ is accompanied by a rapid decrease of the subsequent terms of series (3-86), i.e. the series becomes rapidly converging.

At a certain moment of time $\tau > \tau_1$ the initial conditions begin to be of secondary importance and from then the process is fully determined only by the conditions of cooling at the interface between the body and medium, the physical properties of the body, its geometry and dimensions. The temperature field is described by the first term of series (3-86), i.e.

$$\vartheta = A_1 U_1 e^{-m_1 \tau} \qquad (3\text{-}87)$$

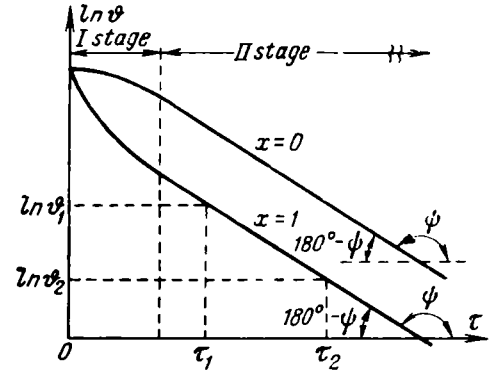

Fig. 3-20. Dependence of ln ϑ on time for cooled (heated) bodies

This formula shows that the variation of excess in temperature both in space and in time no longer depends on the initial temperature distribution. Taking the logarithm of the last equation and omitting the subscripts, we obtain:

$$\ln \vartheta = \ln (AU) - m\tau$$

or

$$\ln \vartheta = -m\tau + C\,(x, y, z) \qquad (3\text{-}88)$$

From Eq. (3-88) it follows that the natural logarithm of excess temperature changes linearly in time at all points of the body considered. Expressed graphically the dependence between ln ϑ and time is a straight line (Fig. 3-20). With prolonged cooling ($\tau \to \infty$ or, what is the same thing, Fo $\to \infty$) all points of the body eventually acquire an equal temperature t_f (i.e. as steady-state process is established).

The entire process of cooling can thus be divided into three stages as follows.

The first stage of the (disordered) regime is characterized by the great effect of the initial temperature distribution; the dependence between ϑ and τ is described by Eq. (3-86).

The second stage is referred to as the *regular regime*, and the dependence between ϑ and τ is described by Eq. (3-87).

The third stage is identical with the process under steady-state conditions, when the temperature of all points of the body concerned is the same as the temperature of surroundings (i.e. there is thermal equilibrium).

Let us proceed with a more detailed consideration of the second stage.

Differentiating both sides of Eq. (3-88) with respect to time, we obtain:

$$\frac{1}{\vartheta}\frac{\partial \vartheta}{\partial \tau} = -m = \text{const} \qquad (3\text{-}89)$$

The left side of Eq. (3-89) is the expression for the relative rate of temperature variation, and is equal to a constant m, depending neither on the coordinates, nor on time.

Constant m is measured in 1/s, and is referred to as the *rate of cooling*. After the regular regime sets in, the rate of cooling depends neither on the coordinates nor on time, and is constant for all points of a body. As follows from Eq. (3-89), the rate of cooling characterizes the relative rate of temperature variation and depends only on the process of surface cooling, the physical properties, geometrical forms, and dimensions of the body.

Thus, the regular regime of cooling (heating) of bodies is distinguished by the fact that variation of the temperature field in time is described by a simple exponential function while the relative rate of cooling m remains constant for points of the body, depending neither on the coordinates nor on time.

If we determine the variation in excess temperature ϑ in time τ by experiment and plot the dependence in semi-logarithmic coordinates, it will follow from Fig. 3-20 that the rate of cooling is the slope of the straight line describing the regular regime:

$$\frac{\ln \vartheta_1 - \ln \vartheta_2}{\tau_2 - \tau_1} = m = \text{const}'$$

The dependence of the rate of cooling m on the physical properties of the body, on its geometry and dimensions, and on the conditions of heat exchange at the surface may be found by analysing the heat balance equation.

The change in internal energy, or enthalpy, of the body is

$$dQ = -cV\rho \frac{\partial \bar{\vartheta}_v}{\partial \tau} d\tau \tag{3-90}$$

where c = specific heat, J/kg·K;
V = volume, m^3;
ρ = density, kg/m^3;
$\bar{\vartheta}_v$ = volumetric mean excess temperature, °C;
τ = time, s.

On the other hand, all the heat must be removed from the surface to the surroundings by conduction during the same interval of time

$$dQ = \bar{\alpha} \bar{\vartheta}_F F \, d\tau, \tag{3-91}$$

where $\bar{\alpha}$ = mean heat-transfer coefficient;
$\bar{\vartheta}_F$ = mean temperature of the surface of the body at the given moment of time:

$$\bar{\vartheta}_F = \frac{1}{F} \int_F \vartheta_F \, dF$$

Equating (3-90) and (3-91), we find:

$$-\frac{\partial \bar{\vartheta}_v}{\partial \tau} = \frac{\bar{\alpha} F}{c\rho V} \bar{\vartheta}_F$$

or, if we divide the expression obtained by $\bar{\vartheta}_v$, taking into account that $c\rho V = C$ [J/K] is the total heat capacity of the body, we obtain:

$$-\frac{1}{\bar{\vartheta}_v} \frac{\partial \bar{\vartheta}_v}{\partial \tau} = \frac{\bar{\vartheta}_F}{\bar{\vartheta}_v} \frac{\bar{\alpha} F}{C} \tag{3-92}$$

The left side of this expression contains the relative rate of cooling, m, 1/s; denoting the ratio $\bar{\vartheta}_F/\vartheta_v$ by Ψ, we can write it in the following form:

$$m = \Psi \frac{\bar{\alpha} F}{C} \tag{3-93}$$

It follows from Eq. (3-93) that *the relative rate of cooling* or, in other words, the cooling rate m of a homogeneous and isotropic body at the final value of the heat-transfer coefficient α *is proportional to the surface heat-transfer coefficient and inversely proportional to its total heat capacity* (this is Kondratyev's first theorem [3.8]).

The factor $\Psi = \bar{\vartheta}_F/\bar{\vartheta}_v$ in Eq. (3-93) is called the *coefficient of non-uniform temperature distribution* and depends on surface cooling conditions. In order to elucidate its dependence on the Biot number that accounts for the conditions under which heat transfer develops at the surface, let us examine two limiting cases:

(a) $Bi \to 0$ (*practically* $Bi < 0.1$).

As already stated, these conditions correspond to an external problem in which temperature distribution in the body is dependent of its size and physical properties and in which, consequently, the surface- and volume-mean temperatures are equal, $\bar{\vartheta}_F = \bar{\vartheta}_v$ (Fig. 3-8). The coefficient of non-uniform temperature distribution in a body is

$$\Psi = \frac{\bar{\vartheta}_F}{\bar{\vartheta}_v} = 1$$

(b) $Bi \to \infty$ (*practically* $Bi > 100$).

Under these conditions the problem becomes an internal one, and the process of cooling is determined solely by the dimensions of the body and its physical properties.

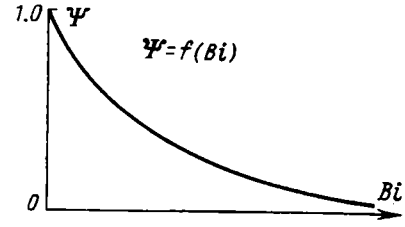

Fig. 3-21. Dependence of $\Psi = f(\mathrm{Bi})$

By virtue of the high intensity of heat transfer the surface temperature of the body becomes constant and equal to that of the surroundings (Fig. 3-7). The coefficient of non-uniform temperature distribution is

$$\Psi = \frac{\bar{\vartheta}_F}{\bar{\vartheta}_v} = 0$$

It follows from the foregoing that Ψ varies between zero and unity (Fig. 3-21).

With $Bi \to \infty$ *or, what is the same thing,* $\alpha \to \infty$ *the rate of cooling* m *becomes directly proportional to the thermal diffusivity* a *of the body*, $\mathrm{m^2/s}$ (Kondratyev's second theorem [3.8]):

$$a = K m_\infty \tag{3-94}$$

The proportionality factor K depends only on the geometry and size of the body. We may show this by considering, for example, the problem of cooling of a uniform infinite plate. It will be recalled that

$$m = \frac{\mu^2 a}{l_0^2}$$

hence

$$\mu = l_0 \sqrt{\frac{m}{a}} \tag{3-95}$$

Consider the characteristic equation derived for an infinite plate

$$\cot \mu = \frac{\mu}{\mathrm{Bi}}$$

With $\mathrm{Bi} \to \infty$, $\cot \mu \to 0$, and μ tends to its limit value of $\frac{\pi}{2}$; with $\mathrm{Bi} \to 0$, $\cot \mu \to \infty$, and μ tends to zero.

Consequently, the criterion μ for a plate changes from zero to its limit value, equal to $\pi/2$ (Fig. 3-22), in the whole range of values of the Biot numbers. Bodies of other geometry have their own limits of variation for μ.

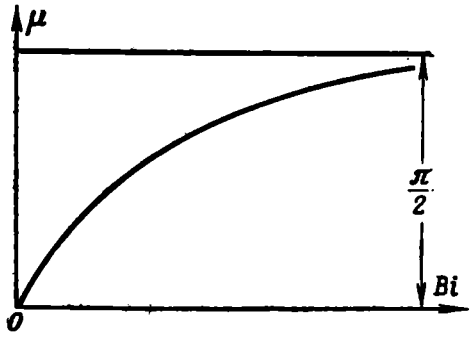

Fig. 3-22. Dependence of $\mu = f_1(\mathrm{Bi})$

Since, with Bi tending to infinity (practically $\mathrm{Bi} > 100$), μ can be assumed equal to $\frac{\pi}{2}$ for a cooled uniform infinite plate, we obtain from Eq. (3-95):

$$m_\infty = \left(\frac{\pi}{2l_0}\right)^2 a \qquad (3\text{-}96)$$

It will be recalled that the characteristic or reference linear dimension of a plate is one-half of its thickness, i.e. $l_0 = \delta$. Then (3-96) becomes:

$$a = \frac{1}{\left(\frac{\pi}{2\delta}\right)^2} m_\infty = K m_\infty$$

where $K = \frac{1}{\left(\frac{\pi}{2\delta}\right)^2}$ is the proportionality factor for an infinite plate which depends solely on its form and geometrical dimensions.

The following are the proportionality factors for solids of other geometrical shapes [3.8]:

for a sphere

$$K = \frac{1}{\left(\frac{\pi}{r_0}\right)^2}$$

for a parallelepiped

$$K = \frac{1}{\left(\frac{\pi}{l_1}\right)^2 + \left(\frac{\pi}{l_2}\right)^2 + \left(\frac{\pi}{l_3}\right)^2}$$

for a finite cylinder

$$K = \frac{1}{\left(\frac{2.405}{r_0}\right)^2 + \left(\frac{\pi}{l}\right)^2}$$

The theory of regular regime has been used to elaborate various experimental methods for determining the thermal and physical properties of various materials [3.9, 3.10]. For determining the physical properties of a solid, the following procedure is adopted.

The thermal diffusivity of a solid is determined by means of an alpha-calorimeter of cylindrical or spherical shape. Conditions of cooling approaching $\bar{\alpha} \to \infty$ are created, the variation of excess temperature in time is measured, and the

dependence is plotted in semi-logarithmic coordinates (Fig. 3-23). Then,

$$m = \frac{\ln \vartheta_1 - \ln \vartheta_2}{\tau_2 - \tau_1}$$

The thermal diffusivity is found from the equation $a = Km_\infty$.

The thermal conductivity of a solid is determined by means of a lambda-calorimeter which is usually spherical. The essence of this method is that the conditions of cooling created ensure a heat-transfer coefficient α of finite value; the rate of cooling is determined under these conditions by the method described above. The thermal conductivity is then found from the characteristic equation for a sphere, which has the following form:

$$\frac{1}{\cot \mu} = -\frac{\mu}{\mathrm{Bi} - 1} \tag{3-97}$$

It will be recalled that the characteristic or reference linear dimension of a sphere is its radius r_0, the value $\mu = r_0\sqrt{m/a}$. Then Eq. (3-97) becomes:

$$\mathrm{Bi} - 1 = -r_0 \sqrt{\frac{m}{a}} \cot\left(r_0 \sqrt{\frac{m}{a}}\right)$$

from which we directly obtain:

$$\lambda = \frac{\bar{\alpha} r_0}{1 - r_0 \sqrt{\frac{m}{a}} \cot\left(r_0 \sqrt{\frac{m}{a}}\right)} \tag{3-98}$$

Here λ is in W/m·K.

The unknown α of Eq. (3-98) is determined by means of a standard or reference calorimeter made of a material of known thermal conductivity.

We have considered the regular regime method for conditions when the temperature of the medium is constant (t_f = const), which has been given the name of the regular regime of the first kind by G.M. Kondratyev.

Lately methods of the regular regime have found wide application for the cases when the temperature of the medium is a linear function of time ($t_f = t_{f0} + b\tau$) and a periodical function of time ($t_f = t_{f0} + t_m \cos \pi\nu\tau$), where ν=oscillation frequency, t_m = = amplitude of oscillation of medium temperature. These two cases have been given the names of regular regime methods of the second and third kind.

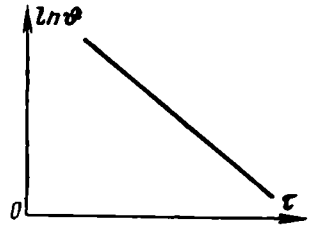

Fig. 3-23. Determining the cooling rate m

A.V. Lykov [3.1] has shown the kinetics of heating of a body to become regular not only in respect to the temperature fields, but also in relation to heat flow. It is not necessary, therefore, to distinguish regular regimes of the first, second and third kinds. The following relation can be considered as the common property of the regular regime

$$-\frac{d\bar{t}}{d\tau} = m(t_f - \bar{t}) \tag{3-99}$$

where $\bar{t}$ = volumetric mean temperature of the body;
t_f = medium temperature;
m = proportionality factor, referred to as the rate of heating (cooling).

It follows from (3-99) that in the stage of the regular regime $dt/d\tau$ the rate of heating of a body is proportional to the difference between the temperature of the medium and mean temperature of the body, with the proportionality factor m being determined not only by the characteristic dimensions of the body, physical properties and nature of heat transfer at the surface, but also by the nature of temperature variation. A detailed treatment of the generalized method considered can be found in Lykov's book [3.1].

The theory of regular regime can be used to solve such practical problems as determination of the time of heating (cooling), evaluation of the thermophysical parameters of a substance, and of the heat-transfer coefficient α, determinations of the radiation coefficient σ and of thermal resistances. Its merits include the simplicity of the experimental work, adequate accuracy of the results obtained, and the quickness of experiment.

3-11. Approximate Methods of Solving of Conduction Problems

At present there are many approximate methods for solving problems in heat conduction which give results satisfactory enough for engineering practice. They are most commonly used where precise analytical calculations are intricate. Some of these methods will be considered below.

(*a*) *Numerical Method*

The analytical solutions obtained by direct integration of the differential equations of conduction make it possible to calculate the temperature at any point of a given system. By contrast, the numerical method of solution is based on an equation expressed in finite differences and used to calculate the temperature at certain previously selected points of the system. This is equivalent to the mathematical method of approximate integration. It will be noted that while a precise analytical solution entails the difficulty of it being necessary to satisfy boundary conditions which are not always to be met, the numerical method always allows the boundary conditions of any practical problem to be satisfied at least approximately.

Of the numerical methods used presently to solve problems in conduction the *finite-differences method* finds the widest application.

In essence, the method consists in replacing in the differential equation the derivatives of the unknown function with approximate relations between the finite differences at individual nodal points. The resulting finite-differences equation is solved by simple algebraic operations. The calculation relationship is reduced to a form in which the future temperature at the nodal point considered is a function of time, true temperature at the point considered and true temperature at the neighbouring points. Equations of this kind are composed for all nodal points of the region considered, including the boundary points. As a result we obtain a closed system of algebraic equations. Owing to the monotypic nature of the calculations involved in solving such a system of equations, modern computers can be widely employed.

To obtain the calculation formulae for numerical integration wide use is made presently of the method of heat balances and of mathematical operations when replacing in the differential equations the derivatives of the functions with finite differences.

By way of example let us obtain the calculation formula for the numerical integration of a one-dimensional problem in transient conduction by the method of heat balances. Let the process of conduction then be described by the equation

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2} \tag{3-100}$$

The first stage of the numerical method consists in dividing the given system into a corresponding number of small volumes and in numbering the nodal points of each of these volumes. The thermal properties of each volume are assumed to be concentrated at its central nodal point. The transfer of heat from one nodal point to another is imagined to be affected through fictitious heat-conducting rods.

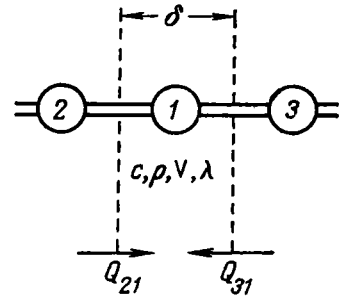

Fig. 3-24. Subdivision and numerical network for a transient one-dimensional system

Under transient conditions heat is not only added to or rejected from each nodal point, but there is also a change in internal energy. The change in internal energy depends on the temperature variation at the nodal point in time, on the heat capacity (specific heat) of the elementary volume which the particular nodal point represents, and on the density of the substance. This approach to the problem of temperature calculation is referred to as the method of approximate numerical iteration.

Consider the application of this method to calculation of the temperature field in a plane wall (Eq. 3-100). Its application to other problems of heat conduction is described in the specialist literature [3.1, 3.4, 3.11, 3.12, 3.13].

Let us divide the plane wall into elementary volumes $V = \delta \cdot \delta \cdot 1 = \delta^2$ (Fig. 3-24). The specific heat c of the substance and its thermal conductivity λ are assumed to be constant within the elementary volume. The quantity of heat conducted by a rod to a nodal point will apparently be determined by Fourier's law: $q = -\lambda \frac{dt}{dx}$. If the distance δ is sufficiently small, the quantity of heat q can be expressed through finite differences, i.e. $q = -\frac{\lambda}{\delta}(\Delta t)$, where Δt is the difference between the temperatures of two adjacent nodal points. The total quantity of heat conducted by a rod in a finite increment of time $\Delta\tau$ is:

$$Q = q\Delta\tau F = -\frac{\lambda}{\delta}\Delta t\,\Delta\tau F \tag{3-101}$$

where the heat transfer surface of a one-dimensional system is $F = \delta \cdot 1$, m^2.

The change in internal energy at the considered nodal point per time $\Delta\tau$ is

$$U = c\rho V\Delta t' = c\rho V(t' - t) \tag{3-102}$$

where t and t' = temperature at the given nodal point at time τ and $\tau + \Delta\tau$ respectively;
c = specific heat;
ρ = density of the substance;
V = elementary volume.

Based on the foregoing, the heat balance equation for nodal point *1* (Fig. 3-24) is:

$$Q_{21}+Q_{31}=\frac{c\rho V}{\Delta\tau}(t_1'-t_1)$$

or

$$\frac{\lambda}{\delta}(t_2-t_1)\,\delta\cdot 1+\frac{\lambda}{\delta}(t_3-t_1)\,\delta\cdot 1=\frac{c\rho V}{\Delta\tau}(t_1'-t_1)$$

Solving the second of these equations with respect to the unknown temperature t_1', we obtain:

$$t_1'=\frac{\Delta\tau\lambda}{c\rho V}\left(t_2+t_3+\frac{1}{\lambda\Delta\tau/c\rho V}-2t_1\right) \tag{3-103}$$

Taking into account that $\lambda/c\rho = a$, the thermal diffusivity of the substance $V = \delta^2$, and $a\Delta\tau/\delta^2 = \mathrm{Fo}$, the Fourier number, Eq. (3-103) acquires the form:

$$t_1'=\mathrm{Fo}\left[(t_2+t_3+t_1)\left(\frac{1}{\mathrm{Fo}}-2\right)\right] \tag{3-104}$$

Equation (3-104) is the basis of the numerical calculation method for processes of transient conduction. Calculation of the temperature t_1' by this formula requires the selection of a certain value of the Fourier number. When this is done, it is important to bear in mind that the choice is limited by the condition

$$\mathrm{Fo}\leqslant 1/2 \tag{3-105}$$

Analysis has shown that Eq. (3-104) is ensured only if condition (3-105) is fulfilled. But if $\mathrm{Fo} > 1/2$ is assumed [i.e. if condition (3-105) is disregarded], the variation in calculated temperature becomes random and irregular, and the calculation is no longer accurate or reliable. It is therefore necessary to ensure that (3-105) is adhered to, when selecting the intervals δ and $\Delta\tau$. If the values of δ and $\Delta\tau$ are selected on the condition $\mathrm{Fo} = 1/2$, Eq. (3-104) becomes:

$$t_1'=\frac{t_2'+t_3}{2} \tag{3-106}$$

Formula (3-106) is widely used when solving graphically problems in stationary conduction, with the expected temperature of the given nodal point being independent of its present temperature.

With other values of Fo Eq. (3-104) gives still more complicated results:

$$\mathrm{Fo}=\frac{1}{3},\quad t_1'=\frac{1}{3}(t_1+t_2+t_3) \tag{3-107}$$

and

$$\mathrm{Fo}=\frac{1}{4},\quad t_1'=\frac{1}{4}(t_3+t_2+2t_1) \tag{3-108}$$

It follows from Eqs. (3-106) through (3-108) that a decrease of the Fourier number increases the number of calculations and the network density; calculation accuracy, however, increases.

The approximate numerical method does not practically differ from precise calculations when one of the plate surfaces is insulated and no heat transfer takes place on it, and the local heat-transfer coefficient of the other surface α tends to infinity even at $\mathrm{Fo} = 1/4$. Comparison of the calculations is illustrated in Fig. 3-25 [3.4]. The described method can also be used to derive the initial equation for numerical calculation of other problems in transient conduction.

For a two-dimensional system, in particular, the pattern of the nodal points of a solid divided into elementary volumes, measuring $\Delta x = \Delta y = \delta$ will be as shown in Fig. 3-26. Working out the heat balance equation for the central nodal point, we get:

$$t_0' = \mathrm{Fo}\left[t_1 + t_2 + t_3 + t_4 + t_0\left(\frac{1}{\mathrm{Fo}} - 4\right)\right] \qquad (3\text{-}109)$$

where t_0, t_1, t_2, t_3, t_4 = temperatures at respective nodal points at time τ;
t_0' = temperature at the central nodal point at time $\tau + \Delta\tau$.

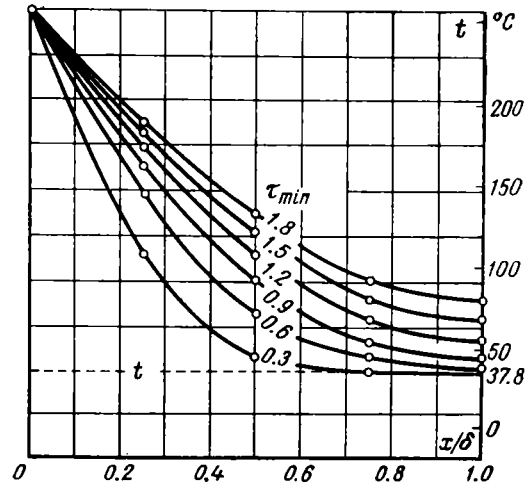

Fig. 3-25. Numerical results compared with exact solutions of temperature distribution (Fo = 1/4) in a plane wall. Lines correspond to the analytical solution, points—to the numerical calculation

For a two-dimensional problem the thickness δ and the time interval $\Delta\tau$ must be selected in accordance with the condition

$$\mathrm{Fo} \leqslant \frac{1}{4} \qquad (3\text{-}110)$$

as with the condition (3-105) for a one-dimensional problem.

With Fo = 1/4 Eq. (3-109) becomes:

$$t_0' = \frac{t_1 + t_2 + t_3 + t_4}{4} \qquad (3\text{-}111)$$

Then the expected temperature at the nodal point is independent of its present temperature.

The derivatives of a function can be replaced in the differential equation with finite differences by means of mathematical operations. Such an approach is not more exact, but it makes it possible to solve problems under various boundary conditions, to evaluate the error involved in passing from the differential equation to a difference equation and to simplify the analysis of the conditions of steadiness and accuracy of the solution.

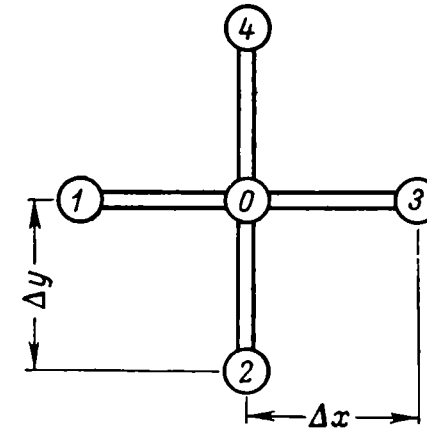

Fig. 3-26. Network of nodal points of a transient two-dimensional problem

Let us obtain an approximate replacement of the first and second derivatives with the finite differences of some function $t = f(x)$, where x may be assumed to be any independent variable.

Let us first of all divide the variation interval for the function into equal sections δ_x. Each point will have its abscissa, differing by δ_x, in other words, let us replace the coordinate of point x_m with $m\delta_x$ ($m = 1, 2, 3, \ldots$). Let

us mark on the curve $t = f(x)$ the points $A(t_m, m\delta_x)$, $B[t_{m-1}, (m-1)\delta_x]$ and $C[t_{m+1}, (m+1)\delta_x]$.

The tangent at point $A(t_m, m\delta_x)$ forms an angle α_m with the positive direction of the x-axis, then the derivative of the function for the considered point $A(t_m, m\delta_x)$

$$t'_m = \tan \alpha_m \tag{3-112}$$

If the distance δ_x is small, the angle α_m can be replaced with sufficient accuracy by angles β_m or γ_m (Fig. 3-27) formed by the secants BA and AC. The derivative at point $A(t_m, m\delta_x)$ then takes the following appearance:

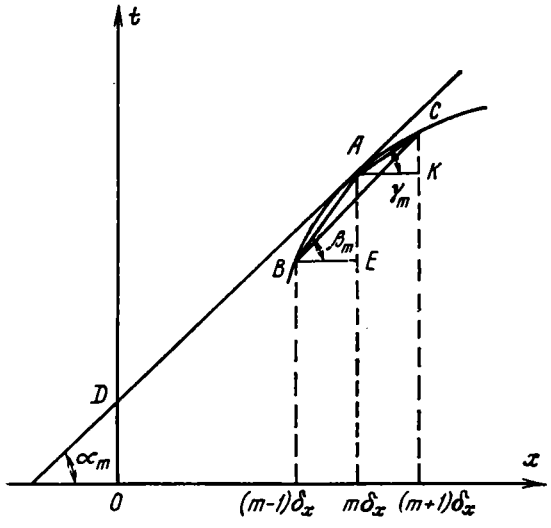

Fig. 3-27. Deriving formulae for replacing first and second derivative by difference quotient

$$t'_m \approx \tan \beta_m = \frac{AE}{BE} = \frac{t_m - t_{m-1}}{\delta_x} \tag{3-113}$$

or

$$t'_m \approx \tan \gamma_m = \frac{CK}{AK} = \frac{t_{m+1} - t_m}{\delta_x} \tag{3-114}$$

Replacing the angular coefficient of tangent AD with the angular coefficient of secant BC, we obtain the following expression for the derivative at point A:

$$t'_m \approx \frac{t_{m+1} - t_{m-1}}{2\delta_x} \tag{3-115}$$

The obtained expressions (3-113) through (3-115) are equally suitable for replacing the first derivative of the function with finite differences, and they are called the preceding, the subsequent and symmetrical finite differences respectively.

Replacing the section BC of the curve with broken line BAC, having two slopes at point A, we obtain the expression for the second derivative of the function $t = f(x)$:

$$t''_m = \frac{1}{\delta_x}\left(\frac{t_{m+1} - t_m}{\delta_x} - \frac{t_m - t_{m-1}}{\delta_x}\right) = \frac{1}{\delta_x^2}(t_{m+1} + t_{m-1} - 2t_m) \tag{3-116}$$

Equations (3-114) through (3-116) are usually used for numerical integration of differential equations of conduction. Let us use the derived formulas for transforming the differential equations into the finite-difference form, considering by way of example a problem involving transient conduction in an infinite wall [Eq. (3-110)]:

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x^2}$$

Since the temperature $t(x, \tau)$ is a function of two variables, it is advantageous to select a rectangular network. Let us divide into equal distances δ_x the section on the x-axis along which x varies from 0 to l and divide the interval of time from $\tau = 0$ to $\tau = \theta$ into one-dimensional intervals δ_τ (Fig. 3-28). The perpendiculars, raised from the points of division intersect and form the calculation nodal points. Then, the temperature for the nodal point 1 with coordinates $x = m\delta_x$ and $\tau = k\delta_\tau$ will be:

$$t_1 = t_1(m\delta_x, k\delta_\tau) = t_{m,k}$$

For point *2* with coordinates $x = m\delta_x$ and $\tau + \delta\tau = (k+1)\,\delta_\tau$, we have

$$t_2 = t_2\,[m\delta_x\,(k+1)\,\delta_\tau] = t_{m.\,k+1}$$

For point *3* with coordinates $x + \delta_x = (m+1)\,\delta_x$ and $\tau + \delta_\tau = (k+1)\,\delta_\tau$ we find that

$$t_3 = t_3\,[(m+1)\,\delta_x,\ (k+1)\,\delta_\tau] = t_{m+1,\,k+1},\ \text{etc.}$$

Let us replace at point *1* $(m\delta_x,\ k\delta_\tau)$ the partial derivatives in the equation of conduction with finite differences:

$$\left(\frac{\partial t}{\partial \tau}\right)_{m,\,k} = \frac{1}{\delta_\tau}\,(t_{m,\,k+1} - t_{m,\,k}) + \varepsilon_1 \qquad (3\text{-}117)$$

$$\left(\frac{\partial^2 t}{\partial x^2}\right)_{m,\,k} = \frac{1}{\delta_x^2}\,(t_{m+1,\,k} - 2t_{m,\,k} + t_{m-1,\,k}) + \varepsilon_2 \qquad (3\text{-}118)$$

In Eqs. (3-117), (3-118) the terms ε_1 and ε_2 are remainder terms, accounting for the transition from derivatives of the function to finite differences. These terms can be shown to tend to zero when the distance δ_x and the time interval δ_τ tend to zero. In finite differences the differential equation acquires the form:

$$\frac{1}{\delta_\tau}\,(t_{m,\,k+1} - t_{m,\,k}) + \varepsilon_1 = \frac{a}{\delta_x^2}\left(\frac{t_{m+1,\,k} - 2t_{m,\,k} + t_{m-1,\,k}}{\delta_x^2}\right) + \varepsilon_2 a \qquad (3\text{-}119)$$

Solving Eq. (3-119) in respect to the expected temperature $t_{m,\,k+1}$ at the point considered, we obtain:

$$t_{m,\,k+1} + \frac{a\delta_\tau}{\delta_x^2}\,[t_{m+1,\,k} + t_{m-1,\,k}] - \left(\frac{2a\delta_\tau}{\delta_x^2} - 1\right) t_{m.\,k} + (a\varepsilon_2 - \varepsilon_1)\,\delta_\tau \qquad (3\text{-}120)$$

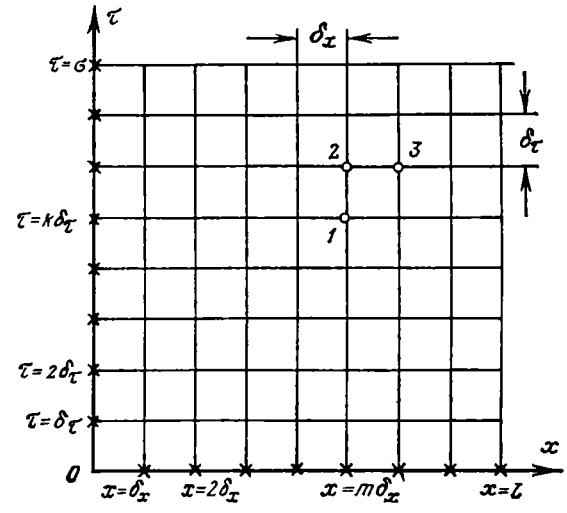

Fig. 3-28. Obtaining calculation network and composing equations for nodal points

It is clear that in Eq. (3-120) the residual term $(a\varepsilon_2 - \varepsilon_1)\,\delta_\tau$ will tend to zero when δ_τ tends to zero. Hence, the smaller the intervals chosen for the network of nodal points, the lesser the error involved in passing from the differential equation of conduction to the difference equation. The errors ε_1 and ε_2 can be evaluated expanding the function t into a Taylor series.

Dropping the remainder term in Eq. (3-120) and denoting the approximate value of the quantity $t_{m,\,k}$ by $T_{m,\,k}$, we obtain an approximate solution for the expected temperature at the nodal point $(m\delta_x,\ k\delta_\tau)$:

$$T_{m,\,k+1} = \frac{a\delta_\tau}{\delta_x^2}\,(T_{m+1,k} + T_{m-1,\,k}) - \left(\frac{2a\delta_\tau}{\delta_x^2} - 1\right) T_{m,\,k} \qquad (3\text{-}121)$$

In equation (3-121) the term $a\delta_\tau/\delta_x^2$ has the meaning of the Fourier number for an elementary space of the region considered, then

$$T_{m,\,k+1} = \mathrm{Fo}\,(T_{m+1,\,k} + T_{m-1,\,k}) - (2\mathrm{Fo} - 1)\,T_{m,\,k} \qquad (3\text{-}121')$$

If the temperature profile (distribution) in the region considered is known for any (for instance, the initial) moment of time, then, using a system of equations of the type derived, we can calculate the temperature at the nodal points for

the subsequent moment of time $\tau + \delta_\tau$. The temperature at the nodal points located on the boundaries of the range of integration is known from the boundary conditions.

It follows from Eq. (3-121) that the calculation temperatures depend on the Fourier number, i.e. on the method of dividing the space-time region. Selecting the space and time intervals, respectively δ_x and δ_τ, we can obtain any Fourier number. Analysis has shown, however, the solution to be persistent not with any value of the Fourier number, and, consequently, the quantities δ_x and δ_τ cannot be chosen arbitrarily. An analysis of the deviation of the numerically calculated solution from an exact solution shows that a persistent solution of the one-dimensional problem considered can be ensured only on condition that in Eq. (3-120′)

$$(2\mathrm{Fo} - 1) \leqslant 0 \tag{3-122}$$

The expression (3-122) is the main condition limiting the arbitrary choice of δ_x and δ_τ for a network. Exactly the same condition for ensuring persistent numerical integration was obtained by the method of heat balances [expression (3-105)].

We can also easily derive a difference equation for a two-dimensional problem in transient conduction $t\,(x,\ y,\ \tau)$. For such a problem the differential equation has the following form:

$$\frac{\partial t}{\partial \tau} = a\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right) \tag{a}$$

The temperature at any nodal point then must be assigned a superscript and two subscripts, $t^k_{m,\,n}$, where the subscripts m, n indicate the coordinates, and k is the superscript indicating time. Let us divide the integration range into equal distances and time intervals. Then, using the relationships derived for a nodal point with coordinates $(m\delta_x,\ n\delta_y,\ k\delta_\tau)$, we obtain:

$$\left.\begin{aligned}
\left(\frac{\partial t}{\partial \tau}\right)^k_{m,\,n} &= \frac{1}{\delta_\tau}\left(t^{k+1}_{m,\,n} - t^k_{m,\,n}\right) + \varepsilon_1 \\
\left(\frac{\partial^2 t}{\partial x^2}\right)^k_{m,\,n} &= \frac{1}{\delta_x^2}\left(t^k_{m+1,\,n} + t^k_{m-1,\,n} - 2t^k_{m,\,n}\right) + \varepsilon_2 \\
\left(\frac{\partial^2 t}{\partial y^2}\right)^k_{m,\,n} &= \frac{1}{\delta_y^2}\left(t^k_{m,\,n+1} + t^k_{m,\,n-1} - 2t^k_{m,\,n}\right) + \varepsilon_3
\end{aligned}\right\} \tag{3-123}$$

Substituting the obtained expressions for the derivatives (less the remainder terms ε_1, ε_2, ε_3) into differential equation (a), we obtain an approximate relationship for the expected temperature at the point $(m\delta_x,\ n\delta_y,\ k\delta_\tau)$:

$$\frac{1}{\delta_\tau}\left(T^{k+1}_{m,\,n} - T^k_{m,\,n}\right) = \frac{a}{\delta_x^2}\left(T^k_{m+1,\,n} + T^k_{m-1,\,n} - 2T^k_{m,\,n}\right) + \\ + \frac{a}{\delta_y^2}\left(T^k_{m,\,n+1} + T^k_{m,\,n-1} - 2T^k_{m,\,n}\right) \tag{3-124}$$

Assuming $\delta_x = \delta_y$ and solving Eq. (3-124) in respect to the expected temperature at the considered point, we obtain:

$$T^{k+1}_{m,\,n} = \frac{a\delta_\tau}{\delta_x^2}\left(T^k_{m+1,\,n} + T^k_{m-1,\,n} + T^k_{m,\,n+1} + T^k_{m,\,n-1}\right) - \left(\frac{4a\delta_\tau}{\delta_x^2} - 1\right) T^k_{m,\,n} \tag{3-124′}$$

Denoting, as in the preceding case, $a\delta_\tau/\delta_x^2 = \mathrm{Fo}$, expression (3-124′) reduces into:

$$T^{k+1}_{m,\,n} = \mathrm{Fo}\left(T^k_{m+1,\,n} + T^k_{m-1,\,n} + T^k_{m,\,n+1} + T^k_{m,\,n-1}\right) - (4\mathrm{Fo} - 1)\, T^k_{m,\,n} \tag{3-124″}$$

It is clear that for such a two-dimensional problem the solution will be persistent only on the condition that

$$(4\mathrm{Fo} - 1) \leqslant 0 \tag{3-125}$$

Assuming Fo = 1/4, Eq. (3-124) becomes:

$$T^{k+1}_{m,n} = \frac{T^{k}_{m+1,n} + T^{k}_{m-1,n} + T^{k}_{m,n+1} + T^{k}_{m,n-1}}{4} \tag{3-126}$$

It follows from Eq. (3-126) that the expected temperature at the point considered is independent of its present temperature and it is determined by the present temperatures at the adjacent points.

Similar calculation formulae for the temperatures at the nodal points can also be derived for three-dimensional problems.

(b) Principle of Stable Heat Flow

There are a number of approximate solutions for problems dealing with the transmission of heat in bodies of arbitrary shape. We shall examine the method based on the principle of stable heat flow. If we assume, and leave, the constant heat flow at the surface of a body, but change the conditions of cooling over a small area of it, this will bring about an essential local change in the temperature field. Changes in the temperature field at points at a distance from the point of disturbance, however, will be insignificant [3.14].

It follows from this that deformation of the surface of the body will affect the temperature field substantially only at points close to the surface, while it will remain constant at points removed from the surface.

Making use of these properties of stable heat flow, the calculation of conduction through bodies of complex geometry can be reduced to calculation of the heating (cooling) of bodies of three classical shapes: a one-dimensional plane plate, or body of the first class; a long cylinder, or body of the second class; and a sphere, or body of the third class. In solving a problem, one has first to determine in a rational way the class to which the body concerned is to be assigned. Then, the temperature field of this body is compared with that of a reference body of that class.

The following conditions must be fulfilled in accordance with the principle of stability:

$$\alpha (t_w - t_f) S \, d\tau = \alpha_0 (t_{w0} - t_{f0}) S_0 \, d\tau_0 \tag{3-127}$$

where α = the surface mean heat-transfer coefficient, W/m·K;
t_w = mean temperature of the surface, °C;
t_f = ambient temperature, °C;
S = cooling surface, m²;
τ = time, s.

Values without the subscript "0" refer to the body considered, and values with the subscript "0" pertain to the reference body. Calculation of the temperature field for the body considered can be resolved into calculation of the temperature field on an equivalent reference body (plate, cylinder, or sphere). This assertion is based on the assumption that the external configuration of the body will affect only the temperature field essentially at points near to its surface, while temperature fields at a distance from the surface will be comparable with those observed in reference bodies of the respective class.

If we denote:

$$t_w - t_f = \vartheta \quad \text{and} \quad t_{w0} - t_{f0} = \vartheta_0,$$

assuming

$$\vartheta = \vartheta_0 \quad \text{and} \quad d\tau = d\tau_0,$$

then Eq. (3-127) acquires the following form:

$$\alpha S = \alpha_0 S_0$$

or

$$\alpha_0 = \alpha \frac{S}{S_0} = \alpha A \tag{3-128}$$

where

$$A = \frac{S}{S_0} \tag{3-129}$$

The dimensionless factor A known as the criterion of form characterizes the surface of the considered body expressed in terms of the surface of the reference body. The factor A is referred to as the shape criterion.

Equation (3-128) expresses the quantitative requirements which have to be fulfilled to ensure equivalence of the two temperature fields.

In calculating these temperature fields, the quantity α in the formulae for the reference body is replaced by α_0 calculated by Eq. (3-128) and an equivalent dimension of a solid of the respective class is taken as the linear dimension l_0. Thus the Biot number has the following form:

$$\mathrm{Bi} = \frac{\alpha A l_0}{\lambda} = \frac{\alpha_0 l_0}{\lambda} \tag{3-130}$$

For bodies of the first class this equivalent linear dimension is

$$l_0 = X_1 = \frac{V}{2S_{mp}} \tag{3-131}$$

where V is the volume of the body in m^3; S_{mp} is the area of middle plane of the body, m^2;

the shape criterion

$$A = A_1 = \frac{S}{S_{mp}} \tag{3-132}$$

where S is the area of one side surface of the wall, m^2.

For bodies of the second class the equivalent linear dimension is

$$l_0 = X_2 = \sqrt{\frac{S_{cs}}{\pi}} \tag{3-133}$$

where S_{cs} is the cross-sectional area of the body;

the shape criterion

$$A = A_2 = \frac{P}{\sqrt{4\pi S_{cs}}} = \frac{P}{P_0} \tag{3-134}$$

where P is the perimeter of the cross section of the body considered, m; P_0 is the perimeter of the cross section of an equivalent cylinder, m.

For bodies of the third class the reference equivalent linear dimension is

$$l_0 = X_3 = \sqrt[3]{\frac{3V}{4\pi}} \tag{3-135}$$

If the surface S_0 of the equivalent sphere is expressed in Eq. (3-129) in terms of its volume V equal to the volume of the considered body, then

$$A = A_3 = \frac{S}{\sqrt[3]{36\pi V_2}} \tag{3-136}$$

The method described for calculation of temperature fields yields satisfactory results when applied to problems of low- and medium-intensity heat transfer at the surface of a body.

With high-intensity heat transfer (Bi $\gg$ 1), Eq. (3-128) is replaced by the expression

$$\lambda_0 = \lambda \frac{S}{S_0} = \lambda A \tag{3-137}$$

Further calculations may be carried out, using the formulae previously derived.

Figures 3-29 and 3-30 show cooling curves for the centre of a cube and the axis of a rectangular bar, both subjected to high-intensity cooling (Bi $\gg$ 1). The

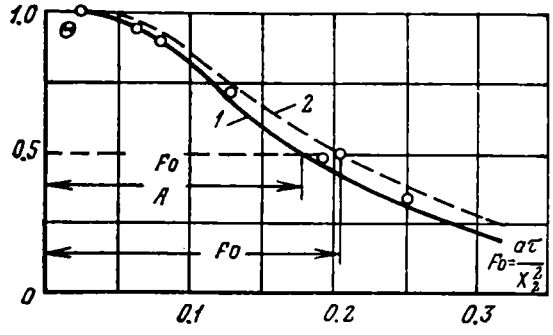

Fig. 3-29. Cooling curves for axes of the second class bodies:
Bi → ∞; *1*—cooling curve of a square bar (A_2 = =1.13); *2*—cooling curve of a main body—circular cylinder (A_2 = 1)

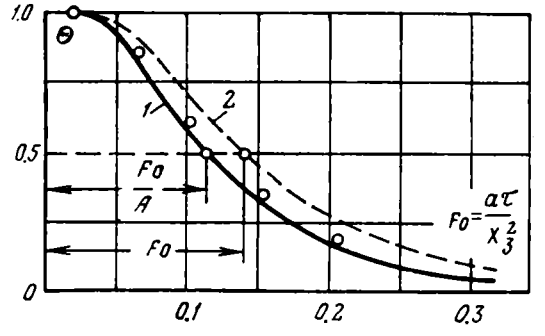

Fig. 3-30. Central temperature profiles for the third class bodies:
Bi → ∞; *1*—cooling curve of cube (A_3 = 1.24); *2*—cooling curve of a main body—sphere (A_3 = 1)

solid curves are calculated by the method of equivalent bodies. The points plotted in Figs. 3-29 and 3-30 are derived from precise solutions of the problem considered. The agreement of results is satisfactory [3.14].

The applicability of the theory is determined by the magnitude of criterion A. The accuracy of calculations increases as A tends to unity.

3-12. Investigation of Conduction Processes by the Method of Analogy

One of the experimental methods used to investigate processes of conduction is that of *analogy*. In this method investigation of the thermal phenomena is replaced by study of similar phenomena, because experimental investigation of the latter often proves simpler than direct study of the thermal processes.

The similarity or resemblance of analogous phenomena consists in the identical nature of the processes concerned. Mathematically analogous phenomena are formally described by similar differential equations and conditions of unambiguity. The physical essence, however, and units of the terms of the equations are different.

Electro-thermal analogy. The phenomena of heat and electrical conduction obey the following equations:

$$dQ = -\lambda \frac{\partial t}{\partial n_t} dF_t \tag{3-138}$$

$$dI = -\sigma \frac{\partial u}{\partial n_e} dF_e \tag{3-139}$$

where dQ and dI are elementary heat flow and electrical current per unit time per area dF_t, dF_e, in the direction of the perpendiculars n_t and n_e; t and u are temperature and electrical potential (electromotive force); λ and σ are the thermal conductivity and electric conductivity.

An application of these equations to a two-dimensional problem of steady-state conduction in time, in which the physical properties (λ, σ) are independent of temperature, leads us to the following Laplace differential equations:

$$\frac{\partial^2 t}{\partial x_t^2} + \frac{\partial^2 t}{\partial y_t^2} = 0 \tag{3-140}$$

$$\frac{\partial^2 u}{\partial x_e^2} + \frac{\partial^2 u}{\partial y_e^2} = 0 \tag{3-141}$$

The equations expressing temperature and electrical potentials are of identical structure. Analogous phenomena should develop in geometrically similar systems. The boundary conditions may be stated in different ways. Let us assume that the boundary conditions are given by the following equations corresponding to boundary conditions of the third kind (Sec. 1-6):

$$-\lambda \operatorname{grad} t = \alpha \, \Delta t$$

or

$$-\operatorname{grad} t = \frac{\Delta t}{\lambda/\alpha} = \frac{\Delta t}{l_t}$$

where

$$l_t = \frac{\lambda}{\alpha}$$

and

$$-\operatorname{grad} u = \frac{\Delta u}{l_e} \tag{3-142}$$

The quantitative relationship between analogous physical parameters (analogues) may be established by reducing their mathematical descriptions to the dimensionless form. For this purpose a certain value Δt_0 can be taken as a scale for the temperature difference, Δu_0 for the electrical potential (similar scale) and similar sections l_{t0} and l_{e0} for the linear dimensions. As before the subscripts "t" and "e" distinguish values pertaining to thermal and electrical phenomena.

Let us denote the quantities expressed in a relative scale

$$\frac{x_t}{l_{0t}} = X_t; \quad \frac{y_t}{l_{0t}} = Y_t; \quad \frac{l_t}{l_{0t}} = L; \quad \frac{\Delta t}{\Delta t_0} = \Theta$$

from which we obtain the relationships

$$x_t = l_{0t} X_t; \quad y_t = l_{0t} Y_t; \quad l_t = l_{0t} L; \quad \Delta t = \Delta t_0 \Theta$$

Analogous relationships are true for the values pertaining to the electrical phenomenon. Having substituted these relationships, Eqs. (3-140) and (3-141) ac-

quire the dimensionless form:

$$\frac{\Delta t_0}{l_{0t}^2}\left(\frac{\partial^2\Theta}{\partial X_t^2}+\frac{\partial^2\Theta}{\partial Y_t^2}\right)=0 \quad \text{or} \quad \frac{\partial^2\Theta}{\partial X_t^2}+\frac{\partial^2\Theta}{\partial Y_t^2}=0 \tag{3-143}$$

$$\frac{\Delta u_0}{l_{0e}^2}\left(\frac{\partial^2 U}{\partial X_e^2}+\frac{\partial^2 U}{\partial Y_e^2}\right)=0 \quad \text{or} \quad \frac{\partial^2 U}{\partial X_e^2}+\frac{\partial^2 U}{\partial Y_e^2}=0 \tag{3-144}$$

which are identical at any choice of similar scales for the temperature and electrical potential.

Upon reduction to the dimensionless form, the equations describing the boundary conditions become

$$-\text{grad}\ \Theta=\frac{\Theta}{L_t}$$

$$-\text{grad}\ U=\frac{U}{L_e}$$

which are identical; hence the solutions of the dimensionless differential equations of heat conduction and electric conduction are identical, provided that either of the conditions

$$L_t = L_e \tag{3-145}$$

or

$$\frac{l_t}{l_{0t}}=\frac{l_e}{l_{0e}}$$

is fulfilled.

Inasmuch as $l_t = \frac{\lambda}{\alpha}$, we obtain a dependence for choosing the linear dimensions of the electrical model of the phenomenon:

$$l_e=\frac{l_{0e}}{l_{0t}}\frac{\lambda}{\alpha}$$

If $l_{0e} = l_{0t}$, then $l_e = \frac{\lambda}{\alpha}$

When α = const and λ = const, then l_e = const.

Where conditions (3-145) are fulfilled, the dimensionless temperature and dimensionless electrical potential are numerically equal at similar points* of the systems considered:

$$\Theta_1=U_1 \quad \text{or} \quad \frac{\Delta t_1}{\Delta t_0}=\frac{\Delta u_1}{\Delta u_0}$$

$$\Theta_2=U_2 \quad \text{or} \quad \frac{\Delta t_2}{\Delta t_0}=\frac{\Delta u_2}{\Delta u_0}$$

. . . . . .

from which we obtain:

$$\frac{\Delta t_1}{\Delta u_1}=\frac{\Delta t_2}{\Delta u_2}=\ldots=\frac{\Delta t_0}{\Delta u_0}=c_{t,u}=\text{const} \tag{3-146}$$

The relationship (3-146) shows that temperature distribution and distribution of the electrical potential are similar under the given conditions, i.e. that there is an analogy.

* Points whose coordinates are interrelated as follows: $x_t = c_l x_e$, $y_t = c_l y_e$, where c_l is a constant.

In investigating transient processes for one-dimensional problems, the initial differential equations of thermal and electrical conduction have the form:

$$\frac{\partial t}{\partial \tau} = a \frac{\partial^2 t}{\partial x_t^2} \tag{3-147}$$

$$\frac{\partial u}{\partial \tau_e} = \frac{1}{R_e C_e} \frac{\partial^2 u}{\partial x_e^2} \tag{3-148}$$

where R_e is the electrical resistance per unit length; C_e is the electrical capacity per unit length.

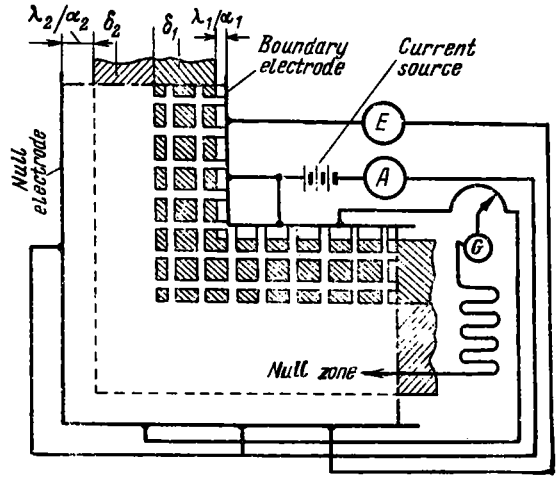

Fig 3-31. Electrical model of the corner of a building

Like thermal diffusivity, these parameters should not depend on temperature.

Comparison of Eq. (3-147) and Eq. (3-148) shows that analogy sets in at once, if the condition

$$a = \frac{1}{R_e C_e}$$

is observed, and the requirement (3-145), determined by the boundary conditions, is also valid.

Variation in heat flow is proportional to the variation of the heat capacity of the system and the variation of temperature in time:

$$dQ = C_t \frac{\partial t}{\partial \tau} d\tau_t$$

Variation of electric current is proportional to the capacity and variation in voltage, i.e. is expressed by an analogous equation

$$dI = C_e \frac{\partial u}{\partial \tau_e} d\tau_e$$

Consequently heat capacities can be reproduced in the model by the corresponding electrical capacities.

Let us consider examples to illustrate the application of these mathematical premises to electrical models.

There are two methods of developing electrical models simulating processes of heat conduction; in one, the geometry of the original thermal system is repeated and a continuous conducting material is used, either solid electric conductors or liquid electrolyte. Models of this type are referred to as geometrical analogues or models with continuous parameters.

In addition to this type, models with concentrated parameters are used, in which the thermal systems are replaced by electrical circuits or network analogues. The properties of the thermal system being studied are concentrated at individual nodal points in the circuits. These network analogues are used to investigate the most complex phenomena.

Models with continuous parameters, or geometrical analogues, are made up of thin conductive sheets or of conductive layers applied on glass or some other kind of plates, of which a flat sample is cut reproducing the geometric shape of the thermal region being studied.

Fig. 3-31 shows a scale model of the corner of a building, whose wall consists of two layers of different thickness and of differing thermal conductivity. The

electrical model must also be made of layers of different thickness and different electric conductivity. If the thermal conductivity of the inner layer, for instance, is smaller than that of the outer, then its electrical resistance is correspondingly increased by holes cut in it, or by the employment of sheets of a higher resistance. The absence of contact resistance between the layers is reproduced by their close union. Permanence of the electrical properties of the sheet conductor is ensured by the employment of suitable materials.

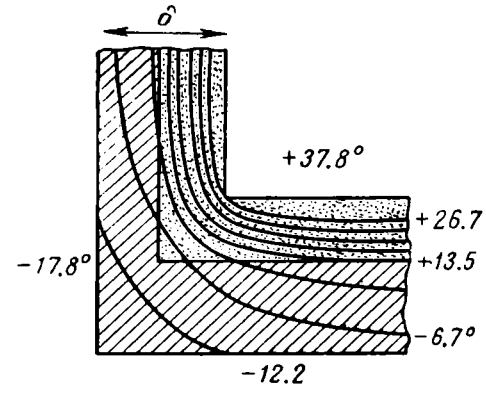

Fig. 3-32. Temperature field in the corner of a building

The thermal resistances of the surfaces of the investigated system are taken into account by means of additional layers $l_{e1} = \frac{\lambda_1}{\alpha_1}$ and $l_{e2} = \frac{\lambda_2}{\alpha_2}$ in the electrical model. Since usually the conditions are λ = const, and α = const, the additional layers must also be of constant thickness. The models are energized by connecting boundary electrodes to a storage battery.

By analogy the voltage at any point of the electrical model corresponds to the temperature at the similar point of the thermal system. The potential or voltage is measured by means of a null contact probe. The voltage at any point may be taken as the reference for readings; the voltage of the inner electrode, for instance, may be used as the zero voltage. The temperature field in the corner, obtained on an electrical model of this kind, is shown in Fig. 3-32. The isotherms plotted in the figure were simulated by equipotential lines in the model.

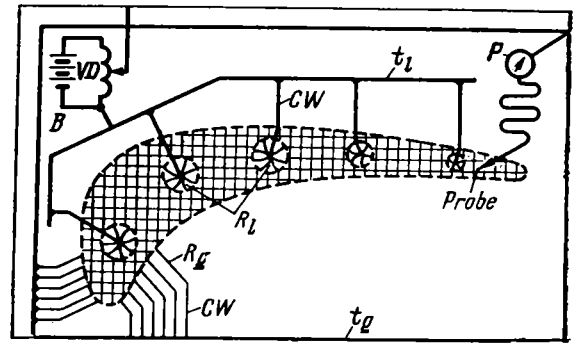

Fig. 3-33. Network analogue of a turbine blade

It is rather difficult in practice to obtain the temperature distribution by direct temperature measurements, as these would entail the employment of a considerable number of thermocouples, whose presence in the system would essentially affect the actual distribution of temperature. Furthermore, direct measurements take much time.

Let us consider an electrical model with concentrated parameters in the form of a network analogue. For this the thermal region being studied is divided into a number of elementary volumes. The initial differential equations and the equations describing the conditions of unambiguity are thereby replaced by equations expressed in finite differences. The corresponding network analogue takes the form of individual resistances which simulate the properties of the elements of the thermal region. Thus the latter, split into several elementary volumes, is replaced by an electrical circuit consisting of corresponding concentrated parameters connected in series.

Figure 3-33 shows a wire network analogue of a turbine blade. The wire model is made to a definite scale in the form of a square network. The analogue

network can be made of calibrated rheostat wires of 0.4 or 0.5 mm in diameter. The wires are first stretched over the pattern of the blade, then their intersections are spot-welded. Electrical resistances are chosen so as to correspond to the thermal resistances of the elements of the thermal system, i.e. of the blade.

The boundary thermal resistances are reproduced by wire-wound resistors R_l and R_g. Bus bars represent the constant temperatures of the gas t_g and cooling water t_l, circulating through channels. The potential differences across these bus bars correspond to the required gas and water temperatures. The leads and connecting wires must be of negligible electrical resistance. The network is

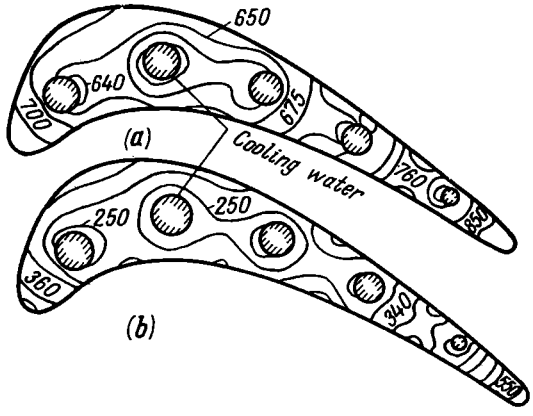

Fig. 3-34. Temperature field in a turbine blade

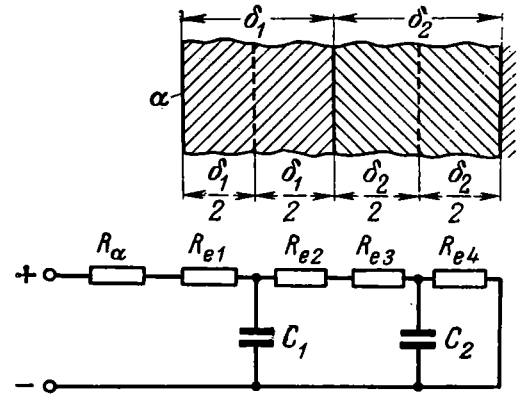

Fig. 3-35. Two-layer plane wall and its network analogue

supplied with current from a storage battery (B) through a voltage divider (VD). The voltage at any point of the electric circuit is measured with the aid of a contact probe and potentiometer (P).

The temperature distribution obtained for the section of the turbine blade is illustrated graphically in Fig. 3-34.

Let us now consider a case of transient conduction. A wall is taken as an example and shown in Fig. 3-35. It consists of two layers made of different materials, and one side is heat-insulated. Temperature distribution is uniform at the initial moment of time, then the wall is instantaneously exposed to a medium of another temperature, which does not vary further in time. The problem is to reproduce this thermal phenomenon by an electrical network analogue. For this purpose each layer of the wall may be subdivided into two layers. Its internal thermal resistances are then represented by four resistances:

$$R_{t1} = \frac{\delta_1}{2\lambda_1}; \qquad R_{t2} = \frac{\delta_2}{2\lambda_2};$$

$$R_{t3} = \frac{\delta_3}{2\lambda_3}; \qquad R_{t4} = \frac{\delta_4}{2\lambda_4}$$

which are simulated by electrical resistors

$$R_{t1} = R_{e1}; \quad R_{t2} = R_{e2}; \quad R_{t3} = R_{e3}; \quad R_{t4} = R_{e4}$$

The heat capacities of the two layers of the wall are reproduced by capacitors C_1 and C_2, and the thermal resistance of its surface is simulated by the external resistor R_α.

The thermal system is thus replaced by an electrical network incorporating resistors, connected in series, and capacitors, connected in parallel (Fig. 3-35, bottom).

The initial thermal state of the system is simulated by an open circuit in the analogue, and subsequent states by applying a voltage difference across the circuit terminals. Then the voltages measured at arbitrary points of the circuit and at arbitrary moments of time will correspond to the temperatures at similar points of the wall on the same scale.

Electrical simulation is now widely used, and a variety of analogue computers are available for solving various physical problems. Some incorporate special non-linear resistors permitting the simulation not only of boundary conditions for convection heat transfer, but also of cases where convection is accompanied by other modes of heat exchange (thermal radiation, for instance). The electrical integrator developed by L.I. Gutenmacher in the USSR belongs to this type of simulating apparatus.

The principle of the *hydro-thermal* or *fluid flow analogy* may also be used to investigate both steady-state and transient processes of conduction, then the similarity of the laws governing the propagation of heat and the flow of fluids is being utilized. As models it is possible to use apparatus with either continuous or concentrated parameters, i.e. in the form of hydraulic analogue circuits. In the latter, the parameters of the thermal process considered are replaced in the analogue circuit by concentrated parameters in the form of hydraulic resistances and capacities.

To illustrate the employment of the fluid-flow analogy, let us consider the transient temperature field in an infinite plane wall with both its dimensions thermal and physical properties given, with an arbitrary cross-sectional temperature distribution at the initial moment of time, and with boundary conditions given by the temperatures of the medium t_{f1} and t_{f2} and by the heat-transfer coefficients α_1 and α_2. A fluid model is the fluid-flow analogue built on the formal similarity between the equation describing the rate of heat flow

$$q = \Delta t / R_t$$

and that expressing the rate of laminar flow of a liquid ($\mathrm{Re} \leqslant \mathrm{Re}_{cr}$)

$$V = \Delta h / R_h$$

where Δh is the hydraulic head in metres of liquid column and R_h is the hydraulic resistance or pressure drop.

Since the phenomena compared vary in time, it is necessary to account for the variations in the rate of flow of both heat and fluid in time, which are expressed by the analogous equations:

$$dQ = C_t \frac{\partial t}{\partial \tau_t} d\tau_t$$

$$dV = f_h \frac{\partial h}{\partial \tau_h} d\tau_h$$

where C_t = heat capacity of the system;
f_h = cross-sectional area of the hydraulic conduit;
Q = amount of heat, J;
V = amount of fluid, m^3.

Comparison of the equations shows that the analogue of the rate of heat flow q is the rate of fluid flow, V, and that of the temperature difference Δt is the hydraulic head Δh; the analogue of the heat capacity is the hydraulic capacity; and that of the thermal resistance R_t is the hydraulic resistance R_h or pressure drop across the model. Let us construct the hydraulic circuit assuming for simplicity equal scales for the hydraulic model and the thermal phenomenon. The wall is divided into a finite number of layers, for example, four (Fig. 3-36). Each layer is represented by a vessel whose cross-section corresponds to the heat capacity of the respective layer of the wall:

$$C_{t1} = f_1; \quad C_{t2} = f_2; \quad C_{t3} = f_3, \quad C_{t4} = f_4$$

The thermal resistance $\frac{\delta_i}{\lambda_i} = R_{ti}$ of each layer of the wall is also represented by a corresponding hydraulic resistance:

$$R_{t1} = R_{h1}; \quad R_{t2} = R_{h2}; \quad R_{t3} = R_{h3}; \quad R_{t4} = R_{h4}$$

These hydraulic resistances are built up by means of capillaries connecting the vessels. The vessels are filled with liquid to levels corresponding to the initial temperature distribution in the wall at $\tau = 0$ (initial moment of time)

$$h_1 = t_1; \quad h_2 = t_2; \quad h_3 = t_3; \quad h_4 = t_4$$

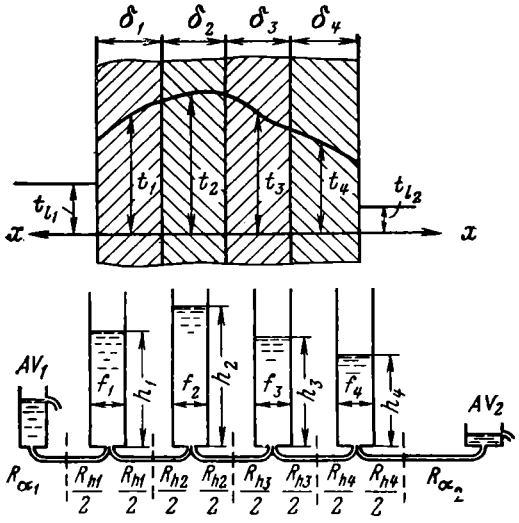

Fig. 3-36. Multi-layer plane wall and its hydraulic analogue

The conditions of heat transfer on the surfaces of the wall are reproduced by means of the constant-level vessels AV_1 and AV_2, in which the levels of liquid correspond to the temperature of the surroundings on both sides of the wall. If these temperatures are constant, the liquid must be maintained at a permanent level. The thermal resistances to heat transfer are simulated by the hydraulic resistance of capillaries $R_{\alpha 1}$ and $R_{\alpha 2}$, connected to the main capillaries that represent the outer layers of the wall. Each capillary has a valve, and all valves are closed at the initial moment of time. The levels of liquid in the vessels reproduce the initial distribution of temperature in the wall. Then all the valves are opened simultaneously, and the variations in levels are recorded at given intervals of time; or they may also be photographed at different moments of time. The levels give the numerical value of the temperature of individual layers of the wall; the more layers there are, the more exact is the reproduction of temperature distribution across the wall.

If different scales have been chosen for the model and the thermal phenomenon, the transition from the model to the actual process is effected by means of scale conversion. For this purpose the original mathematical descriptions must be reduced to dimensionless form by the method previously described.

The so-called hydrointegrator developed by the Soviet scientist V.S. Lukyanov is based on hydro-thermal analogy, and can be used to solve one-, two-, or three-dimensional problems of heat conduction.

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2 HEAT TRANSFER BY CONVECTION IN A HOMOGENEOUS MEDIUM

CHAPTER
4

FUNDAMENTALS OF HEAT TRANSFER BY CONVECTION

4-1. Basic Concepts and Definitions

The concept of convection embraces the process of heat transfer through the motion of a liquid or gas. Heat is transported simultaneously during the process by convection and by conduction. By convection is meant the process of heat transport occurring through the movement of the macroparticles of the liquid or gas in space from a region of one temperature to that of another. Convection is possible only in a fluid medium, and the transport of heat is directly linked here with the transport of the medium itself.

If per unit time a mass of fluid $\rho\vec{w}$, kg/m^2·s, where $\vec{w}$ is the velocity of flow and ρ—the density of the fluid, enthalpy is transported together with the fluid in the amount J/m^2·s:

$$\vec{q}_{conv} = \rho\vec{w}i \tag{4-1}$$

The transfer of heat by convection is always accompanied by heat transfer by conduction, since when a gas or liquid is in motion individual particles which are at different temperatures come inevitably into contact with each other. As a result, heat transfer by convection is described by equation

$$\vec{q} = \vec{q}_{cond} + \vec{q}_{conv} = -\lambda\nabla t + \rho\vec{w}i \tag{4-2}$$

In formula (4-2) $\vec{q}$ is the local rate of heat flow due to heat transfer by convection. The first term in the right-hand side of Eq. (4-2) describes the transport of heat by conduction and the second—by convection.

The convection between a flow of liquid or gas and the surface of a solid is called the *heat transfer by convection* or *heat transfer*. In practice, engineers have to deal often with heat transfer from a wall to the surroundings or from the latter to a wall; then, the transfer of heat by convection inside the fluid medium may be of indirect interest, inasmuch as the transport of heat inside the fluid involved also affects local heat transfer, i.e. heat transfer from a wall to a medium or vice versa.

Practical calculations are based on Newton's law

$$dQ_w = \alpha\,(t_w - t_f)\,dF \tag{4-3}$$

according to which the amount of heat dQ_w, W, transferred from the fluid to an elemental area dF of the wall exposed to the fluid (or from the surface element dF to the fluid) is directly proportional to dF and the temperature difference $\Delta t = t_w - t_f$, where t_w is the surface temperature of the wall and t_f is the temperature of the surrounding liquid or gaseous medium. The temperature difference $t_w - t_f$ is also referred to as the temperature drop.

The proportionality factor α in Eq. (4-3) is called the local heat-transfer coefficient; it accounts for the condition under which a practical process of heat transfer occurs, affecting its intensity. According to Eq. (4-3)

$$\alpha \equiv \frac{dQ_w}{(t_w - t_f)\, dF} = \frac{q_w}{t_w - t_f} \tag{4-4}$$

The identity (4-4) should be considered as the definition of the local heat-transfer coefficient which is measured in $W/m^2 \cdot K$.

Thus, the heat-transfer coefficient is the rate of heat flow q_w at the fluid-and-body interface related to the temperature difference between the surface of the body concerned and its surroundings.

In the general case the local heat-transfer coefficient varies over the surface F. If α and Δt do not vary over the surface F, then Newton's law takes the following form:

$$Q_w = \alpha\, (t_w - t_f)\, F$$

The local heat-transfer coefficient depends on many factors. In the general case α is a function of the shape and size of the body, of the pattern of fluid flow, fluid velocity and temperature, physical properties of the fluid and of other quantities. The process of heat transfer from a solid to a fluid and vice versa proceeds in different ways, depending on the nature and origin of fluid motion.

To impart motion to a fluid, a force must be applied to it. The forces acting upon any element of the fluid can be classified as *mass* (or volumetric) and *surface forces*. By mass forces are meant those applied to all fluid particles and due to external fields of forces (for instance, gravitational or electric). Surface forces originate due to the action of the surrounding fluid or solids; they are applied to the surface of the considered volume of the fluid. Such forces are represented by external pressure forces and by frictional forces.

Two types of convection are distinguished, *free*, or *natural*, and *forced convection*. In the event of free, or natural, convection the origin of motion in the considered volume of fluid is due to the heterogeneity of the mass forces acting upon the volume. If a fluid with a non-uniform temperature distribution, and, as a result, characterized by a non-uniform distribution of density, is subject to the terrestrial gravitational field, free, gravitational motion may originate in it. Below, we shall consider in the main gravitational free convection due to a non-uniform temperature field.

The forced motion of the volume of fluid considered is due to the action of the external surface applied at its boundaries at the expense of the preliminarily added kinetic energy (for instance, at the expense of a pump, fan, wind). Also considered as forced is the flow of the investigated volume of fluid under the action of an internal homogeneous field of mass forces. Forced motion of this kind can be illustrated by the flow of an isothermal film of liquid on a wall under the action of gravity.

In the general case forced motion may be accompanied by free, or natural, motion. The relative effect of the latter is the greater the larger the temperature difference between the particles of the fluid and the lower the velocity of forced

convection (motion). With high-velocity forced motion the effect of free convection becomes negligible.

Only steady-state processes of flow and heat transfer will be considered here. The condition ensuring a steady state consists in the velocity and temperature of the fluid remaining invariable in time at any point of the fluid (gas).

4-2. Physical Properties of Fluids

The process of heat transfer develops in a different way peculiar to each of these substances, depending on its physical properties. Heat transfer is especially affected by the following physical properties: thermal conductivity λ, specific heat c_p, density ρ, thermal diffusivity a, already applied when considering conduction, and viscosity μ. These properties have definite magnitudes for each substance, and as a rule are a function of temperature, and some are also functions of pressure. Physical properties may change essentially in the near-critical region of thermodynamic states and in the low-temperature region.

The processes which shall be investigated below in the book are proceeding at monotone and not too significant changes of the physical properties of a certain substance. Heat transfer in the near-critical region will also be treated.

In the theoretical analysis of heat transfer by convection it will be assumed for the sake of simplicity that the physical properties of the fluid (gas) involved are constant within the investigated temperature range.

All real fluids are viscous; a force of internal friction, offering resistance to flow, always arises between the layers of a fluid moving at different velocities in relation to one another. According to Newton's law, the tangential force s, Pa (reduced to unit area) active at any point of the flow in the plane oriented in the direction of flow is proportional to the velocity gradient (i.e. to the variation of velocity in the direction perpendicular to flow):

$$s = \mu \frac{dw}{dn} \tag{4-5}$$

The factor μ is known as the *dynamic viscosity* or, simply, viscosity; it is measured in N·s/m^2. With $dw/dn = 1$, $s = \mu$ numerically.

The equations of hydrodynamics and heat transfer often incorporate a ratio of viscosity μ to density ρ, referred to as *kinematic viscosity* and denoted by ν, m^2/s:

$$\nu = \frac{\mu}{\rho}$$

The two viscosity coefficients μ and ν are physical parameters and depend essentially on temperature*.

The viscosity of liquids is almost independent of pressure, but declines significantly with rising temperature. Its dependence on temperature, $\mu = f(t)$, is illustrated graphically in Fig. 4-1.

The viscosity μ of gases increases with temperature (Fig. 4-2). An increase in pressure is also accompanied by greater viscosity in gases, but not quite as essentially [4.1].

* Some fluids do not obey Newton's law (4-5). Should we attempt to describe the viscous properties of these fluids with the aid of Eq. (4-5), we shall find that the viscosity factor is not only a function of the state of a substance, but it also depends on the process parameters, on the variation of velocity and temperature. Such fluids are referred to as non-newtonian (or simple) liquids.

The kinematic viscosity of liquids diminishes with rising temperature almost to the same extent as μ, since the density ρ depends but slightly on temperature; in contrast, the kinematic viscosity of gases, whose density diminishes greatly with rising temperature, increases rapidly with rising temperature. Internal friction in a flowing viscous liquid or gas brings about a process of *dissipation of energy*. In essence this consists in a fraction of the kinetic energy of the fluid

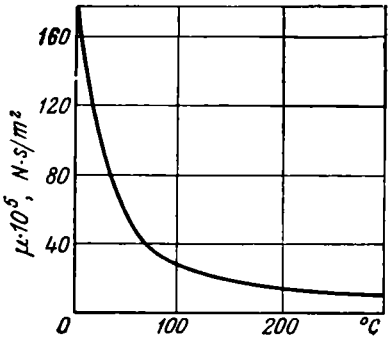

Fig. 4-1. Dependence of the dynamic viscosity of water on temperature

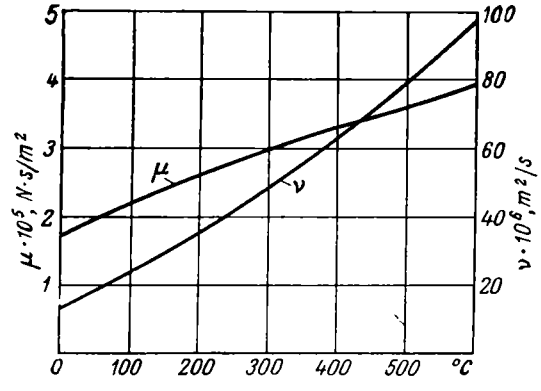

Fig. 4-2. Dependence of the dynamic and kinematic viscosity of air on temperature at a pressure $p = 760$ mm Hg

being converted into thermal energy and heating it. The heating, however, will be insignificant, if the viscosity and flow velocity of the fluid are small.

The quantity of heat liberated through friction in most processes considered here is insignificant and may be ignored.

The process of heat transfer is affected by the *compressibility* of fluid. The isothermal compressibility of a body at $t = \text{const}$ is expressed by

$$\varepsilon = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_t \qquad (4\text{-}6)$$

which represents the relative change in density with a change of unity in pressure.

The isothermal compressibility of liquids is very small. For water, for example, $\varepsilon \cong 5 \cdot 10^{-10}$ Pa^{-1}, i.e. a rise in pressure of 1 bar brings about a relative decrease in density of 1/20 000. The same is true for all other liquids. Thus their compressibility is so small that it can be ignored.

For air under standard conditions $\varepsilon = 10^{-5}$ Pa^{-1}. Thus, its compressibility is 20 000 times that of water. A similar relationship is true for other gases.

The main point, however, is not the compressibility of a gas, but the extent to which it is actually compressed in any particular flow. Any significant compression of a gas requires a considerable change in pressure. If the differences in pressure appearing during gas flow are small compared with the absolute pressure of the gas, the observed variation in volume is small; these gas flows can be considered incompressible, as a first approximation.

Considerable variations in pressure appear at high flow velocities. The heat of friction and compressibility must then be taken into account; as a result heat transfer at high velocities has a number of special features which will lead to considerable errors if ignored.

In the main we shall be concerned here with heat transfer in incompressible fluids, the word "fluid" being used for both liquids and gases. The process of heat transfer in compressible gases will be considered separately.

There is no sharp line between flows of compressible and incompressible gases. It is usually assumed that the laws of flow and heat transfer derived for an incompressible fluid are applicable to a gas flow with a velocity less than one-quarter of the speed of sound. Apart from isothermal compressibility for convection, the *thermal expansion of fluids* is of great importance. It is characterized by the volume coefficient of expansion determined under constant pressure by the equation (p = const)

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial t}\right)_p \tag{4-7}$$

By definition the volume coefficient of expansion β, [1/K], is the relative change in volume caused by a change in temperature by 1 K provided p = const.

The volume coefficient of expansion of liquids is comparatively small (except in the region near the critical point). The coefficient β may be negative for some liquids, for example, for water at temperatures below 4 °C.

The volume coefficient of expansion of an ideal gas is the reciprocal of its absolute temperature:

$$\beta = \frac{1}{T}$$

A non-uniform density field appears in a non-uniformly heated fluid due to thermal expansion, and in the long run this may lead to the free motion of its particles.

4-3. Differential Equations of Heat Transfer by Convection (Statement of Boundary-value Problems in Heat Transfer by Convection)

From Eq. (4-2)

$$\vec{q} = -\lambda \nabla t + \rho \vec{w} i$$

it follows that the rate of heat flow is determined unambiguously at any point in the fluid for any moment of time, if the temperature field, unit enthalpy and velocity fields are known.

Temperature and enthalpy can be related in the following way. For a real liquid $i = i\,(T,\ p)$ and according to the notion of the total differential

$$di = \left(\frac{\partial i}{\partial T}\right)_p dT + \left(\frac{\partial i}{\partial p}\right)_T dp$$

Whence

$$i = \int_T c_p\, dT + \int_p \left(\frac{\partial i}{\partial p}\right)_T dp$$

It follows from the differential equations of thermodynamics and from the definition of the coefficient of volume expansion that

$$\left(\frac{\partial i}{\partial p}\right)_T = \frac{1}{\rho}\left[1 + \frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p\right] = \frac{1}{\rho}(1 - \beta T)_p$$

In solving many problems based on the assumption of the non-compressibility of liquid (ρ = const), it may be assumed with sufficient accuracy that $(\partial i/\partial p)_T = 0$, i.e. we may make use of the relationship which is valid for a thermodyna-

mically real, or perfect, gas:

$$di = c_p\,dT \quad \text{and} \quad i = \int_T c_p\,dT$$

The above equations permit us to relate the temperature field and the enthalpy field. To find analytically the temperature (enthalpy) and velocity fields and to determine $\vec{q}$, we must have available the following equations.

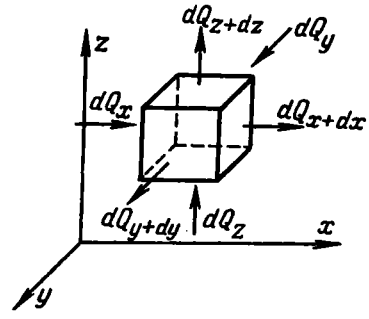

Fig. 4-3. Deriving the differential equation of energy

Equation of energy. Let us derive the differential equation describing the temperature field in a moving fluid, assuming the fluid to be homogeneous and isotropic, also assuming that its physical properties are constant and that the potential strain energy is small compared with the variation of internal energy.

Let us consider in the flow of fluid an elementary parallelepiped which is stationary in respect to the coordinate system (Fig. 4-3) with sides dx, dy and dz. Heat is transferred through the sides of the parallelepiped by conduction and convection; in the general case heat may be evolved in the volume considered from inner heat sources at the expense of the energy which is external in respect to the fluid considered.

In accordance with the conditions assumed here, the equation of energy was already derived in Sec. 1-6. We obtained Eq. (1-25):

$$\rho \frac{\partial i}{\partial \tau} = -div\,\vec{q} + q_v$$

where

$$div\,\vec{q} \equiv \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

According to Eq. (4-2) the projections of the rate of heat flow $\vec{q}$ on the coordinate axes Ox, Oy and Oz are:

$$q_x = -\lambda \frac{\partial t}{\partial x} + \rho w_x i, \quad q_y = -\lambda \frac{\partial t}{\partial y} + \rho w_y i \quad \text{and} \quad q_z = -\lambda \frac{\partial t}{\partial z} + \rho w_z i \qquad (4\text{-}8)$$

Substituting the values of q_x, q_z and q_z in Eq. (1-25), we get:

$$\rho \frac{\partial i}{\partial \tau} = \lambda \left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \right) - \rho \left(w_x \frac{\partial i}{\partial x} + w_y \frac{\partial i}{\partial y} + w_z \frac{\partial i}{\partial z} \right) -$$
$$- \rho i \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} \right) + q_v$$

For incompressible fluids ρ = const [Eq. (4-20)]

$$\operatorname{div} \vec{w} \equiv \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} = 0$$

Then

$$\frac{\partial i}{\partial \tau} + w_x \frac{\partial i}{\partial x} + w_y \frac{\partial i}{\partial y} + w_z \frac{\partial i}{\partial z} = \frac{\lambda}{\rho} \left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \right) + \frac{q_v}{\rho} \qquad (4\text{-}9)$$

or, if $i = \int_T c_p \, dT$,

$$\frac{\partial t}{\partial \tau} + w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} + w_z \frac{\partial t}{\partial z} = a \left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} \right) + \frac{q_v}{\rho c_p} \qquad (4\text{-}10)$$

Equation (4-10), just as Eq. (4-9), is the sought for equation of energy describing temperature distribution in a moving fluid.

The polynomial in the left-hand side of Eq. (4-10) is the total derivative of temperature with respect to time. Indeed, if $t = t(\tau, x, y, z)$, then based on the concept of the total, or perfect, derivative we have:

$$\frac{dt}{\partial \tau} = \frac{\partial t}{d\tau} + \frac{\partial t}{\partial x} \frac{dx}{d\tau} + \frac{\partial t}{\partial y} \frac{dy}{d\tau} + \frac{\partial t}{\partial z} \frac{dz}{d\tau}$$

where

$$\frac{dx}{d\tau}, \quad \frac{dy}{d\tau} \text{ and } \frac{dz}{d\tau}$$

have the meaning of velocity components, w_x, w_y and w_z.

The ratio $\partial t/\partial \tau$ characterizes here the variation of temperature with time at any point in the fluid, i.e. it represents the local variation of temperature t; the term

$$w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} + w_z \frac{\partial t}{\partial z}$$

defines the variation of temperature from point to point, i.e. it represents the variation of t due to convection.

Using the notation

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = \nabla^2 t$$

the equation of energy may be presented in the following form:

$$\frac{dt}{d\tau} = a \nabla^2 t + \frac{q_v}{\rho c_p} \qquad (4\text{-}10')$$

If $w_x = w_y = w_z = 0$, the equation of energy becomes the equation of conduction.

In processes of steady-state convection $\partial t/\partial \tau = 0$. Eq. (4-10) is still more simplified, if temperature varies only along one or two coordinate axes. For the case of a steady-state one-dimensional temperature field all derivatives with respect to τ, x and y equal zero.

As follows from Eq. (4-10), the temperature field in a moving fluid depends on the velocity components w_x, w_y and w_z. A closed set of equations can be obtained by adding equations which would describe the variation of velocity in time and space. The differential equations of flow are equations of this kind.

Equations of flow. Derivation of the differential equation of flow of a viscous fluid involves cumbersome calculations. A simplified derivation, however, may be obtained for the one-dimensional flow of an incompressible viscous fluid [4.2]. This conclusion is not very strict, its main advantage consists in that the conclusion can be visualized. The equation for three-dimensional flow will be given here without derivation. Flow equations will be considered in detail in courses on hydrodynamics and in the specialized literature on heat transfer [4.3].

Consider an elementary volume with sides dx, dy and dz in the flow of viscous fluid (Fig. 4-4). The velocity of flow varies only in the direction of the y-axis, and in a random manner.

The equation of flow is derived from Newton's second law: force equals mass times acceleration.

The forces acting on the considered element of fluid may be classified as mass (or volumetric) and surface forces. The mass forces are characterized by the vector $\vec{F}$, m²/s, which is numerically equal to the ratio of the force acting upon a given particle to its mass. If only gravity is taken into consideration, then $\vec{F} = \vec{g}$, where $\vec{g}$ is the acceleration due to gravity. Only the force of gravity will be considered here. The magnitude of the surface forces is equal to the ratio of the force acting on an element of the surface to the area of the element. Pressure and friction forces are surface forces.

In short, three forces act on the considered element of fluid: gravity, and the resultants of pressure and friction. Let us find their projections on the Ox-axis.

The force of gravity df_1 is applied at the centre of gravity of the element. Its projection on the Ox-axis is the product of the projection of the acceleration force g_x and the mass of the element:

$$df_1 = \rho g_x \, dV$$

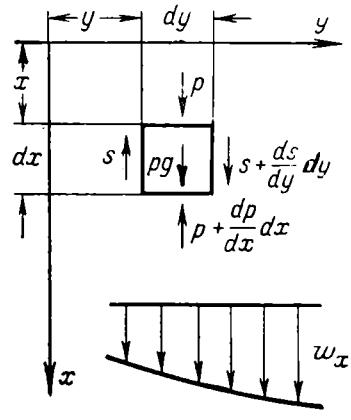

Fig. 4-4. Deriving the differential equation of fluid flow

The resultant of the applied pressures df_2 is determined as follows. If the specific pressure of the fluid on the upper face of the element equals p, then a force $p \, dy \, dz$ acts on the area $dy \, dz$. The lower face of the element is subjected to a specific pressure equal to $p + \frac{dp}{dx} dx$, with an accuracy to the second term of the Taylor series; and the force acting upon this face is equal to $-\left(p + \frac{dp}{dx} dx\right) \times$ $\times \, dy \, dz$. The minus sign here signifies that this force acts in the direction opposite to the fluid flow. The resultant of the pressure forces is equal to their algebraic sum:

$$df_2 = -\frac{dp}{dx} dV$$

The resultant of friction forces df_3 is determined from the following considerations. Since velocity varies only in the direction of the Oy-axis, the force of friction appears only on the side faces of the elementary volume of fluid (Fig. 4-4). The velocity of the particles near its left face is less than that of the element itself; therefore, in the cross-section y, the friction force is directed against the flow and is equal to $s \, dx \, dz$. The reverse is true of the right face; here the particles of the fluid move at a velocity exceeding that of the element, and therefore in the cross-section $y + dy$, the force of friction acts in the direction of flow. The resultant of these forces is equal to their algebraic sum:

$$df_3 = \left(s + \frac{ds}{dy} dy\right) dx \, dz - s \, dx \, dz = \frac{ds}{dy} dV$$

Substituting $s = \mu\,(dw_x/dy)$, we find that

$$df_3 = \mu \frac{d^2w_x}{dy^2}\,dV$$

Summing the expressions for df_1, df_2 and df_3, we obtain the projection on the Ox-axis of the resultant of all the forces acting on the considered volume:

$$df = \left(\rho g_x - \frac{dp}{dx} + \mu \frac{d^2w_x}{dy^2}\right) dV \qquad \text{(a)}$$

By the second law of mechanics, this resultant force is equal to the product of the mass of the considered element and its acceleration $\frac{dw_x}{d\tau}$ and accounts for the forces of inertia:

$$df = \rho \frac{dw_x}{d\tau}\,dV \qquad \text{(b)}$$

Equating the right sides of Eqs. (a) and (b) and cancelling, we finally obtain the following equation along the Ox-axis:

$$\rho \frac{dw_x}{d\tau} = \rho g_x - \frac{dp}{dx} + \mu \frac{d^2w_x}{dy^2}$$

The mathematical description of the fluid flow becomes more complicated when the velocity varies in three directions. For the general case of three-dimensional flow of an incompressible fluid the velocity field is described mathematically by three equations of flow, each corresponding respectively to the projections upon the Ox-, Oy- and Oz-axes.

for the Ox-axis

$$\rho \frac{dw_x}{d\tau} = \rho g_x - \frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 w_x}{\partial x^2} + \frac{\partial^2 w_x}{\partial y^2} + \frac{\partial^2 w_x}{\partial z^2}\right) \qquad \text{(4-11)}$$

for the Oy-axis

$$\rho \frac{dw_y}{d\tau} = \rho g_y - \frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 w_y}{\partial x^2} + \frac{\partial^2 w_y}{\partial y^2} + \frac{\partial^2 w_y}{\partial z^2}\right) \qquad \text{(4-12)}$$

for the Oz-axis

$$\rho \frac{dw_z}{d\tau} = \rho g_z - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w_z}{\partial x^2} + \frac{\partial^2 w_z}{\partial y^2} + \frac{\partial^2 w_z}{\partial z^2}\right) \qquad \text{(4-13)}$$

These equations are also known as the Navier-Stokes differential equations.

The units of all their component terms are expressed by force per unit volume.

In the general case the velocity components w_x, w_y and w_z vary in time and space. The term present in the left side of Eqs. (4-11), (4-12), and (4-13) is the total derivative of velocity with respect to time.

The concept of total derivative gives us

$$\frac{dw_x}{d\tau} = \frac{\partial w_x}{\partial \tau} + w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} + w_z \frac{\partial w_x}{\partial z} \qquad \text{(4-14)}$$

Similarly, we obtain for the two other axes:

$$\frac{dw_y}{d\tau} = \frac{\partial w_y}{\partial \tau} + w_x \frac{\partial w_y}{\partial x} + w_y \frac{\partial w_y}{\partial y} + w_z \frac{\partial w_y}{\partial z} \qquad \text{(4-15)}$$

$$\frac{dw_z}{d\tau} = \frac{\partial w_z}{\partial \tau} + w_x \frac{\partial w_z}{\partial x} + w_y \frac{\partial w_z}{\partial y} + w_z \frac{\partial w_z}{\partial z} \qquad \text{(4-16)}$$

The derivatives $\frac{\partial w_x}{\partial \tau}$, $\frac{\partial w_y}{\partial \tau}$ and $\frac{\partial w_z}{\partial \tau}$ characterize the variation of velocity in time at any point in the fluid, i.e. the local variation of velocity; the other three terms of the right side of the equations define the variation of velocity in passing from point to point.

Presented in vector form, Eqs. (4-11) to (4-13) will be:

$$\rho \frac{d\vec{w}}{d\tau} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{w} \tag{4-17}$$

Eq. (4-17) was derived disregarding the dependence of the physical parameters on temperature, particularly, the dependence of fluid density on temperature was not taken into account, although free motion of a fluid, or natural convection, is due to the difference in the density of its cold and heated particles.

Let us confine ourselves to an approximate allowance for the variation of density*. For this we use the volume coefficient of expansion β assuming it to be constant and independent of temperature in the given temperature range. These conditions are quite readily met in gases but not so well in liquids.

From the definition of the volume coefficient of expansion, given in Sec. 4-2, it follows that with β = const

$$\beta = \frac{\rho_0 - \rho}{\rho_0 \vartheta}$$

where ρ and ρ_0 are the densities corresponding to temperatures t and t_0 and $\vartheta = t - t_0$. t_0 is some fixed temperature (fixed, or reference, point).

From this relationship it follows that

$$\rho = \rho_0 (1 - \beta\vartheta)$$

Substituting the value for density from the last equation in flow equation (4-17), accounting for the mass forces, we obtain:

$$\rho \frac{d\vec{w}}{d\tau} = \rho_0 (1 - \beta\vartheta)\vec{g} - \nabla p + \mu\nabla^2\vec{w}$$

Let us examine the term $\rho_0 (1 - \beta\vartheta)\vec{g} = \rho_0\vec{g} - \rho_0\vec{g}\beta\vartheta$, which may be treated as the sum of the force of gravity $\rho_0\vec{g}$, taken at a definite density, and the buoyant force $\rho_0\vec{g}\beta\vartheta$. The term $\rho_0 g$ can be presented as the hydrostatic pressure gradient p_0 in a still fluid with a density ρ_0. Then, the term $-(\nabla p - \rho_0\vec{g})$ can be replaced with $-\nabla p_1$, where $p_1 = p - p_0$. When replacing p with p_1 the equation of flow will also account for the term $\rho_0\vec{g}$.

Omitting the subscript "0" at ρ and the subscript "1" at p, and dividing the left- and right-hand sides by ρ, we obtain the following equation of flow:

$$\frac{d\vec{w}}{d\tau} = -\vec{g}\beta\vartheta - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{w} \tag{4-18}$$

The system of equations obtained is not closed, because they still include the unknown p in addition to w_x, w_y, w_z and ϑ. One further equation must be added to the system—the differential equation of continuity.

Equation of continuity. Consider a stationary elementary parallelepiped with sides dx, dy and dz in a fluid flow, and calculate the mass of fluid passing through

* In the general case at $\rho \neq$ const potential strain energy must also be taken into account.

this parallelepiped per time $d\tau$ in the direction of the Ox-, Oy- and Oz-axes (Fig. 4-5).

The mass of fluid flowing into the parallelepiped along the Ox-axis is

$$dM_x = \rho w_x \, dy \, dz \, d\tau \qquad \text{(a)}$$

The product ρw_x is the quantity of fluid flowing per unit time per unit cross-sectional area. The mass of fluid leaving the element through the opposite face is equal to

$$dM_{x+dx} = \rho w_{x+dx} \, dy \, dz \, d\tau \qquad \text{(a)}$$

Confining ourselves to the first two terms of the expansion in series, we see that the mass of fluid dM_{x+dx}, leaving the parallelepiped along the Ox-axis, is equal to:

$$dM_{x+dx} = \left[\rho w_x + \frac{\partial (\rho w_x)}{\partial x} dx \right] dy \, dz \, d\tau \qquad \text{(b)}$$

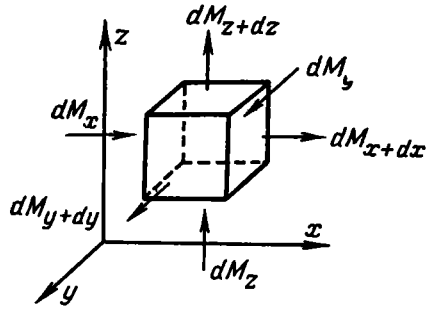

Fig. 4-5. Deriving the differential equation of continuity

Subtracting (a) from (b), we obtain the excess mass of fluid leaving the parallelepiped in the direction of the Ox-axis:

$$dM_{x+dx} - dM_x = \frac{\partial (\rho w_x)}{\partial x} dV \, d\tau \qquad \text{(c)}$$

Similarly, we find the excess amounts of fluid leaving the parallelepiped in the direction of the Oy- and Oz-axes:

$$dM_{y+dy} - dM_y = \frac{\partial (\rho w_y)}{\partial y} dV \, d\tau \qquad \text{(d)}$$

$$dM_{z+dz} - dM_z = \frac{\partial (\rho w_z)}{\partial z} dV \, d\tau \qquad \text{(e)}$$

The total quantity of excess fluid is equal to the sum of expressions (c), (d) and (e). It is due to the change in density of the fluid in volume dV, and is equal to the change in mass of that volume in time: $\frac{\partial \rho}{\partial \tau} dV \, d\tau$. Eliminating dV and $d\tau$ and transposing all terms to the left side of the equality we finally obtain the different equation of *continuity* for compressible fluids:

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial (\rho w_x)}{\partial x} + \frac{\partial (\rho w_y)}{\partial y} + \frac{\partial (\rho w_z)}{\partial z} = 0 \qquad (4\text{-}19)$$

For incompressible fluids we assume ρ = const, and obtain:

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} = 0 \qquad (4\text{-}20)$$

or what is the same thing

$$\operatorname{div} \vec{w} = 0 \qquad (4\text{-}20')$$

The equation of continuity is the equation of mass conservation.

Thus the process of convection in an incompressible single-phase medium with constant physical parameters (values) is described by the system of differential equations: (4-2),(4-10), (4-18) and (4-20).

Some of the peculiarities of the mathematical expression of the differential equations for turbulent flows, using mean values of the variables, will be treated in Sec. 4-5.

Conditions of unambiguity (single-valuedness). The differential equations of convection derived here describe an infinite number of actual processes. In order to single out any one of them and to determine it uniquely the equations must be supplemented by conditions of unambiguity or single-valuedness. These conditions give a mathematical description of all special peculiarities of the phenomenon considered and consist of the following:

(1) *geometrical conditions* characterizing the shape and size of the body or system in which the process occurs;

(2) *physical conditions* defining the physical properties of the environment;

(3) *time or initial conditions* characterizing the peculiarities of the process at the initial moment of time (not needed for steady-state problems);

(4) *boundary conditions* characterizing the peculiarities of the process taking place at the boundaries of the fluid medium.

The last should include given boundary values of the dependable (unknown) variables or of their derivatives. For instance, given for any moment of time is the temperature distribution or the distribution of heat flows over the surface of the body [in the simplest case $t_w = \text{const}$ or $q_w = -\lambda\,(\partial t/\partial n)_{n=0} = \text{const}$], the distribution of temperature and fluid velocities at the conduit inlet or at a distance from heat-transfer surface considered, the values of velocity on the wall, etc. Obviously the results of integration, presented as formulae of numerical values, may be different, depending on the way in which the boundary and other conditions are given.

Combined with the conditions of unambiguity (single-valuedness), the set of differential equations is the mathematical statement of boundary-value problems.

Difficulties may often be encountered while assuming the distribution $t_w\,(\tau, x_w, y_w, z_w)$ and $q_w\,(\tau, x_w, y_w, z_w)$ where x_w, y_w, z_w are the coordinates of the surface, since both t_w and q_w depend in the general case on heat transfer in the wall and on its opposite side. Strictly speaking, the thermal boundary conditions cannot then be appointed before since they are a complex function of a whole set of individual heat-transfer processes. It is necessary to combine the system of differential equations of the considered process of convection heat transfer with the differential equations describing conduction in the wall and convection heat transfer on its other side, and assume *conjugation conditions.*

In the presence of continuous temperature fields the conditions of conjugation can be given as temperature equality on the media contact surface, and in the absence of heat liberation at the interface at the expense of inner sources, as the equality of heat flows, described by Fourier's law.

With a conjugated problem the differential equations, describing heat transfer in adjoining systems, and the conditions of conjugation can be treated as the boundary conditions for the problem considered. The boundary conditions will then be very complicated. Problems in heat transfer by convection are usually solved with the aid of given boundary conditions.

A physical analysis of heat transfer by convection shows that the mathematical statement of a problem can be simplified in a number of cases, not introducing essential errors. The mathematical statement can be simplified, for instance, when using the concept of the boundary layer, considered in the following paragraph. As a result mathematically exact solutions can be obtained.

The intricacy of heat transfer by convection often necessitates the application of experimental methods of investigation in studying processes of convection. Experimental work yields synthesized data on processes and sometimes the influence of individual factors cannot be identified. These shortcomings can be overcome with the aid of the theory of similarity considered in Chapter 5.

The theory of similarity is based on the mathematical statement of a boundary-value problem.

Heat transfer by convection is often investigated making use of the fact that the process is analogous to processes of another physical nature. The analogy is established on the basis of the mathematical description of these processes.

The introduction of electronic computers has led to a wider application of numerical calculation methods. Many problems for which no exact solution is possible can be calculated with the aid of electronic digital computers. The calculation then is based on the mathematical statement of the problem presented as a set of differential equations and given boundary conditions.

4-4. Hydrodynamic and Thermal Boundary Layers

Of great practical interest is the exchange of heat between a fluid and a solid exposed to it. Let us consider the peculiarities of flow and the transport of heat in the layer of fluid adjoining a wall.

Conditions of "adhesion". At the present time a hypothesis gained recognition in the hydrodynamics of viscous liquids, consisting in that the particles of the liquid which are in direct contact with the solid are adsorbed by the latter, the particles as though adhere to the surface of the solid, i.e. their velocity is equal to that of the solid (i.e. equals zero, if the solid is stationary).

This layer of "adhered" fluid should be considered infinitely thin. The hypothesis that fluid velocity at the wall is zero has found indirect confirmation, being in good agreement with the result of much theoretical work based on it.

The velocity of a fluid at the wall remains zero so long as it can be considered a continuous medium. The interaction between a gas and a wall weakens as the density of the gas diminishes (or as its rarefaction increases) and a rarefied gas begins to slip near the wall.

The degree of rarefaction of a flow is characterized by the magnitude of the Knudsen number $\overline{l}/l_0$, which is the ratio of the mean free path of gas molecules $\overline{l}$ to the reference dimension of the solid l_0 (for instance, the diameter of a tube or wire).

If $\overline{l}/l_0 > 0.001$ the gas can no longer be considered as a continuous medium for which the conditions of adhesion are valid.

With Knudsen numbers larger than 10 a gas must be considered to be a free molecular flow. Its interaction with a solid is described by the laws of the kinetic theory of gases.

With Knudsen numbers between 0.001 and 10 a rarefied gas can be regarded neither as a completely continuous medium nor as a fully free molecular one. Special methods of calculation of flow and of heat transfer are being elaborated for this range of the Knudsen numbers.

We shall be concerned below mainly with continuous media and will depart from the condition that the velocity of an infinitely thin layer of fluid in direct contact with the surface of a solid is equal to zero.

Heat transfer equation. Since there is a thin layer of stationary fluid at the surface of the solid, it follows from Eq. (4-2) that the rate of heat flow on the wall can be determined with the aid of Fourier equation

$$q_w = -\lambda (\partial t/\partial n)_{n=0} \tag{4-21}$$

where n is the normal to the surface of the solid.

Thus, if the temperature field is known, q_w can be calculated without applying to Newton's law:

$$q_w = \alpha (t_w - t_f) \qquad (4\text{-}21')$$

If necessary, the known temperature field can be used to determine the local heat-transfer coefficient. From Eqs. (4-21) and (4-21′) it follows that

$$\alpha = -\frac{\lambda}{t_w - t_f}\left(\frac{\partial t}{\partial n}\right)_{n=0} \qquad (4\text{-}22)$$

Equation (4-22) shall be referred to as the *equation of heat transfer* from a solid to a fluid or vice versa.

From the condition that the relative velocity of a fluid on the surface of a wall is equal to zero other important conclusions follow, which are of great practical interest and facilitate identification of the temperature field and, hence, the determination of the rate of heat flow q_w and of the heat-transfer coefficient α.

Hydrodynamic boundary layer. Let us consider the flat surface of a solid exposed to an infinite flow of fluid. The velocity and temperature of the free-stream flow are constant and equal to w_0 and t_0 respectively. As the particles of the fluid come into contact with the surface of the solid they "adhere" to it. As a result, due to the action of viscosity forces, a thin layer of stagnated fluid forms near the wall and within this layer of fluid flow velocity changes from zero at the surface of the body to the velocity of undisturbed flow (at a distance from the solid). This layer of stagnated fluid has been given the name of the hydrodynamic boundary layer. The theory of the hydrodynamic boundary layer was first suggested by L. Prandtl in 1904.

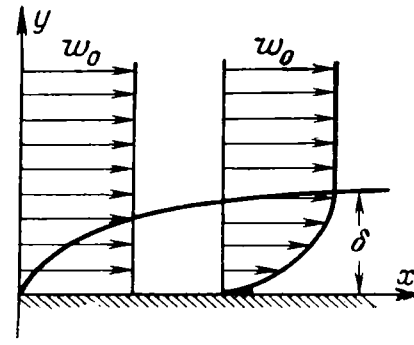

Fig. 4-6. Velocity variation in a hydrodynamic boundary layer

The greater the distance x from the leading edge of the plate, the thicker the boundary layer, since the effect of viscosity penetrates deeper and deeper into the undisturbed flow of the fluid along the plate. This peculiarity of the boundary layer is illustrated in Fig. 4-6 which shows the distribution of velocity at different values of x.

The condition $\partial w_x/\partial y \neq 0$ is true for the flow of fluid with the boundary layer, and the following expression holds for the flow of fluid beyond the boundary layer and at its outer boundary

$$\partial w_x/\partial y = 0 \quad \text{and} \quad w_x = w_0$$

The concepts of "boundary layer thickness" and "outer border of the boundary layer" are quite fictitious, because there is no abrupt transition from the boundary layer to the flow beyond it. The velocity within the boundary layer tends to w_0 asymptotically with increasing y.

By thickness of the boundary layer, δ, therefore, is meant the distance from the wall at which the velocity differs from that of the flow along the wall by a certain small prescribed value $\varepsilon \ll 1$ (by 1 per cent, for instance): at $y = \delta$ $w_x = (1 - \varepsilon)\, w_0$.

Thus, as fluid flows past a solid the flow as though separates into two parts: *the boundary layer* and *the external flow* in which inertia forces predominate and

the viscous forces do not manifest themselves. In contrast, in the boundary layer the viscous forces and the inertial forces are commensurable.

We can then write the following system of differential equations describing the stationary velocity field for a plate, which is infinitely long in the direction of the Oz-axis, placed in longitudinal fluid flow. The equations of flow:

$$w_x\frac{\partial w_x}{\partial x}+w_y\frac{\partial w_x}{\partial y}=\nu\left(\frac{\partial^2 w_x}{\partial x^2}+\frac{\partial^2 w_x}{\partial y^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{4-23}$$

$$w_x\frac{\partial w_y}{\partial x}+w_y\frac{\partial w_y}{\partial y}=\nu\left(\frac{\partial^2 w_y}{\partial x^2}+\frac{\partial^2 w_y}{\partial y^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4-24}$$

The equation of continuity

$$\frac{\partial w_x}{\partial x}+\frac{\partial w_y}{\partial y}=0 \tag{4-25}$$

Let us see whether it is possible to simplify the above presented system of differential equations for the boundary layer and let us designate the boundaries within which the simplified description of the stationary velocity field is valid.

Due to the small thickness of the boundary layer we assume that the pressure does not change across the layer, i.e. that $\partial p/\partial y = 0$. With the surface of a flat plate exposed to an infinite flow, when velocity is constant in the external flow and equal to w_0, it follows from Bernoulli's equation

$$p+\frac{\rho w_0}{1}=\text{const}$$

that pressure does not change in the external flow either. Then $\partial p/\partial x = 0$ (in hydrodynamics such flow is often referred to as "gradientless flow"). The conditions $\partial p/\partial y = 0$ for the boundary layer and $\partial p/\partial x = 0$ for the external flow lead to the conclusion that the derivative $\partial p/\partial x$ is zero in the region of the boundary layer (in the case considered).

Velocity w_x changes from zero to w_0 and let us estimate the order of magnitude of w_x as w_0. Let us take the scale l for the longitudinal coordinate. Then (O designates the order of a given quantity)

$$\frac{\partial w_x}{\partial x}=O\left(\frac{w_0}{l}\right)$$

According to the equation of continuity (4-25) the order of the derivatives $\partial w_x/\partial x$ and $\partial w_y/\partial y$ is the same. Then

$$\frac{\partial w_y}{\partial y}=O\left(\frac{w_0}{l}\right)=O\left(w_0\frac{\delta}{l}\frac{1}{\delta}\right)$$

where δ is the order of the cross coordinate y for the boundary layer. The order of magnitude of w_y then can be estimated as

$$w_y=O\left(w_0\frac{\delta}{l}\right)$$

Let us evaluate the individual terms of the inertia (convection) and viscosity parts of the equation of flow projected on the Ox-axis:

$$w_x\frac{\partial w_x}{\partial x}=O\left(\frac{w_0^2}{l}\right);\quad w_y\frac{\partial w_x}{\partial y}=O\left(w_0\frac{\delta}{l}\frac{w_0}{\delta}\right)=O\left(\frac{w_0^2}{l}\right)$$

$$\nu\frac{\partial^2 w_x}{\partial x^2}=\nu\frac{\partial}{\partial x}\frac{\partial w_x}{\partial x}=O\left(\nu\frac{w_0}{l^2}\right)$$

$$\nu\frac{\partial^2 w_x}{\partial y^2}=\nu\frac{\partial}{\partial y}\frac{\partial w_x}{\partial y}=O\left(\nu\frac{w_0}{\delta^2}\right)$$

It follows from the above evaluation that the order of the individual items of the inertia part is the same and equal to w_0^2/l. The ratio of the viscosity items gives

$$\frac{\partial^2 w_x/\partial x^2}{\partial^2 w_x/\partial y^2} = O\left(\frac{w_0/l^2}{w_0/\delta^2}\right) = O\left(\frac{\delta^2}{l^2}\right)$$

For the boundary layer $\delta \ll 1$, whence $\frac{\partial^2 w_x}{\partial y^2} \gg \frac{\partial^2 w_x}{\partial x^2}$, the last derivative can be ignored. Then presented in projections on the Ox-axis the equation of flow can be written in the following form:

$$w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} = \nu \frac{\partial^2 w_x}{\partial y} \tag{4-26}$$

The order of magnitude of the left-hand side of this equation is equal to $O\left(\frac{w_0^2}{l}\right)$, and of the right-hand side, $O\left(\nu \frac{w_0}{\delta^2}\right)$. Equating the two orders of magnitude, we obtain:

$$O\left(\frac{w_0}{l}\right) = O\left(\nu \frac{w_0}{\delta^2}\right) \text{ or } \frac{\delta}{l} = O\left(\frac{1}{\sqrt{w_0 l/\nu}}\right) = O\left(\frac{1}{\sqrt{\mathrm{Re}}}\right) \tag{4-27}$$

where $\mathrm{Re} \equiv w_0 l/\nu$ is the *Reynolds* number characterizing the ratio of the inertia and viscous forces.

When $\mathrm{Re} \ll 1$, then $\frac{\delta}{l} \gg 1$ ($\delta \gg l$). In principle, then the flow does not separate into two regions, the entire fluid adjoining the wall is acted upon by viscous forces.

If $\mathrm{Re} \gg 1$, then $\delta \ll l$, i.e. a comparatively thin layer of stagnated fluid forms at the surface of the solid and for this layer the simplifications made above are true as a first approximation. Thus, the theory of the boundary layer can be used as a method of simplifying the mathematical statement of the boundary-value problem and of the associated possibility of solving the problem.

Let us estimate the order of magnitude of the projections on the Oy-axis present in the equation of flow. Accounting for Eq. (4-27), we find that the order of magnitude of the terms

$$w_x \frac{\partial w_y}{\partial x}, \quad w_y \frac{\partial w_y}{\partial y} \quad \text{and} \quad \nu \frac{\partial^2 w_y}{\partial y^2}$$

is $O\left(\frac{w_0^2}{l}\frac{\delta}{l}\right) = O\left(\frac{w_0^2}{l}\frac{1}{\sqrt{\mathrm{Re}}}\right)$, and the order of magnitude of the term $\nu \frac{\partial^2 w_y}{\partial x^2} = O\left(\frac{w_0^2}{l}\frac{1}{\mathrm{Re}\sqrt{\mathrm{Re}}}\right)$.

Thus, the projections on the Oy-axis present in the equation of flow are small compared with the items of Eq. (4-23). For the boundary layer Eq. (4-24) can be omitted. Then, for the case of flat gradientless steady-state flow of a viscous fluid in the boundary layer near a flat surface we can write:

$$w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} = \nu \frac{\partial^2 w_x}{\partial y^2} \tag{4-28}$$

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} = 0 \tag{4-29}$$

In Eqs. (4-28) and (4-29) the dependent variables are w_x and w_y. The right-hand side of Eq. (4-28) can be presented as $\frac{1}{\rho}\frac{\partial s}{\partial y}$, where s is the tangential stress due to friction in the plane parallel to the plane xz.

Thermal boundary layer. By analogy with the hydrodynamic boundary layer G.N. Kruzhilin introduced the concept of the thermal boundary layer (Fig. 4-7) as being the layer of fluid at a wall, within which the temperature changes from that of the wall to the temperature of the fluid at a distance from the wall. The condition $\partial t/\partial y \neq 0$ is true for the zone within the thermal boundary layer, while the conditions for the outer border of this layer and beyond it* are

$$\partial t/\partial y = 0 \quad \text{and} \quad t = t_0$$

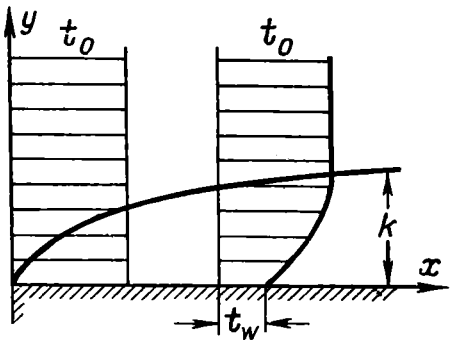

Fig. 4-7. Temperature variation in the thermal boundary layer

Thus, all variation in fluid temperature is concentrated in a comparatively thin layer which is in direct contact with the surface of the wall. Considering below the phenomenon of heat transfer (Chapter 7) for a flat plate in forced longitudinal flow, we shall elucidate the condition at which the inequality $k \ll l$ holds, in which k is the thickness of the thermal boundary layer. The thicknesses of the hydrodynamic and thermal boundary layers δ and k in general do not coincide, this depends on the kind of fluid involved and on certain characteristics and properties of the pattern of flow and heat transfer. Let us assume that they are of the same order of magnitude: $k = 0$ (δ). Due to the small thickness of the thermal boundary layer, heat transfer by conduction along the layer can be ignored compared with the transport of heat in the transverse direction, i.e. we can assume

$$\frac{\partial^2 t}{\partial x^2} = 0 \quad \left(\frac{\partial^2 t}{\partial x^2} \ll \frac{\partial^2 t}{\partial y^2}, \text{ because } k^2 \ll l^2 \right)$$

Then, the equation of energy for the case considered acquires the following appearance:

$$w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} = a \frac{\partial^2 t}{\partial y^2} \tag{4-30}$$

Taking into account that $q_y = -\lambda (\partial t/\partial y)$ and, hence, $\lambda (\partial^2 t/\partial y^2) = -\partial q_y/\partial y$, the right-hand side of Eq. (4-30) can be presented in the form $\frac{1}{\rho c_p} \frac{\partial q_y}{\partial y}$.

To close the problem we must add to Eq. (4-30) the equation of flow (4-28) and the equation of continuity (4-29).

It will be recalled that the system of differential equations (4-28), (4-29) and (4-30) was derived for the case of steady-state gradientless flow of a flat surface by fluid with constant physical properties; there are no inner heat sources in the fluid and the evolution of heat due to friction is negligibly small. It will be noted that with the conditions assumed here, the field of velocities does not depend on the temperature field.

With natural convection caused by the difference in density of the more and less heated particles of the fluid the boundary layer is built up in a special way. The definition of the boundary layers previously given remains true for the case of natural convection but the velocity of flow w_0 is often zero at a distance from

* More exact, at $y = k$ $t = (1 - \varepsilon)\, t_0$, where $\varepsilon \ll 1$, since temperature t must tend to t_0 asymptotically.

the body near which the natural, or free, convection originated. Figure 4-8 shows the distribution of temperature and velocity over a certain section of flow for the case of a free-stream flow near a hot body. Here, too, the thicknesses of the thermal and hydrodynamic boundary layers may not coincide.

For cases of natural convection ($w_0 = 0$) the term $\vec{g}\beta\vartheta$ must be accounted for in the differential equation of flow (4-28). The velocity field then is inexorably associated with the temperature field, i.e. with the process of heat transfer.

The shape and size of the heat-transfer surface have an essential effect on the process of heat transfer. They can sharply change the pattern of flow past the surface, so that the boundary layer is built up in a different manner. A great variety of heating surfaces are used in engineering practice. Each such surface creates specific conditions for motion (convection) and heat transfer.

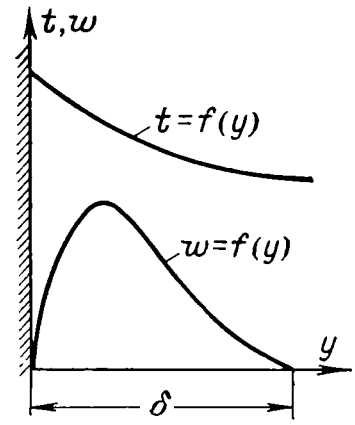

Fig. 4-8. Hydrodynamic and thermal boundary layers in free motion

It is generally known that there are two main basic types or patterns of flow: laminar and turbulent. The particles of a fluid in laminar flow move without intermixing, in layers; with a fluid in turbulent flow individual particles have a random disordered motion and the direction and velocity of motion of individual particles vary incessantly. These patterns of flow are observed in the boundary layer. With small values of x fluid flow in the boundary layer may be of the laminar kind. With increasing x the thickness of the boundary layer increases, the layer becomes unsteady and flow in the boundary layer becomes turbulent.

It will be shown below that heat transfer depends materially on the pattern of flow. The derived system of differential equations (4-28) through (4-30) describes the process of heat transfer occurring only in a laminar boundary layer.

4-5. Turbulent Transport of Heat and Momentum

Turbulent flow differs essentially from laminar. Shown in Fig. 4-9 is the oscillogram of velocity fluctuation at a certain fixed point in a turbulent flow with an invariable mean velocity of flow. The instantaneous velocity fluctuates about a certain time-average velocity. Apart from the variation in absolute velocity w, shown graphically in Fig. 4-9, there is also a change in the direction of the instantaneous velocity. The deviation of the latter from the time-average velocity $\bar{w}$ is referred to as the fluctuations of velocity or *fluctuating velocities* w', with $w = \bar{w} + w'$. Thus, the turbulent motion as though consists of a regular flow, described by mean velocities, and of a superposed disordered pulsating flow.

In the presence of velocity fluctuation there is a transport of mechanical energy. If there is a temperature difference in the flow, velocity fluctuation also leads to the transport of heat, resulting in that fluctuation of temperature originates in the flow (Fig. 4-9). The temperature at a certain fixed point in a turbulent flow fluctuates about some time-average temperature $\bar{t}$. The fluctuation of temperature t' is related to t and $\bar{t}$ by equation $t = \bar{t} + t'$.

Thus, turbulent flow, strictly speaking, is a transient process. But, if the time-average velocities and temperatures, $\overline{w}$ and $\overline{t}$, do not change, this type of flow and the associated transport of heat may be regarded as steady-state (quasistationary) processes. The interval of time selected for averaging must be sufficiently large, compared with the period of fluctuation, but at the same time sufficiently small compared with any interval of time characteristic of the ave-

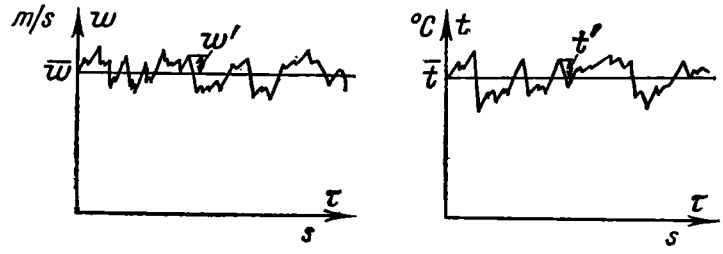

Fig. 4-9. Variation of velocity w and temperature t at a stationary point in a turbulent flow

raged flow, to permit account for the possible variation of the mean velocities and mean temperatures in time. It will be assumed below that the mean values of w and t are integral mean values.

In the general case the fluctuations of velocity and temperature lead to the fluctuation of pressure and physical properties.

The differential equations of heat transfer by convection derived in Sec. 4-3 are believed to be true for individual streams of a pulsating flow. These equations can be written down in mean velocities and temperatures, if we replace t with $\overline{t} + t'$, w_x with $\overline{w}_x + w_x'$, w_y with $\overline{w}_y + w_y'$ etc. Having made certain transformations and suggesting some supplementary hypotheses, we can obtain a system of differential equations describing as a first approximation averaged turbulent flow and heat transfer. Formulated sufficiently strict this problem has not been solved to the end.

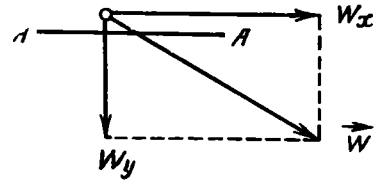

Fig. 4-10. Instantaneous value of velocity in a flat turbulent flow

Let us consider first of all the qualitative side of the phenomena involved in the transport of energy in a turbulent flow. The results of the consideration will be used to formulate a number of relationships required for solving simplest problems.

Let at a certain moment of time $\tau + d\tau$ the components of velocity at a fixed point (small region) in a turbulent flow be w_x and w_y (Fig. 4-10). The temperature of the fluid at that point is t. Let us arrange the fictitious reference surface AA close to the point considered and parallel to surface xz. In a time interval $d\tau$ a mass equal to $\rho w_y\, d\tau$, kg/m^2 passes through unit area of surface AA. At the same time, in particular, momentum $\rho w_y w_x\, d\tau$ is transported in the direction of the Oy-axis in respect to the Ox-axis, and correspondingly enthalpy $\rho w_y i\, d\tau \approx$ $\approx \rho c_p w_y t\, d\tau$ (ρ and c_p are assumed constant).

During the next periods of time velocity components may differ. The integral mean enthalpy q_y, J/m^2·s, transported in the direction of the Oy-axis per unit time and through unit reference area will be:

$$q_y = \frac{1}{\Delta\tau} \int_{\tau}^{\tau+\Delta\tau} \rho c_p w_y t\, d\tau = \rho c_p \overline{w_y t} \qquad (4\text{-}31)$$

The quantity $\rho c_p \overline{w_y t}$ can be presented in the form

$$q_y = \rho c_p \overline{w_y t} = \rho c_p \overline{(\bar{w}_y + w'_y)(\bar{t} + t')} =$$
$$= \rho c_p \overline{(\bar{w}_y \bar{t} + \bar{w}_y t' + w'_y \bar{t} + w'_y t')} = \rho c_p \bar{w}_y \bar{t} + \rho c_p \overline{w'_y t'} \quad (4\text{-}32)$$

In Eq. (4-32) use is made of the properties of integral mean averaging

$$\bar{\varphi} = \frac{1}{\Delta\tau} \int_{\tau}^{\tau+\Delta\tau} \varphi \, d\tau \quad (4\text{-}33)$$

of the variable in time quantities φ and ψ (for instance, of w_x and t):

$$\overline{\varphi + \psi} = \bar{\varphi} + \bar{\psi}, \quad \overline{\bar{\varphi}\psi} = \bar{\varphi}\bar{\psi}, \quad \overline{\bar{\varphi}} = \bar{\varphi} \quad (4\text{-}34)$$

Below, use will also be made of the property

$$\frac{\overline{\partial\varphi}}{\partial y} = \frac{\partial\bar{\varphi}}{\partial y}$$

following from Eq. (4-33), due to it being possible to change the sequence of integration in respect to τ and differentiation in respect to y. It is assumed that the averaging interval $\Delta\tau$ is selected in accordance with the previously stated conditions. Indeed, averaging $\varphi = \varphi + \varphi'$, we obtain:

$$\bar{\varphi} = \overline{\bar{\varphi} + \varphi'} = \bar{\bar{\varphi}} + \overline{\varphi'}$$

It follows from the above that $\overline{\varphi'} = 0$. It will be noted that $\overline{\varphi'^2} \neq 0$, which follows from equation $\overline{\varphi'^2} = \overline{(\varphi - \bar{\varphi})^2}$ (the trivial case $\varphi = \bar{\varphi} = 0$ is excluded).

The integral mean value of momentum in respect to axis Ox, transported in the direction Oy in unit time and through unit area, can be obtained proceeding as in deriving Eq. (4-32). As a result

$$s = \frac{1}{\Delta\tau} \int_{\tau}^{\tau+\Delta\tau} \rho w_x w_y \, d\tau = \rho \overline{w_x w_y} = \rho \bar{w}_x \bar{w}_y + \rho \overline{w'_x w'_y} \quad (4\text{-}35)$$

In the general case similar expressions can be obtained for the transport of momentum with respect to any coordinate axes in the direction of the Ox-axis, Oy-axis and Oz-axis.

Thus, in accordance with Eqs. (4-32) and (4-35), the transport by convection is the sum of two components: the averaged and the fluctuation (turbulent) transport. Let us denote:

$$q_{y,\,t} = q_t = \rho c_p \overline{w'_y t'} \quad (4\text{-}36)$$

$$s_{xy,\,t} = s_t = -\rho \overline{w'_x w'_y} \quad (4\text{-}37)$$

In the general case q_t and s_t are not equal to zero, and what is more in different regions of a turbulent flow to which a solid is exposed q_t and s_t may acquire great values.

Let us consider the flow of fluid near a wall, but at a certain distance from it. For the sake of simplicity let us assume that the averaged velocities and temperatures change only in the direction of the Oy-axis (Fig. 4-11). Also assume that due to the fluctuations of w'_y enthalpy $c_p \bar{t}\,(y_1)$ is transported from layer y_1 into layer y_2, where $\bar{t}\,(y_1)$ is the averaged (mean) temperature with $y = y_1$.

The planes y_1 and y_2 are parallel to plane xz. The difference between the enthalpies $c_p\,[\bar{t}\,(y_1) - \bar{t}\,(y_2)]$ will be assumed equal to the amount of heat transported on the section $y_2 - y_1 = l'$. Fluctuation as though does not dissipate along the section l'. With $y = y_2$ the dissipation of fluctuating flow leads to the transfer of enthalpy to layer y_2. In the case of quasistationary flow considered this transfer of enthalpy gives rise to fluctuation of temperature in the layer y_2 [temperature $\bar{t}\,(y_2)$ is fixed]. And so forth.

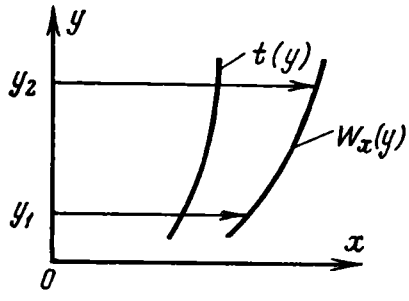

Fig. 4-11. Deriving the formulae for averaged turbulent transfer of heat and momentum

Sometimes an analogy is drawn between l' and the molecular free path (from collision). As a result of this analogy, the quantity l' is called the *mixing path*. Similar to the simplest idea of molecular motion the volume of fluid as if moves through a distance l', and the motion of the mass of fluid is accompanied by the transport of enthalpy, in particular. The analogy between molecular and turbulent motion is sufficiently fictitious. Its merit consists in its obviousness. It will be noted that from the meaning of turbulent flow it follows that the mixing path l' should not be constant. We may speak of the statistical value of l'.

The difference $[\bar{t}\,(y_1) - \bar{t}\,(y_2)]$ can be presented in the following way:

$$\bar{t}\,(y_1) - \bar{t}\,(y)_2 = \bar{t}\,(y_1) - \bar{t}\,(y_1 + l') = \bar{t}\,(y_1) - \bar{t}\,(y_1) - l'\frac{d\bar{t}}{dy} - \frac{l'^2}{2}\frac{d^2\bar{t}}{dy^2} - \ldots \approx -l'\frac{d\bar{t}}{dy}$$

Then, for turbulent (pulsating) transport of heat we can write:

$$q_t = \rho c_p \overline{w'_y\,[\bar{t}\,(y_1) - \bar{t}\,(y_2)]} = -\rho c_p \overline{w'_y l'}\,\frac{d\bar{t}}{dy} \tag{4-38}$$

Starting from assumptions, similar to these made above, turbulent transport in the y direction of momentum in respect to the Ox-axis can be described by equation

$$s_t = -\rho \overline{w'_y\,[\bar{w}_x\,(x_1) - \bar{w}_x\,(y_2)]} = \rho \overline{w'_y l'}\,\frac{d\bar{w}_x}{dy} \tag{4-39}$$

Thus, the quantities q_t and s_t are proportional to the derivatives $d\bar{t}/dy$ and $d\bar{w}_x/dy$. Allowing for this important conclusion, let us write down the following defining equations:

$$q_t = -\rho c_p \varepsilon_q \frac{\partial \bar{t}}{\partial y} = -\lambda_t \frac{\partial \bar{t}}{\partial y} \tag{4-40}$$

and

$$\bar{s}_t = \rho \varepsilon_s \frac{\partial \bar{w}_x}{\partial y} = \mu_t \frac{\partial \bar{w}_x}{\partial y} \tag{4-41}$$

where λ_t and μ_t = respectively the coefficients of turbulent transfer of heat and momentum;

$\varepsilon_q = \lambda_t/\rho c_p$, $\varepsilon_s = \mu_t/\rho$ = respectively kinematic coefficients of turbulent transport of heat and momentum.

The dimensions of these coefficients correspond to these of similar coefficients λ, μ, a, ν, accounting for the molecular transport of heat and momentum.

The coefficients λ_t and μ_t are not physical parameters of a medium. They depend, as it follows from Eqs. (4-40), (4-41) and (4-36), (4-37), on the parameters of a process and, consequently, may vary in the space considered.

Heat and momentum are transported in the direction of the Oy-axis also by a molecular mechanism, permitting us to write the following equations:

$$q_y = -(\lambda + \lambda_t)\frac{\partial \bar{t}}{\partial y} \tag{4-42}$$

and

$$s_{xy} = (\mu + \mu_t)\frac{\partial \bar{w}_x}{\partial y} \tag{4-43}$$

A continuous solid wall is impermeable to transverse pulsations w_y'; hence, at $y = 0$, $w_y' = 0$. It follows from this that directly on the wall $\lambda_t = 0$ and $\mu_t = 0$. At a distance from the wall the turbulent-transport coefficients λ_t and μ_t can exceed several times λ and μ respectively; for this region, it can be assumed, on the contrary, that $\lambda = 0$ and $\mu = 0$ (more exact, $\lambda_t \gg \lambda$, $\mu_t \gg \mu$).

As follows from Eqs. (4-32) and (4-35), when writing equations with mean velocities and temperatures account must also be taken of the turbulent (pulsating) transport of heat and momentum. With the limitations assumed above (Sec. 4-4), when applied to the turbulent boundary layer, the equations of energy (4-30), flow (4-28) and continuity (4-29) can be presented in the following form:

$$\rho c_p\left(\bar{w}_x\frac{\partial \bar{t}}{\partial x} + \bar{w}_y\frac{\partial \bar{t}}{\partial y}\right) = \frac{\partial}{\partial y}\left[(\lambda + \lambda_t)\frac{\partial \bar{t}}{\partial y}\right] \tag{4-44}$$

$$\rho\left(\bar{w}_x\frac{\partial \bar{w}_x}{\partial x} + \bar{w}_y\frac{\partial \bar{w}_x}{\partial y}\right) = \frac{\partial}{\partial y}\left[(\mu + \mu_t)\frac{\partial \bar{w}_x}{\partial y}\right] \tag{4-45}$$

$$\frac{\partial \bar{w}_x}{\partial x} + \frac{\partial \bar{w}_y}{\partial y} = 0 \tag{4-46}$$

An allowance is made here for the fact that turbulent transport in the direction of the Ox-axis is much less than the turbulent transport in the direction of the Oy-axis, since $\delta \ll l$ and $k \ll l$, where l is the length of the plate.

The coefficients μ_t and λ_t are believed to depend on the same factors (variables) on which depend the fields of mean velocity and temperature. The system of differential equations can be closed by adding equations relating μ_t and λ_t with these variables.

Many methods were suggested, permitting as a first approximation to close the system of differential equations for turbulent flow, but we shall consider only the simplest one. From Eq. (4-37)

$$s_t = -\rho\overline{w_x' w_y'}$$

and equation (4-39) for one-dimensional turbulent transport

$$s_t = \rho\overline{w_y' l'}\frac{d\bar{w}_x}{dy}$$

it follows that

$$w_x' = l'\frac{d\bar{w}_x}{dy}$$

Let us assume that the proportionality

$$w_y' \sim l'\frac{d\bar{w}_x}{dy}$$

is satisfied.

Then

$$\overline{w'_x w'_y} \sim \overline{l'^2} \left(\frac{d\bar{w}_x}{dy}\right)^2$$

Including the coefficient of proportionality into the newly introduced quantity l, equation (4-37) yields*

$$s_t = \rho l^2 \left(\frac{d\bar{w}_x}{dy}\right)^2 \tag{4-47}$$

The quantity l is often referred to as the mixing path, although it is only proportional to l'. In the last few years l is preferably called the *turbulence scale.* The quantity l is assumed to characterize the internal geometric structure of a turbulent flow, some average dimension of a turbulently moving masses of fluid. With a fixed value of the derivative $d\bar{w}_x/dy$ the tangential stress of turbulent friction s_t is proportional to l^2.

Comparing Eqs. (4-39) and (4-47), we obtain:

$$\overline{w'_y l'} = l^2 \frac{d\bar{w}_x}{dy} \tag{4-48}$$

Substituting the last value into Eq. (4-38), we have:

$$q_t = -\rho c_p l^2 \left|\frac{d\bar{w}_x}{dy}\right| \frac{d\bar{t}}{dy} \tag{4-49}$$

Formulae (4-47) and (4-49) were suggested by L. Prandtl. In the region of turbulent flow adjoining a wall the turbulence scale (just as turbulent transport of momentum and heat) must diminish in the direction of the wall due to the effect of the latter. According to Prandtl

$$l = \varkappa y \tag{4-50}$$

Measurements and calculations have shown that in the wall-adjoining region of turbulent flow (but in the region where molecular friction can be ignored) the dimensionless quantity $\varkappa$ may be assumed equal to 0.4.

Thus, as a first approximation the problem is closed, the values of ε_s and ε_q (or of λ_t and μ_t) are determined:

$$\varepsilon_s = \varepsilon_q = l^2 \left|\frac{d\bar{w}_x}{dy}\right| = (\varkappa y)^2 \left|\frac{d\bar{w}_x}{dy}\right| \tag{4-51}$$

[compare formulae (4-40), (4-41) and (4-47), (4-49)].

Formula (4-51) shows that there exists an analogy between the transport of momentum and heat. The formal analogy, following from Eq. (4-51), reflects the concept according to which one and the same volumes of fluid, being involved in pulsating motion, transport simultaneously momentum and heat and do not interact along the path l' with the surroundings. Indeed, during the transport of heat, for instance, heat exchange may take place. The pulsating transport of momentum may be associated with the dissipation of mechanical energy due to the viscosity of fluid. All this necessitates the introduction of corrections

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* To determine correctly the sign of s_t, formula (4-47) should be presented in the form

$$s_t = \rho l^2 \left|\frac{d\bar{w}_x}{dy}\right| \frac{d\bar{w}_x}{dy} \tag{4-47'}$$

The sign of s_t is determined by that of derivative $d\bar{w}_x/dy$.

into the previously described theory, in particular, introduce various values of l for the description of the transport of momentum and heat.

In spite of the certain imperfection of the theory described above, it may yield results suitable for practical applications.

Extensive literature has been devoted to the theory of turbulent transport of energy and substance. One's knowledge may be widened from books [4.3-4.7].

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CHAPTER

5

SIMILARITY AND SIMULATION (MODELLING) OF CONVECTIVE HEAT TRANSFER

5-1. General

Heat transfer by convection is described by a system of differential equations and conditions of unambiguity including a great number of variables. Endeavours to solve them analytically run into many difficulties. Experimental studies make it possible to obtain numerical values of the unknown variables for certain values of the arguments, and then to select equations that describe the experimental results obtained. However, when studying a process as complex as heat transfer by convection, it is not always easy to conduct experimental work.

In order to investigate the effect of any one of the properties on the process, all the others must be maintained invariable, and this is not always possible or is hampered by the sheer number of variables involved. In addition, the investigator must be sure that the results obtained by means of the test apparatus or scale models are applicable to similar processes (full-size equipment). The theory of similarity helps to resolve difficulties of this kind. The theory of similarity allows the dimensional physical parameters to be united into dimensionless groups or terms with fewer groups than the number of parameters composing them. The dimensionless groups obtained in this way can be regarded as new variables.

The introduction of dimensionless groups into the equations formally reduces the number of values under the sign of a function, and so simplifies investigation of the physical processes. Furthermore, the new dimensionless variables take into account the effect not only of individual factors, but also of their sets, which makes it easier to determine the physical links in the process.

The theory of similarity also establishes the conditions under which the results of laboratory research are applicable to phenomena similar to that studied.

Thus, the theory of similarity is above all a theoretical basis for experiments, but it is also of great help in theoretical research. Although the type of the unknown function cannot be determined by the methods of the theory of similarity, the theory frequently facilitates the analysis of a process and description of the results obtained.

The theory of similarity has been progressing mainly owing to the efforts of Soviet scientists. In this field, well known are the works of A.A. Gukhman, M.V. Kirpichev, M.A. Mikheev, L.S. Eigenson, P.K. Konakov, B.S. Petukhov [5.1-5.7].

Practical application of the conclusions of the theory of similarity calls for knowledge of how to reduce the mathematical descriptions of a process to the dimensionless form.

Several methods are available for this operation. We shall use that known as the method of scale conversion.

5-2. Presenting the Mathematical Statement of the Boundary-value Problem in Dimensionless Variables

Let the surface of a body be exposed to the flow of an incompressible fluid whose temperature and velocity at a distance from the body are constant and equal respectively to t_0 and w_0. The dimensions of the solid l_0 are also given. Its surface temperature is t_w. For the sake of certainty we assume that $t_w > t_0$. We also assume that the physical properties of the fluid are constant (taking into account only the buoyant force resulting from the dependence of density on temperature). Heat released through friction is neglected. The process is of the steady-state kind.

The origin of the coordinates is taken as shown in Fig. 5-1. For the sake of simplicity we assume that the Oy-axis is normal to the surface of the body, and that the Ox-axis runs at right angles to it along the body.

In this case $g_x = g$, and the projections of the gravity vector (or buoyant force) on the Oy- and Ox-axes are equal to zero ($g_y = g_z = 0$).

The dimension of the body along the Oz-axis exceeds greatly l_0.

With the conditions assumed the temperature and velocity fields can be described by differential equations when approaching the boundary layer (see Sec. 4-4). Let us also allow for the buoyancy (lift) force $\rho g \beta \vartheta$, assuming it to be commensurable with the term representing viscosity, $\mu (\partial^2 w_x / \partial y^2)$. Let us also introduce the notation $\vartheta = t - t_0$, where t is fluid temperature (it will be noted that $dt = d\vartheta$, because $t_0 = \text{const}$).

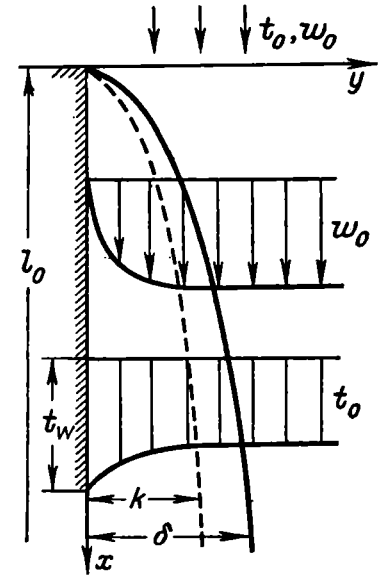

Fig. 5-1. Stating a boundary-value problem in heat transfer by convection

The equation of energy

$$w_x \frac{\partial \vartheta}{\partial x} + w_y \frac{\partial \vartheta}{\partial y} = a \frac{\partial^2 \vartheta}{\partial y^2}$$

equation of flow

$$w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} = \nu \frac{\partial^2 w_x}{\partial y^2} + g\beta\vartheta$$

equation of continuity

$$\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} = 0$$

The boundary conditions are as follows:

(1) at a distance from the body ($y = \infty$)

$$\vartheta = \vartheta_0 \equiv 0, \quad w_x = w_0 \quad \text{and} \quad w_y = 0$$

(2) at the surface ($y = 0$, $0 \leqslant x \leqslant l_0$), $-\infty \leqslant z \leqslant +\infty$

$$\vartheta = \vartheta_w \equiv t_w - t_0 = \text{const}, \quad w_x = w_y = w_z = 0$$

(a)

Three kinds of values may be distinguished in the equations and conditions of unambiguity:

independent variables—the coordinates x, y;

dependent variables—ϑ, w_x, and w_y, which are uniquely determined by the independent variables, provided the values entering the conditions of unambiguity are given;

constants—w_0, t_0, l_0, ϑ_w, ν, a, $g\beta$, etc. which are given by the conditions of unambiguity and for any definite problem are independent of the variables, though they may themselves vary from problem to problem; these values are referred to as constants because they are not functions of independent variables.

Thus the unknown dependent variables ϑ, w_x, w_y depend on many quantities, being functions of independent variables and constants of the conditions of unambiguity.

The terms present in the equations and the conditions of unambiguity can be united in dimensionless groups, the number of which will be smaller than the number of dimensional quantities.

To reduce the expressions to dimensionless form, we select scales of reduction, for which it is convenient to adapt the constants present in the conditions of unambiguity. For linear values we select any reference dimension, for instance, the length of the heat-transfer surface l_0, for velocity w_0, and for temperature ϑ_w.

Let us denote the dimensionless quantities as follows:

$$\left.\begin{aligned} X &= \frac{x}{l_0}, \quad Y = \frac{y}{l_0}, \quad W_x = \frac{w_x}{l_0} \\ W_y &= \frac{w_y}{w_0}, \quad \Theta = \frac{\vartheta}{\vartheta_w} \end{aligned}\right\} \tag{b}$$

Then

$$\left.\begin{aligned} x &= l_0 X, \quad y = l_0 Y, \quad w_x = w_0 W_x \\ w_y &= w_0 W_y, \quad \vartheta = \vartheta_w \Theta \end{aligned}\right\}. \tag{c}$$

Substitute the values of Eq. (c) into the equations. Rearrange the energy equation. Since, for instance,

$$\frac{\partial^2 \vartheta}{\partial y^2} = \frac{\partial}{\partial (l_0 Y)}\left[\frac{\partial (\vartheta_w \Theta)}{\partial (l_0 Y)}\right] = \frac{\vartheta_w}{l_0^2}\frac{\partial^2 \Theta}{\partial Y^2}$$

substituting equalities (c), and multiplying the left and right sides of the energy equation by l_0^2/a, we obtain:

$$\frac{x_0 l_0}{a}\left(W_x \frac{\partial \Theta}{\partial X} + W_y \frac{\partial \Theta}{\partial Y}\right) = \frac{\partial^2 \Theta}{\partial Y^2} \tag{5-1}$$

The flow equations are rearranged in the same way. After substitution of equalities (c), we multiply both sides of the flow equation by $l_0^2/\nu w_0$, obtaining thereby:

$$\frac{w_0 l_0}{\nu}\left(W_x \frac{\partial W_x}{\partial X} + W_y \frac{\partial W_x}{\partial Y}\right) = \frac{\partial^2 W_x}{\partial Y^2} + \frac{g\beta\vartheta_w l_0^2}{\nu w_0}\Theta$$

Rearranging the individual components of this equation as follows:

$$\frac{g\beta\vartheta_w l_0^2}{\nu w_0}\Theta = \frac{g\beta\vartheta_w l_0^3}{\nu^2}\frac{\nu}{w_0 l_0}\Theta$$

and taking these rearrangements into account we finally obtain:

$$\frac{w_0 l_0}{\nu}\left(W_x \frac{\partial W_x}{\partial X} + W_y \frac{\partial W_x}{\partial Y}\right) = \frac{\partial^2 W_x}{\partial Y^2} + \frac{g\beta\vartheta_w l_0^3}{\nu^2}\frac{\nu}{w_0 l_0}\Theta \tag{5-2}$$

Rearrangement of the continuity equation gives:

$$\frac{w_0}{l_0}\left(\frac{\partial W_x}{\partial X}+\frac{\partial W_y}{\partial Y}\right)=0$$

or, since w_0/l_0 does not equal zero,

$$\frac{\partial W_x}{\partial X}+\frac{\partial W_y}{\partial Y}=0 \tag{5-3}$$

Reducing the boundary conditions to dimensionless form, we obtain:

(1) at a distance from the body ($Y=\infty$)

$$\Theta=\Theta_0=0,\ W_x=1,\ W_y=0$$

(2) on its surface ($Y=0,\ 0\leqslant X\leqslant 1$)

$$\Theta=\Theta_w=1,\ W_x=W_y=0 \tag{d}$$

From conditions (d) it follows that the dimensionless terms Θ_0, Θ_w, W_x, etc. have definite numerical values in the case considered, even though the quantities w_0, t_0, t_w, etc. present in the dimensional boundary conditions, may have other numerical values.

As follows from Sec. 4-4 when the temperature field is known the local coefficient of heat transfer can be determined from equation

$$\alpha=-\frac{\lambda}{t_w-t_0}\left(\frac{\partial t}{\partial y}\right)_{y=0}$$

Presenting the equation in dimensionless variables, we obtain:

$$\frac{\alpha l_0}{\lambda}=-\left(\frac{\partial\Theta}{\partial Y}\right)_{Y=0} \tag{5-4}$$

The dimensionless term $\alpha l_0/\lambda$ is fully determined by the derivative $(\partial\Theta/\partial Y)_{Y=0}$.

5-3. Dimensionless Terms (Criteria of Similarity) and Dimensionless Equations of Similarity

Apart from the dimensionless terms Θ, W_x, W_y, and the dimensionless coordinates, composed of homogeneous physical quantities, the convection equations and conditions of unambiguity also contain dimensionless terms with dissimilar physical parameters:

$$\frac{\alpha l_0}{\lambda},\quad \frac{w_0 l_0}{\nu},\quad \frac{w_0 l_0}{a},\quad \frac{g\beta\vartheta_w l_0^3}{\nu^2}$$

These terms referred to as the criteria of similarity, are called after scientists who made great contributions to the development of hydrodynamics and heat transfer.

The first of them is denoted:

$$\mathrm{Nu}\equiv\frac{\alpha l_0}{\lambda} \tag{5-5}$$

and is called the *Nusselt* number, or criterion of heat transfer. The Nusselt number characterizes the process of heat transfer at the "wall-fluid" boundary as follows from Eqs. (4-3) and (5-1). It is usually an unknown in problems of convection, since it includes the heat-transfer coefficient α, which is being determined.

In spite of its seeming resemblance to the Biot number, considered when studying conduction, the Nusselt number is essentially different. The Biot number includes the thermal conductivity of a solid, the Nusselt number, the

thermal conductivity of a fluid. Furthermore, the heat-transfer coefficient of the Biot number is given under the conditions of unambiguity, while that in the Nusselt number is considered as an unknown parameter.

The dimensionless term

$$\mathrm{Re} \equiv \frac{w_0 l_0}{\nu} \tag{5-6}$$

is called the *Reynolds* number, and characterizes the relation between the forces of inertia and viscosity. Indeed, the Reynolds number is obtained by dividing the term accounting for inertia in the equation of flow by the term allowing for friction in the same equation:

$$\frac{w_x \partial w_x/\partial x}{\nu \partial^2 w_x/\partial y^2} = \frac{w_0^2/l_0}{\nu w_0/l_0^2} \frac{W_x \partial W_x/\partial X}{\partial^2 W_x/\partial Y^2} = \frac{w_0 l_0}{\nu} \frac{W_x \partial W_x/\partial X}{\partial^2 W_x/\partial Y^2}$$

We performed a similar operation in Sec. 5-2, when reducing the equations of flow to dimensionless form

The Reynolds number is a very important characteristic of both isothermal and non-isothermal processes of fluid flow

The third dimensionless term is denoted:

$$\mathrm{Pe} \equiv \frac{w_0 l_0}{a} \tag{5-7}$$

and is called the *Peclet* number. It can be rearranged as follows:

$$\frac{w_0 l_0}{a} = \frac{\rho c_p w_0 \vartheta}{\frac{\lambda}{l_0} \vartheta}$$

where the numerator characterizes the amount of heat transported by convection, and the denominator, the quantity of heat transported by conduction.

Essentially, we obtained the Peclet number earlier when we divided the term in the equation of energy accounting for convection by the term in the same equation accounting for conduction

The dimensionless term

$$\mathrm{Gr} \equiv \frac{g \beta \vartheta_w l_0^3}{\nu^2} \tag{5-8}$$

is called the *Grashof* number and characterizes the buoyancy force appearing in the fluid due to differences in density. Since deriving the equation of flow (4-18) it was assumed that $\beta\vartheta = \frac{\rho_0 - \rho}{\rho_0}$, the Grashof term Gr may be replaced with its general modification which is the Archimedean number:

$$\mathrm{Ar} = \frac{g l_0^3}{\nu^2} \frac{\rho_0 - \rho}{\rho_0} \tag{5-9}$$

The Ar term is called the Archimedean number. It is identical with the Grashof number on condition that $\beta = \mathrm{const}$.

Making use of dimensionless terms, the system of differential equations can be presented in the following form:

$$\mathrm{Nu} = -(\partial\Theta/\partial Y)_{Y=0} \tag{5-10}$$

$$\mathrm{Pe}\left(W_x \frac{\partial \Theta}{\partial X} + W_y \frac{\partial \Theta}{\partial Y}\right) = \frac{\partial^2 \Theta}{\partial Y^2} \tag{5-11}$$

$$\mathrm{Re}\left(W_x \frac{\partial W_x}{\partial X} + W_y \frac{\partial W_x}{\partial Y}\right) = \frac{\mathrm{Gr}}{\mathrm{Re}} \Theta + \frac{\partial^2 W_x}{\partial Y^2} \tag{5-12}$$

$$\frac{\partial W_x}{\partial X} + \frac{\partial W_y}{\partial Y} = 0 \tag{5-13}$$

This system of dimensionless differential equations and dimensionless conditions of unambiguity (d) (see Sec. 5-2) offer a mathematical description of our problem.

The dimensionless terms Θ, W_x, W_y, X, Y, Nu, Re, Pe, Gr may be considered as new variables and fall into three groups:

independent variables—the dimensionless coordinates X, Y;

dependent variables—Nu, Θ, W_x, W_y which are uniquely determined by the independent variables at definite values of the quantities in the conditions of unambiguity;

constants—Pe, Re, Gr which are given by the conditions of unambiguity and are constant for the particular problem [indeed, as follows from equations (5-6) to (5-8), the Reynolds, Peclet and Grashof numbers are composed of values entering the given conditions of the problem].

From the foregoing it follows that .

$$\mathrm{Nu} = f_1(X_w, Y_w, \mathrm{Pe}, \mathrm{Re}, \mathrm{Gr}) \tag{5-14}$$

$$\Theta = f_2(X, Y, \mathrm{Pe}, \mathrm{Re}, \mathrm{Gr}) \tag{5-15}$$

$$W_x = f_3(X, Y, \mathrm{Pe}, \mathrm{Re}, \mathrm{Gr}) \tag{5-16}$$

$$W_y = f_4(X, Y, \mathrm{Pe}, \mathrm{Re}, \mathrm{Gr}) \tag{5-17}$$

Equations of this kind are called *criterion equations*, or dimensionless characteristic equations of similarity.

In Eq. (5-14) X_w, Y_w correspond to the heat-transfer surface (wall). There is no sense in finding α (or Nu) for space points not located on the surface of the wall. In the problem considered $Y_w = 0$.

If account is taken of the term $\frac{1}{\rho}\frac{\partial p}{\partial x}$ in the equation of flow, reduction to the dimensionless form would result in the appearance of another term

$$\frac{l_0}{\rho w_0}\frac{\partial P}{\partial X} = \frac{\partial}{\partial X}\left(\frac{p}{\rho w_0^2}\frac{w_0 l_0}{\nu}\right) = \frac{\partial}{\partial X}(\mathrm{EuRe})$$

The dimensionless term

$$\mathrm{Eu} = \frac{p}{\rho w_0^2} \tag{5-18}$$

is called the *Euler* number, and characterizes the relation between the forces of pressure and those of inertia. This term appears in the equation of convection heat transfer only under the sign of derivative, which means that when an incompressible fluid of constant physical properties is considered, it is not the absolute pressure that is essential, but the variation in pressure*. The Euler number, therefore, is usually written as follows:

$$\mathrm{Eu} = \frac{p - p_0}{\rho w_0^2}$$

where p_0 is any fixed value of pressure, e.g. the channel inlet pressure, which may be unknown.

Other reference dimensions than l_0 are also essential for many processes of flow and heat transfer.

The reference dimensions for fluid flow in a straight plain tube, for instance, are its diameter and length; with a bent tube, however, the radius of curvature is an additional reference dimension. With fluid flow inside rough tubes the

* In case of compressible flow it is necessary to account for the dependence of density on pressure; and the absolute pressure is then of interest.

height of the surface irregularities and their distribution over the surface are of interest. All the reference dimensions l_0, l_1, l_2, etc. must be given when stating a problem. The function symbol of equations (5-14) to (5-17) must then include the values

$$L_1 = \frac{l_1}{l_0}, \quad L_2 = \frac{l_2}{l_0}, \text{ etc.}$$

The introduction of these values L_1, L_2 . . ., L_n under the symbol of the function is clearly essential here. The list of dimensionless terms must always correspond to the mathematical description of the problem, and arbitrary elimination or introduction of new variables is absolutely inadmissible; any operation of the kind must be substantiated.

With the mathematical statement of the problem invariable, new dimensionless terms can obviously be obtained by multiplying the dimensionless terms by one another in a certain way; but in doing so the number of variables under the function symbol must not change.

The Peclet number obtained by reducing the equation of energy to dimensionless form can be represented as the product of two dimensionless terms:

$$\mathrm{Pe} = \mathrm{Re} \cdot \mathrm{Pr} = \frac{w_0 l_0}{\nu} \frac{\nu}{a} \tag{5-19}$$

The term $\mathrm{Pr} \equiv \frac{\nu}{a}$ is a new criterion, called the *Prandtl* number, which is composed wholly of physical parameters and is itself, therefore, a physical parameter. It may be presented in the following form:

$$\mathrm{Pr} = \frac{\nu}{a} = \frac{\mu c_p}{\lambda} \tag{5-20}$$

The Prandtl number can be given a certain physical meaning. The energy equation (4-30)

$$w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} = \alpha \frac{\partial^2 t}{\partial y^2}$$

and the equation of motion (4-28)

$$w_x \frac{\partial w_x}{\partial x} + w_y \frac{\partial w_x}{\partial y} = \nu \frac{\partial^2 w_x}{\partial y^2}$$

are similarly presented. At $a = \nu$ the calculated temperature and velocity fields will be similar, provided the conditions of unambiguity are also similar. For condition $a = \nu$ $\mathrm{Pr} = \frac{\nu}{a} = 1$. Thus, at certain conditions the Prandtl number can be considered to be the measure of similarity of temperature and velocity fields.

The Prandtl number of *dropping* liquids depends greatly on temperature, the dependence for most liquids being in the main similar to that of viscosity μ (t), since specific heat c_p and thermal conductivity λ depend less on temperature. As a rule, the Prandtl number diminishes rapidly with rising temperature (Fig. 5-2). For saturated water, its dependence on temperature is illustrated graphically in Fig. 5-3. Its value at temperatures between 0° and 180 °C drops sharply with rising temperature (from 13.7 to 1.0); and such a drop is traceable to the sharp reduction in the viscosity of water and increase of λ in this temperature range, where specific heat depends only sligh-

tly on temperature. With water temperatures between 130° and 310 °C the Prandtl number for water varies insignificantly and is close to unity. The nature of its dependence on temperature changes drastically only at pressures and

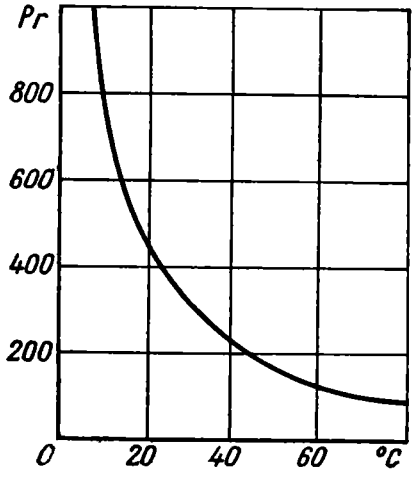

Fig. 5-2. Change of the Prandtl number for transformer oil depending on temperature variation

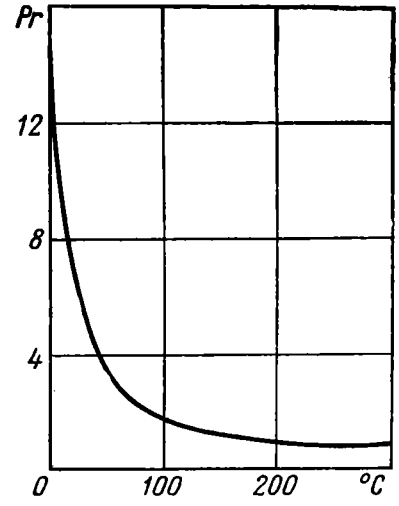

Fig. 5-3. Change of the Prandtl number for water depending on temperature variation in the range 0°-300 °C

temperatures approaching the critical region. Heat transfer in the near-critical region will be considered separately.

The Prandtl number of *gases* is practically independent of either temperature or pressure, is constant for any gas and is determined by its valency.

From the kinetic theory of gases it follows that

for monatomic gases	Pr = 0.67
for biatomic gases	Pr = 0.72
for triatomic gases	Pr = 0.80
for tetratomic gases	Pr = 1.0

The actual Prandtl numbers of real gases however differ somewhat from the values indicated.

The Prandtl number of heavy and alkali *liquid metals*, used as heat-carrying agents, changes approximately from about 0.005 to 0.05. Its smallness is due to the high conductivity of liquid metals.

Depending on the magnitude of the Prandtl number, liquids are divided into three groups: liquids with $Pr \ll 1$ (liquid metals), heat-carrying liquids with $Pr \approx 1$ (nonmetallic liquids at high temperatures and gases), liquids with $Pr > 1$ (nonmetallic liquids).

Considering that $Pe = Re \cdot Pr$, dimensionless characteristic equations (5-14) to (5-17) can be written in the form:

$$Nu = F_1(X_w, Y_w, Re, Pr, Gr) \quad (5\text{-}21)$$

$$\Theta = F_2(X, Y, Re, Pr, Gr) \quad (5\text{-}22)$$

$$W_x = F_3(X, Y, Re, Pr, Gr) \quad (5\text{-}23)$$

$$W_y = F_4(X, Y, Re, Pr, Gr) \quad (5\text{-}24)$$

Proceeding from Eqs. (5-14) to (5-17) and (5-21) to (5-24) the dimensionless terms may be divided into two kinds:

unknowns—terms containing unknown dependent variables in the case considered—α, ϑ, w_x, w_y; hence, Nu, Θ, W_x, W_y are known and must be determined;

reference terms—criteria composed wholly of independent variables or constants present in the conditions of unambiguity. In the case under consideration these are X, Y, Re, Pr (or Pe) and Gr.

The dimensionless terms composed of given parameters (constants) of a mathematical description of a process are also referred to as *criteria of similarity.*

5-4. Conditions of Similarity for Physical Processes

Like the original system of dimensional equations the derived system of dimensionless differential equations (5-10) to (5-13) describes an infinite number of particular processes of convection. They hold for any process of heat transfer between a solid and an incompressible fluid, provided that the process satisfies the assumptions made in deriving the equations of heat transfer. Thus the system of equations obtained describes a large class of phenomena, i.e. a whole set of physical processes characterized by a similar mechanism of occurrence.

In Part I of this book we found that the differential equation of heat conduction $\nabla^2 t = 0$ described an infinite number of practical processes of one class, the feature uniting them being the similar mechanism of heat propagation. There are, however, other differential equations of a form similar to that of the differential equation of conduction, for example, the equation of electrical potential (see Sec. 3-12). The two equations are similar in appearance, if we denote temperature and electrical potential by like symbols. But though they are similar in form, they describe different classes of phenomena, since the values of both equations are physically different. The natural phenomena described by differential equations which are similar in form, but different in physical content, are referred to as *analogous.*

The differential equations reflect the most general features of phenomena, and do not account for special, qualitative peculiarities, such as the geometry and dimensions of the system in which the physical process occurs; the special features of a process also include the physical properties of its operative bodies or the conditions governing its development at the boundaries of the system, etc. The special features of the various phenomena pertaining to the same class are determined from the conditions of unambiguity.

The foregoing analysis of the dimensionless differential equations and conditions of unambiguity makes clearer the general conditions of similarity of physical processes which are formulated in the following three rules:

(1) *similar processes must be qualitatively alike, i.e. they must be of a similar physical nature and be described by differential equations of similar form;*

(2) *the conditions of unambiguity of similar processes must be similar in all respects, excepting only the numerical values of their constants;*

(3) *like reference criteria of similar processes must be numerically equal.*

These rules determine the conditions of similarity of physical processes.

The first condition states that similar processes must belong to the same class of physical phenomena, and apart from their similar physical nature, must be described by differential equations similar in form.

Gravity can be ignored in many problems of forced convection. Obviously a nil value for gravity changes the mechanism and mathematical description of the process under consideration. In considering free convection in a bulk volume the pressure gradient in the fluid may be neglected; its exclusion from

the flow equation changes the form of the equation describing the process, and the class of phenomena considered therefore changes as well.

Thus, for processes of heat transfer by convection to be similar, they must have both a similar nature and identical motivating forces. Individual processes may be described by different differential equations (if they are only particular cases of more general equations); they will then belong to different classes of phenomena.

A change in the original differential equations changes the system of dimensionless terms required for the process being studied.

The second condition of similarity requires that the conditions of unambiguity of similar processes be alike in all respects except the numerical values of their constants, i.e. similar processes must belong to the same group of phenomena*.

Thus, the dimensional conditions of unambiguity of similar processes must be identically described in the general form (in symbols). The concrete values of the velocity of free-stream flow w_0, wall temperature t_w, etc. may then differ numerically. Comparison of the boundary conditions (a) and (d) (Sec. 5-2) shows that the dimensionless boundary conditions will be the same for all these processes in spite of the different magnitudes of w_0, t_0, t_w, etc.

From these two rules it follows that similar processes must be described by identical dimensionless differential equations and dimensionless boundary conditions.

When expressed in dimensionless form, the mathematical description of the similar processes concerned here is the same. They may thus be described by a single formula, for instance

$$\mathrm{Nu} = f_1 (X_w, \mathrm{Re}, \mathrm{Pr})$$

or

$$\Theta = f_2 (X, Y, \mathrm{Re}, \mathrm{Pr}), \text{ etc.}$$

The function f_1 is identical for all similar processes. The same may be said of the function f_2, etc. Considerable mathematical difficulties may be encountered in finding functions f_1 and f_2, if the system of dimensionless equations and boundary conditions is too complex. It can always be maintained, however, that these functions exist.**

The processes being investigated will depend on the same criteria of similarity, provided that the first two rules are observed. This follows inevitably from what has been said about similar processes being described by identical dimensionless equations and boundary conditions.

The first two rules alone are not sufficient to establish physical similarity. It is necessary to add the condition that the like reference terms of similar processes must be numerically equal, i.e.

$$X = \text{idem}, \; Y = \text{idem}, \; \mathrm{Re} = \text{idem}, \; \mathrm{Pr} = \text{idem}, \; \mathrm{Gr} = \text{idem}, \text{ etc.}$$

Since similar processes have like functions f_1 and f_2 and numerically equal reference terms, the unknown like dimensionless terms of similar processes will also be numerically equal, i.e.

$$\mathrm{Nu} = \text{idem}, \; \Theta = \text{idem}, \; W_x = \text{idem}, \; W_y = \text{idem}, \text{ etc.}$$

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* Processes are identical in the particular case, where the dimensional constants of the conditions of unambiguity are numerically equal (provided that the other conditions of similarity are satisfied).

** The problem is assumed to be precisely formulated.

Assume that a system of dimensional differential equations is considered in conjunction with dimensional boundary conditions. Their solution would yield a certain formula. This can be illustrated by considering the solutions of the problems of heat conduction examined above. Substitution of concrete numerical values of the arguments λ, δ, and Δt in the formula $q = \frac{\lambda}{\delta}\Delta t$ would yield a definite numerical value of the dependent variable q. At the same values of λ, δ, and Δt all processes of heat conduction described by this formula will clearly be identical, i.e. it will be the same process.

It is another matter when the formula is expressed in dimensionless variables. The constancy of each of the reference criteria X, Y, Re, Pr, and Gr, for instance, in equation $\Theta = f(X, Y, \text{Re}, \text{Pr}, \text{Gr})$ yields one value for the dimensionless temperature $\Theta = \frac{t - t_0}{t_w - t_0}$, but the temperatures of the fluid and wall may differ numerically.

For equal values of Θ there will be a corresponding multitude of physical processes, differing in their dimensional temperature characteristics, and possibly identical only in a special case.

The Kirpichev-Gukhman theorem (1931) embodies three conditions of similarity.

From the foregoing it follows that, apart from fulfilment of the first conditions, similarity also requires that like reference criteria be numerically equal. Thus similarity of *processes as a whole* requires that like reference terms consisting of constants, given by the conditions of unambiguity, be numerically equal. Two processes of heat transfer in a fluid flowing inside tubes, for instance, will be similar if the first two conditions of similarity are fulfilled, and the like reference criteria (consisting of constants), are numerically equal. The processes will be similar *in toto*. At the same time the local (point) values of the unknown variables must be considered at the points where the like dimensionless coordinates are equal.*

Thus, only dimensionless reference terms are essentially *criteria of similarity*, and, above all, dimensionless terms consisting of constants.

From what has been said in this chapter it follows that the theory of similarity can be considered as one of generalized dimensionless variables characteristic of each given process. The basic feature of the theory is replacement of ordinary dimensional variables by generalized ones.

We have considered the conditions of similarity of physical processes, taking the example of convective heat transfer in the flow of an incompressible fluid. The conditions of similarity, of course, are true not merely for convection, but also for other physical processes.

We can compose criteria of similarity for any physical phenomenon, provided that its full mathematical description is available. Knowing the mathematical description of a process is a prerequisite of the theory of similarity.

The previously formulated conditions of similarity can be used to work out an *analogy* between two physical, heterogeneous processes. For this purpose the first condition of similarity should only call for a formal identity of differential equations.

Thus, the concept of similarity, or similitude, can be spread over physically heterogeneous (analogous) processes.

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* For transient processes dimensionless times must be equal as well, e.g. the Fourier numbers.

5-5. Corollaries of the Theory of Similarity

Consider two similar processes of convective heat transfer, for instance, in a fluid flowing along conduits of arbitrary cross-section. One of the processes is denoted by the letter A and the other by the letter B.

For the units of linear dimensions we may take any dimension of the conduits, for instance, their heights h_A and h_B. Then,

$$X_A=\frac{x_A}{h_A}, \quad Y_A=\frac{y_A}{h_A}, \quad Z_A=\frac{z_A}{h_A}$$

and

$$X_B=\frac{x_B}{h_B}, \quad Y_B=\frac{y_B}{h_B}, \quad Z_B=\frac{z_B}{h_B}$$

Let us consider the processes A and B at the points characterized by the equations:

$$X_A = X_B,\ Y_A = Y_B, \text{ and } Z_A = Z_B \tag{5-25}$$

The points satisfying these conditions are known as *corresponding points.*

The following relationships are true for corresponding points:

$$x_A = x_B \frac{h_A}{h_B} = x_B c_l$$

$$y_A = y_B \frac{h_A}{h_B} = y_B c_l$$

$$z_A = z_B \frac{h_A}{h_B} = z_B c_l$$

where

$$c_l = \frac{h_A}{h_B}$$

If the equalities of (5-25) are satisfied for two similar processes, the following must evidently also hold for corresponding points:

$$W_{xA} = W_{xB}$$

or

$$\frac{w_{xA}}{w_{0A}} = \frac{w_{xB}}{w_{0B}}$$

where w_{0A} and w_{0B} are velocities given by the conditions of unambiguity; this can be, for instance, the inlet velocities for respectively A and B conduits.

From the last equality it follows that

$$\frac{w_{xA}}{w_{xB}} = \frac{w_{0A}}{w_{0B}} = c_w = \text{const}$$

i.e. the ratio of velocities is constant at any corresponding point of the two similar processes.

Similarly we can write:

$$\frac{\alpha_A}{\alpha_B} = \frac{\lambda_A}{\lambda_B}\frac{h_A}{h_B} = c_\alpha = \frac{c_\lambda}{c_l} = \text{const}, \qquad \frac{\vartheta_A}{\vartheta_B} = c_\vartheta = \text{const}, \text{ etc.}$$

Thus, if processes A and B are similar, any physical value φ at a given point of A is proportional to the value at the corresponding point of B, i.e.

$$\varphi_A = c_\varphi \varphi_B \tag{5-26}$$

The proportionality factors c_φ are called the constants of similarity. They are dimensionless, not equal to unity in the general case, are not dependent on coordinates, time, or direction, and are different for all values of different physical sense. When all similarity constants are equal to unity processes are said to be identical; assume that a process C is also similar to A and B. Then we can write:

$$\varphi_A = c'_\varphi \varphi_B$$

with c_φ and c'_φ not equal in the general case.

Thus similar processes may be regarded as identical but taken on a different scale, and the scales of the unlike or dissimilar values may not be equal.

Similarity constants cannot be chosen arbitrarily. Let us illustrate this by an example.

For two similar processes of forced convection A and B the condition $\mathrm{Re}_A = \mathrm{Re}_B$ holds, where

$$\mathrm{Re}_A = \frac{w_{0A} l_{0A}}{\nu_A} \quad \text{and} \quad \mathrm{Re}_B = \frac{w_{0B} l_{0B}}{\nu_B}$$

The like values in these equations are related by means of the similarity constants:

$$w_{0A} = c_w w_{0B}, \quad l_{0A} = c_l l_{0B}, \quad \text{and} \quad \nu_A = c_\nu \nu_B$$

Introducing these equalities in Re_A, we obtain:

$$\mathrm{Re}_A = \frac{c_w c_l}{c_\nu} \frac{w_{0B} l_{0B}}{\nu_B} = \frac{c_w c_l}{c_\nu} \mathrm{Re}_B$$

or

$$\frac{\mathrm{Re}_A}{\mathrm{Re}_B} = \frac{c_w c_l}{c_\nu} = 1$$

This is the condition that limits arbitrary choice of the constants c_w, c_l and c_ν.

Similarly

$$\frac{\mathrm{Pr}_A}{\mathrm{Pr}_B} = \frac{c_\nu}{c_\alpha} = 1; \qquad \frac{\mathrm{Nu}_A}{\mathrm{Nu}_B} = \frac{c_\alpha c_l}{c_\lambda} = 1, \text{ etc.}$$

5-6. Methods of Dimensionality

The prerequisite of the theory of similarity is the existence of a mathematical description of the investigated process in the form of differential equations (or integral-differential) equations and conditions of unambiguity.

From the mathematical statement of the process follows a list of physical quantities which are essential for the process investigated. If the list is made up, the dimensionless terms or criteria of similarity can be composed by the method of dimensional analysis.

The list of the dimensional quantities is sometimes made up by intuition, without strict formulation of a problem, then errors are possible.

Dimensional analysis is treated in greater detail in special literature [5.1, 5.8, 5.9, for instance]. We shall confine ourselves here to certain conclusions following from it, which are of interest for the practical application of generalized variables.

Two kinds of physical quantities can be distinguished: *primary* (fundamental) and *secondary* (derived).

The primary quantities characterize some physical phenomenon directly, independently of other quantities. The secondary quantitites, or parameters,

are expressed in terms of the primary in accordance with definitions or physical laws. So, for instance, if length and time are primary quantities, i.e. if length cannot be expressed in terms of time (and vice versa), velocity which is defined as the ratio of length (distance covered) to time is a secondary, derived quantity.

The selection of primary quantities is in general arbitrary. The primary quantities of the SI system of units are length (L), mass (M), time (T), temperature (Θ), current intensity (I), luminous intensity (J); here the letters L, M, T, Θ, I and J are symbols corresponding to the primary quantities. Also known are other systems of primary quantities, used or suggested for usage. Gauss, for instance, suggested to use as primary quantities length, mass and time; other conceivable quantities must be derived ones. When selecting primary quantities of great importance is their convenient usage

The symbolic expression of a derived quantity in terms of primary ones is called the *dimension*. We can speak of dimension only as applied to a definite system of primary quantities. The dimension can be presented in the form of a power formula. As applied to the SI system of units the dimensional formula has the appearance:

$$[\varphi] = L^{n_1} M^{n_2} T^{n_3} \Theta^{n_4} I^{n_5} J^{n_6} \tag{5-27}$$

where $[\varphi]$ = derived unit of measurement;
n_i = real numbers.

The dimension of a secondary quantity in respect to a given primary i may be characterized by the exponent n_i at that primary quantity. Dimensionless numbers, therefore, are often referred to as quantities with zero dimension, since for them all exponents in the dimensional formula (5-27) are zero. In accordance with formula (5-27), the dimension of a primary quantity can be assumed equal to unity (taken in respect to self).

In addition to dimension, physical quantities are characterized by numerical values which are obtained for primary quantities by direct measurement, i.e. by correlating the measured quantity with some quantity of the same physical nature. The choice of units of measurement for primary quantities (basic units of measurement) is arbitrary and is determined by their convenient usage.

The numerical value of a secondary quantity is determined indirectly, being found from the numerical values of primary quantities. The numerical values of both the primary and secondary quantities depend on the selection of the units of measurement. And only the numerical values of dimensionless quantities do not depend on the selection of units of measurement (quantities with zero dimension).

The selection of the list of primary quantities and their units of measurement is the main and necessary step in creating a system of units.

Let us consider an example involving the use of the method of dimensional analysis. Determine the dimensionless variables corresponding to the mathematical statement of the problem considered in Sec. 5-1. It follows from this problem that

$$\vartheta = f(x, y, \vartheta_w, l_0, w_0, \nu, a, g\beta) \tag{a}$$

The list of quantities essential for the process being considered includes nine variables ($n = 9$). Used in this problem are three primary quantities measured in units of the SI system of units: length, time, temperature ($k = 3$).

Making use of the possibility of choosing arbitrarily the basic units of measurement, let us divide the variables in equation (a) into two groups, one inclu-

ding quantities with an independent dimension (primary) and another including quantities with a dependable dimension (derived). We as if create a new system of units (especially for the problem being considered). The first step on this way consists in the selection of the list of primary quantities (quantities with an independent dimension).

Let us select such quantities:

$$[l_0] = L, \quad [\vartheta_w] = \Theta \text{ and } [\nu] = L^2T^{-1}$$

The number of quantities with an independent dimension corresponds to the number of primary units in the SI system of units, used in the problem considered ($k = 3$).

Let us express the dimension of the other quantities in terms of $[l_0]$, $[\vartheta_w]$, and $[\nu]$ in accordance with the dimensional formula:

$$[x] = [l_0], \quad [y] = [l_0], \quad [\vartheta] = [\vartheta_w], \quad [w_0] = [l_0]\,T^{-1} = [l_0]^{-1}\,[\nu],$$
$$[a] = [l_0]^2\,T^{-1} = [\nu], \quad [g\beta] = [l_0]\,[\Theta]^{-1}\,T^{-2} = [l_0]^{-3}\,[\nu]^2\,[\Theta]^{-1}$$

Let us assign units of measurement to quantities with an independent dimension. In the present case it is convenient to take as the basic units of measurement the numerical values of the constants l_0, ϑ_w and ν given in conditions of unambiguity. The new numerical values of the physical quantities x', ϑ', etc. are obtained comparing with the new standard, i.e. $x' = x/l_0$, $\vartheta' = \vartheta/\vartheta_w$, etc. A physical process is independent of the choice of units of measurment, therefore, Eq. (a) must preserve its structure at various scale values. Using the new numerical values of the variables, equation (a) may be presented in the following form:

$$\frac{\vartheta}{\vartheta_w} = f\left(\frac{x}{l_0}, \frac{y}{l_0}, \frac{\vartheta_w}{\vartheta_w}, \frac{l_0}{l_0}, \frac{w_0}{a l_0^{-1}}, \frac{\nu}{\nu}, \frac{a}{\nu}, \frac{g\beta}{l_0^{-3}\nu^2\vartheta_w^{-1}}\right)$$

Here all the terms are dimensionless ones. The quantities ϑ_w/ϑ_w, l_0/l_0, ν/ν, equal to unity, may be taken out of the function.

Let us make use of the criteria of similarity derived in Sec. 5-2. Then

$$\Theta = f\,(X, Y, \text{Pe}, \text{Pr}, \text{Gr}) \qquad \text{(b)}$$

A similar result was previously obtained by means of the method of scale transformations, formula (5-22).

According to Eq. (4-22), in the present case

$$\frac{\alpha\vartheta_w}{\lambda} = -\left(\frac{\partial\vartheta}{\partial y}\right)_{y=0}$$

i.e. the dimensionless term $\alpha\vartheta_w/\lambda = q_w/\lambda$ depends on the same variables as ϑ. Then, inasmuch as

$$\left[\frac{\alpha\vartheta_w}{\lambda}\right] = \Theta L^{-1} = [\vartheta_w]\,[l_0]^{-1} \quad \text{and} \quad \left(\frac{\alpha\vartheta_w}{\lambda}\right)' = \frac{\alpha\vartheta_w/\lambda}{\vartheta_w l_0^{-1}} = \frac{\alpha l_0}{\lambda} = \text{Nu}$$

we find that the Nusselt number depends on the same dimensionless terms Θ which is evident from equation (b).

It follows from equation (a) and (b) that after passing to dimensionless quantities the number of variables reduced formally from nine to six. This conclusion is in agreement with the so-called π-theorem.

According to this theorem a *physical equation containing* $n \geqslant 2$ *denominate quantities, of which* $k \geqslant 1$ *quantities have an independent dimension will contain upon reduction to a dimensionless form* $n - k$ *dimensionless quantities.*

The method of scale transformations used in Sec. 5-1 does not show the number of dimensionless variables that we must obtain: their number is shown by the π-theorem. The error admitted in determining the number of dimensionless variables which are of importance to the process being considered may lead to serious errors when experimental data are described by similarity equations.

In conclusion the following will be noted. The mathematical statement of the problem given in Sec. 5-1 is written for a laminar boundary layer, since no account is made of coefficients of turbulent transport of heat and momentum. The coefficients λ_t and μ_t are believed to depend on the same quantities as the fields of averaged velocity and temperature. Then, according to dimensional analysis, the obtained system of similarity criteria is also valid for turbulent flow. Naturally, the temperatures and velocities which are present in the dimensionless terms, or criteria of similarity, will now be averaged in time.

5-7. Simulating Processes of Convective Heat Transfer

In simulating a process of heat transfer, study of the process in full-size equipment is replaced by investigation of the same process in a scale model. With a model, the process must obviously be realized in a way that makes it possible to apply the results to the full-size equipment. The conditions of simulation, i.e. the conditions to be met by the scale model and by the process developing in it, are given by the theory of similarity. If the process developing in the scale model is similar to that occurring in full-size pilot equipment, the test results obtained in the model can be applied to the pilot equipment.

Simulation or modelling includes in essence two independent problems. First, the process in the model must be similar to that taking place in the full-size equipment or installation; and second, all the measurements and observations required must be made on the model. Here we shall concern with the first problem as the measurement and observation techniques are adequately described in the specialist literature [5.10, 5.4, 5.6]. So that the processes in the model will be similar to that in the full-size equipment, the conditions of similarity formulated earlier must be realized.

The first condition of similarity states that qualitatively similar processes must be simulated, i.e. processes of an identical physical nature, and described by identical differential equations.

The second condition requires that the conditions of unambiguity be identical in all respects (both in the full-size equipment and the model), except in the numerical magnitude of the constants concerned.

For steady-state processes the conditions of unambiguity consist of:

(1) the *geometrical conditions*, characterizing the form and size of the body in which the process occurs;

(2) the *physical conditions*, characterizing the physical properties of the system considered;

(3) the *boundary conditions*, characterizing the special feature of the investigated process at the boundary of the fluid medium.

Thus, geometrical similarity must be ensured between the equipment to be studied and its scale model. All dimensions of the full-size equipment and of the model which are essential for the process of heat convection must be linked to one another by the relationship $l_{fs} = c_l l_m$, i.e. the model must be constructed as an exact copy of the full-size equipment, but reduced c_l times in size. It is not, of course, the external shape of the equipment that must be

copied, but the internal configuration of the passages along which the flow of gas or liquid takes place.

It is not usually difficult to ensure geometrical similarity. It must only be remembered that a change in dimensions should not lead to a qualitative change in the process occurring in the model, and, hence, violate the first condition of similarity. A gas, for instance, should not be considered a continuum; and the differential equations of heat transfer by convection previously used cannot be taken to study the flow and transfer of heat in the gas, if the Knudsen number $\overline{l}/l_0$ is fairly large (see Sec. 4-4). For gas flow in a pipe, the diameter d of the pipe can be taken as the reference dimension l_0. If the mean free path of molecules $\overline{l}$ is larger than $0.001d$, then the properties of gas flow differ from those in a continuum.

When the physical parameters are constant, as was assumed earlier in deriving the differential equations of convection, there are no particular difficulties in realizing similarity of the physical conditions. Like physical parameters of the model and of the full-size equipment must also be linked by a corresponding conversion scale c_φ; if the physical properties of the fluid in the scale model and in the equipment are identical, then $c_\varphi = 1$.

The problem is much more complicated when the physical parameters are variable, and the variability manifests itself in the investigated process. This problem will be examined later.

It is also necessary to ensure similarity of the processes at the boundaries of the fluid system considered. This condition is usually expressed by requiring similarity between the inlet conditions of the equipment and of the scale model (so as to ensure similar distribution of inlet velocities), and between the temperature fields at the inlet of the equipment and at the surface of the bodies involved in the process of heat transfer. Similarity of fluid inlet flow can be obtained by making the inlet section of the model geometrically similar to that of the full-size apparatus. If the inlet temperature of the fluid in the latter does not vary over the section of the channel, it is not difficult to ensure conditions of similarity of the temperature fields, by seeing that there is no heat transfer in the passage delivering the liquid or gas to the model.

If the inlet temperature field, however, is complicated, it is more difficult to realize its distribution in the model. There are also often difficulties in realizing similarity of temperature fields at the heat-transfer surface. Strict satisfaction of boundary conditions then becomes a task of paramount importance to the experimenter.

The third condition of similarity requires that like reference terms of similar processes be numerically equal. The unknown like dimensionless terms will then also be equal.

Heat transfer by convection depends essentially on the pattern of liquid or gas flow. With forced convection, the nature of flow depends above all on the Reynolds number. When these processes are simulated the Reynolds numbers must therefore be equal at the inlets of the equipment and the scale model:

$$\frac{w_{0m}l_{0m}}{\nu_m} = \frac{w_{0fs}l_{0fs}}{\nu_{fs}}$$

From this it follows that the inlet velocity of the fluid in the scale model must be equal to:

$$w_{0m} = w_{0fs}\frac{l_{0fs}}{l_{0m}}\frac{\nu_m}{\nu_{fs}}$$

Let us assume that the same fluid flows through the model and equipment. Then, if we ignore differences in the temperature of the fluid, $\frac{\nu_m}{\nu_{fs}} = 1$. Let the model be constructed to a 1/10 scale. Then $\frac{l_{0fs}}{l_{0m}} = 10$ and consequently $w_{0m} = 10w_{0fs}$.

This means that to satisfy the requirement of equality of Reynolds numbers, the velocity of fluid flow in the model must be increased by as many times as its geometrical dimensions are reduced. Apart from equality of the Reynolds numbers, equality of other reference terms must also be ensured, in particular, we must have:

$$\mathrm{Pr}_m = \mathrm{Pr}_{fs}$$

This condition, permitting in principle the substitution of one fluid for another, in fact significantly restricts this possibility. Thus, for example, the Prandtl numbers for water are numerically close to those for gases only in the temperature range between 150 ° and 300 °C (and, consequently, at pressures exceeding 5×10^5 Pa). To use water to simulate the flow of incompressible gases, therefore, it would be necessary to maintain too high pressure in the model.

The problem of replacing one working fluid by another is even more complicated, owing to variations in the physical parameters. To account for these variations, it is necessary to change the system of differential equations describing the process of convective heat transfer obtained earlier. In deriving the equations, the variable magnitudes of the physical parameters must not be taken out of the sign of the derivatives. Moreover, equations of the kind

$$\lambda = f_1(t), \quad \mu = f_2(t), \quad c_p = f_3(t), \quad \text{and} \quad \rho = f_4(t)$$

describing variation of the physical parameters with temperature, must be added to the basic system of differential equations.*

According to the first condition of similarity, these equations, written in dimensionless form, must be identical for like parameters. Only then can we speak of exact similarity. The physical parameters will thus change in the space considered, i.e. they will depend on the coordinates (and on time, in transient processes), and consequently be dependent variables.

The theory offers no general uniform equation to describe the variation of a given physical parameter with temperature and suitable for all fluids currently used in engineering. The equations available are at best suitable for individual groups of heat-carrying agents within a definite range of temperature variation.

This circumstance imposes a grave limitation on the possibilities of precise simulation, since it is not possible to obtain exact similarity of the convection processes of heat transfer within a wide range of fluid and of temperature variations. This results, in particular, in it being practically impossible to replace a gas by a dropping liquid, owing to lack of similarity between the field of physical parameters in the full-size equipment (gas) and in the model (dropping liquid).

Thus realization of an exact similarity of convection processes and, consequently, their precise simulation often encounters insuperable difficulties.

In this connection a need arises for methods of *approximate* simulation. One possibility is offered by the phenomenon of *self-similarity* of the process with

* In some problems dependence of the physical parameters on pressure, where it exists, must also be taken into consideration.

respect to any of the reference terms. The unknown dimensionless term is said to be self-similar of the reference term, if the unknown term is independent of the latter.

If a process is self-similar in respect of any reference terms, there is no need, in simulating a process, for this dimensionless term to be equal for the full-size equipment and the model.

The phenomenon of self-similarity permits simplification of the differential equations and conditions of unambiguity. The terms in the equations (or conditions of unambiguity) accounting for the factors in respect to which the process happens to be self-similar may be omitted or altered.

Owing to the difficulties encountered in the attempts at exact analogy, the approximate *method of local thermal simulation* is often used. The essence of this method is that an attempt is made to obtain similarity of processes only at the spot or section where heat transfer is being studied. If, for example, the object investigated is a bank of tubes exposed to fluid flow, it is possible to use only one of the tubes in heat-transfer tests. The remaining tubes merely serve to give the model a geometry similar to that of the full-size equipment, while the data on heat transfer are obtained from the measurements taken on a single tube.

It is assumed that heat transfer of the tested tube depends in the main on the way in which it is exposed to fluid flow, which is determined by the arrangement of the bank of tubes and not by thermal conditions.

The method of local simulation is comparatively simple and often enables adequate results to be obtained. It should, however, be borne in mind that the unfounded application of the method of local thermal simulation may lead to considerable errors.

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CHAPTER

6

TREATMENT AND GENERALIZATION OF RESULTS OF MEASUREMENT AND CALCULATION OF HEAT TRANSFER BY CONVECTION

6-1. Local Heat-transfer Coefficient

The local heat-transfer coefficient is determined by Eq. (4-4)

$$\alpha = \frac{dQ_w}{(t_w - t_f)\, dF} = \frac{q_w}{t_w - t_f}$$

The quantities q_w and t_w are taken for a surface element dF. But the selection of the calculation temperature of fluid is not predetermined by Newton's law. In the general case of heat transfer by convection the temperature of fluid varies in the space considered. It becomes necessary to settle on the temperature of fluid to be used for calculation purposes, i.e. to be used in the equation describing Newton's law.

It is common practice to use various values of temperature in calculations. For instance, with fluid flowing in tubes, taken as the calculation temperature may be the mean temperature of the fluid $\bar{t}$ over the cross-section considered or the fluid mean inlet temperature $\bar{t}_{in}$. Depending on the choice of the calculation temperature of the fluid, the numerical values of the local heat-transfer coefficient α may differ as well as the laws governing the variation of α along the tube.

In the book, taken as the calculation temperature will be the mean temperature of fluid over the cross-section of a tube. When considering heat transfer for a solid exposed to an infinite flow the calculation temperature will be the temperature of the fluid beyond the borders of the thermal boundary layer.

6-2. Mean Fluid Temperature over the Cross-section of Flow

In the general case the temperature and velocity of fluid vary over the cross-section of flow. The possible distribution of t and w_x in a certain cross-section of a tube is shown in Fig. 6-1.

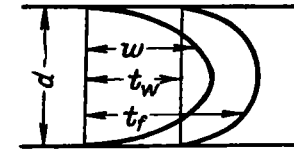

Fig. 6-1. Temperature and velocity distribution of fluid over the cross-section of a channel

Let us consider in the cross-section of a conduit an elementary area df. The mass flow rate through df is $dG = \rho w_x\, df$, kg/s. The amount of heat transferred by convection through df per unit time will be equal to:

$$dQ_x = \rho w_x\, i\, df$$

Integrating over the entire cross-section, we obtain the amount of heat transported per unit time through the given cross-section with coordinate x:

$$Q_x = \int_0^{f_0} \rho w_x i\, df \qquad \text{(a)}$$

Let us select the mean specific enthalpy $\bar{i}$ so that the equality

$$Q_x = \bar{i} \int_0^{f_0} \rho w_x \, df = \bar{i} G \tag{b}$$

will be valid.

It follows from equations (a) and (b) that

$$\bar{i} = \frac{\int_0^{f_0} \rho w_x i \, df}{\int_0^{f_0} \rho w_x \, df} = \frac{1}{G} \int_0^{f_0} \rho w_x i \, df \tag{6-1}$$

The mean enthalpy determined by Eq. (6-1) is called the mass mean enthalpy over the cross-section of flow, and the temperature $\bar{t}$, corresponding to that enthalpy, is the mass mean temperature of fluid over the cross-section of flow.

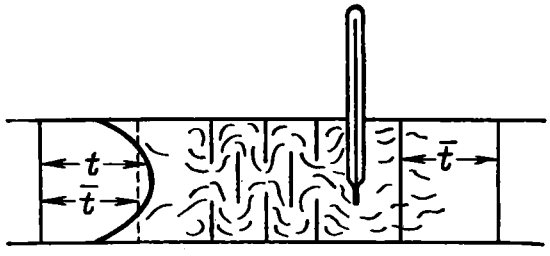

Fig. 6-2. Experimental determination of the enthalpy-mean temperature of a fluid

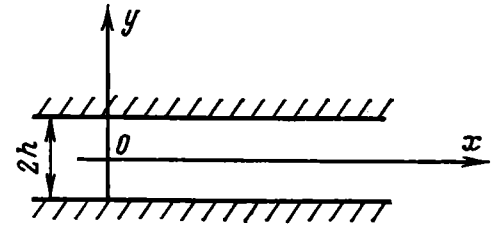

Fig. 6-3. Determining heat flux due to the energy equilibrium in fluid

If the variation of ρ and c_p in the given cross-section is negligible, formula (6-1) acquires the following form:

$$\bar{t} = \frac{1}{V} \int_0^{f_0} w_x t \, df \tag{6-2}$$

where $V = G/\rho$ is the volumetric rate of fluid flow in cubic metres per second.

If the velocity of flow is also constant over the cross-section, the averaging formula acquires the following form:

$$\bar{t} = \frac{1}{f_0} \int_0^{f_0} t \, df \tag{6-3}$$

The enthalpy-mean temperature can be determined more simply by experiment, by placing a mixer in the duct. Temperature evens out downstream of the mixer, becoming equal to the enthalpy-mean temperature, and can be determined by taking measurements at a point (see Fig. 6-2).

6-3. Determining the Heat Flux from the Fluid Energy Balance

Let us consider laminar flow of an incompressible liquid in a flat slot whose height $2h$ is much smaller than its width b. The enthalpy and velocity fields will be assumed to be symmetrically arranged with respect to plane xz (Fig. 6-3). The

distribution symmetry for enthalpy and velocity predetermines the symmetry of the temperature field. It also follows from the assumed symmetry of the problem that at $y = 0$ the components of the vector representing the rate of heat flow, $g_{y\,cond} = -\lambda \partial t/\partial y$ and $q_{y\,conv} = \rho w_y i$, are equal to zero. In the regions $(0, +h)$ and $(0, -h)$ the components of the rate of flow q have respectively different signs, but they are equal modulo at the same value of $|y|$. Plane xz is an adiabatic surface.

The assumed conditions also permit us to ignore the derivatives with respect to z (we are considering the so-called "plane flow").

For the case being considered the equation of energy takes the form:

$$\rho\left(\frac{\partial i}{\partial \tau} + w_x \frac{\partial i}{\partial x} + w_y \frac{\partial i}{\partial y}\right) = \frac{\partial}{\partial x}\left(\lambda \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial t}{\partial y}\right) + q_v$$

Adding to the left-hand side $\rho i\,(\partial w_x/\partial x + \partial w_y/\partial y) = 0$, we get:

$$\rho \frac{\partial i}{\partial \tau} + \rho w_x \frac{\partial i}{\partial x} + \rho i \frac{\partial w_x}{\partial x} + \rho w_y \frac{\partial i}{\partial y} + \rho i \frac{\partial w_y}{\partial y} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial t}{\partial y}\right) + q_v$$

or

$$\rho \frac{\partial i}{\partial \tau} + \frac{\partial}{\partial x}(\rho w_x i) + \frac{\partial}{\partial y}(\rho w_y i) = \frac{\partial}{\partial x}\left(\lambda \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial t}{\partial y}\right) + q_v$$

Let us multiply the left- and right-hand sides of the last equation by dy and integrate between the limits $y = 0$ and $y = h$:

$$\int_0^h \rho \frac{\partial i}{\partial \tau}\,dy + \int_0^h \frac{\partial (\rho w_x i)}{\partial x}\,dy + \int_0^h \frac{\partial (\rho w_y i)}{\partial y}\,dy =$$

$$= \int_0^h \frac{\partial}{\partial x}\left(\lambda \frac{\partial t}{\partial x}\right) dy + \int_0^h \frac{\partial}{\partial y}\left(\lambda \frac{\partial t}{\partial y}\right) dy + \int_0^h q_v\,dy \qquad (6\text{-}4)$$

The third integral in the left-hand side is equal to zero since at $y = h$ $w_y = 0$ due to wall impermeability, at $y = 0$ $w_y = 0$ owing to the symmetry of fields.

Let us calculate now the second integral in the right-hand side of Eq. (6-4):

$$\int_0^h \frac{\partial}{\partial y}\left(\lambda \frac{\partial t}{\partial y}\right) dy = \lambda \frac{\partial t}{\partial y}\Big|_0^h = \lambda \left(\frac{\partial t}{\partial y}\right)_{y=h} = -q_w$$

Since τ, x and y are independent variables, the sequence of differentiation with respect to τ and x and integration with respect to y can be changed. As a result we can write:

$$q_w = -\left[\frac{\partial}{\partial \tau}\int_0^h \rho i\,dy + \frac{\partial}{\partial x}\int_0^h \left(\rho w_x i - \lambda \frac{\partial t}{\partial x}\right) dy - \int_0^h q_v\,dy\right] \qquad (6\text{-}5)$$

Let us multiply and divide the right-hand side by the perimeter $u \approx b$. Taking into account that an elementary area of the cross-section df is equal to $b \cdot dy$, the latter equation can be presented in the form:

$$q_w = -\frac{1}{u}\left[\frac{\partial}{\partial \tau}\int_0^{f_0} \rho i\,df + \frac{\partial}{\partial x}\int_0^{f_0} \left(\rho w_x i - \lambda \frac{\partial t}{\partial x}\right) df - \int_0^{f_0} q_v\,df\right] \qquad (6\text{-}6)$$

where f_0 is the total area of the cross-section corresponding to the calculation perimeter u.

As distinguished from Eq. (6-5), Eq. (6-6) is true for conduits of any lengthwise-constant cross-sectional area. The first item in the right-hand side of Eq. (6-6) accounts for the accumulation of heat in a transient process, the second allows for axial heat transport by convection and conduction, the third – for the evolution of heat from inner heat sources.

The heat flux flowing through the walls of a tube l long is determined in the following way:

$$Q_w = \int_0^l q_w u \, dx \tag{6-7}$$

If $q_v = 0, | \rho w_x i | \gg \left| \lambda \frac{\partial t}{\partial y} \right|$ and the process is of the steady-state kind, it follows from Eq. (6-6) that

$$q_w = -\frac{1}{u} \left(\frac{d}{dx} \int_0^{f_0} \rho w_x i \, df \right) = -\frac{1}{u} dQ_x/dx$$

Then

$$Q_w = -\int_0^l \frac{1}{u} \frac{dQ_x}{dx} \quad \text{and} \quad dx = Q_{x=0} - Q_{x=l} = (\bar{i}G)_{x=0} - (\bar{i}G)_{x=l}$$

or, since $G = \text{const}$,

$$Q_w = G (\bar{i}_{x=0} - \bar{i}_{x=l}) \tag{6-8}$$

If $c_p = \text{const}$, Eq. (6-8) can be presented in the following form

$$Q_w = Gc_p (\bar{t}_{x=0} - \bar{t}_{x=l}) \tag{6-9}$$

The local rate of heat flow is

$$q_w = \frac{Gc_p}{u} \frac{d\bar{t}}{dx} \tag{6-10}$$

Equations (6-8) and (6-9) are widely used in practical calculations. They are valid only for comparatively simple processes, and in more general cases use should be made of Eqs. (6-6) and (6-7). It will be noted that the application of Eqs. (6-8) and (6-9) is limited not only by the requirement that the cross-sectional area of flow should be constant. Equation (6-8) is also valid for turbulent flow of fluid.

6-4. Measuring the Rate of Heat Flow, Fluid and Wall Temperature Along a Tube

Let us find the distribution of the mass mean temperature t, averaged over the cross-section of flow, along a tube (the averaging sign is omitted). The distribution of the local heat-transfer coefficient $\alpha = \alpha(x)$ and of wall temperature $t_w = t_w(x)$ is assumed to be known. According to Eq. (6-10), for an element of the tube dx long we can write:

$$q_w(x) \, u dx = \alpha(x) [t_w(x) - t] \, u dx = Gc_p dt \tag{6-11}$$

where, as assumed, G, c_p and u do not depend on the coordinate x, measured as before, from the beginning of the tube. Eq. (6-11) will be presented in the

following form:

$$\frac{dt}{dx}+f(x)\,t=g(x);\quad f(x)\equiv\frac{\alpha(x)\,u}{Gc_p};\quad g(x)\equiv f(x)\,t_w(x) \tag{6-12}$$

Let us use the Lagrangian method of variation of the arbitrary constant $c(x)$, according to which the solution must be of the form

$$t=c(x)\exp\Big[-\int_0^x f(x)\,dx\Big] \tag{6-13}$$

Substituting the value of t from Eq. (6-13) into Eq. (6-12), we obtain

$$\exp\Big[-\int_0^x f(x)\,dx\Big]\frac{dc(x)}{dx}-c(x)\,f(x)\exp\Big[-\int_0^x f(x)\,dx\Big]+$$

$$+c(x)\,f(x)\exp\Big[-\int_0^x f(x)\,dx\Big]=g(x)$$

or after cancellations

$$\frac{dc(x)}{dx}=g(x)\exp\Big[\int_0^x f(x)\,dx\Big]$$

Separating the variables and integrating between the limits from $x=0$ to x, we obtain

$$c(x)=\int_0^x g(x)\exp\Big[\int_0^x f(x)\,dx\Big]dx+c(0)$$

where $c(0)$ is the value of the arbitrary constant at $x=0$. Upon substitution of $c(x)$ into Eq. (6-13) we find that

$$t=\Big\{c(0)+\int_0^x g(x)\exp\Big[\int_0^x f(x)\,dx\Big]\Big\}\exp\Big[-\int_0^x f(x)\,dx\Big]$$

Let us designate fluid temperature at the tube inlet $t(0)$ by t_0.
At $x=0$ from the last equation it follows that

$$t_0=\Big\{c(0)+\int_0^0 g(x)\exp\Big[\int_0^0 f(x)\,dx\Big]dx\Big\}\exp\Big[-\int_0^0 f(x)\,dx\Big]=c(0)$$

Substituting the value of $c(0)$, we obtain:

$$t=\Big[t_0+\int_0^x g(x)\,e^{\varphi(x)}\,dx\Big]e^{-\varphi(x)} \tag{6-14}$$

The notations in Eq. (6-14) are:

$$\varphi(x)=\int_0^x f(x)\,dx=\int_0^x\frac{\alpha(x)\,u}{Gc_p}dx,\qquad g(x)=\frac{\alpha(x)\,u}{Gc_p}\,t_w(x) \tag{6-15}$$

Let us consider some private cases of Eq. (6-14). Let $t_w = \text{const}$. Allowing for

$$\frac{d\varphi(x)}{dx} = \frac{d}{dx}\int_0^x f(x)\,dx = f(x) \quad \text{and} \quad \varphi(0) = \int_0^0 f(x)\,dx = 0$$

we obtain from Eq. (6-14):

$$t = \left[t_0 + \int_0^x f(x)\,t_w e^{\varphi(x)}\,dx\right] e^{-\varphi(x)} = \left[t_0 + t_w \int_0^x \frac{d\varphi}{dx} e^{\varphi(x)}\,dx\right] e^{-\varphi(x)} =$$

$$= \left[t_0 + t_w \int_0^{\varphi(x)} e^{\varphi(x)}\,d\varphi\right] e^{-\varphi(x)} = [t_0 + t_w (e^{\varphi(x)} - 1)]\, e^{-\varphi(x)} = t_w + (t_0 - t_w)\, e^{-\varphi(x)}$$

Denoting $\vartheta = t - t_w$ (and, consequently, $\vartheta_0 = t_0 - t_w$, where ϑ_0 is the initial temperature difference), the obtained result can be presented in the following form:

$$\vartheta = \vartheta_0 e^{-\varphi(x)} = \vartheta_0 \exp\left[-\int_0^x \frac{\alpha(x)\,u}{Gc_p}\,dx\right] \tag{6-16}$$

Equation (6-16) describes the variation along the tube both of the mass fluid temperature, averaged over the cross-section of flow and of the temperature difference (at $t_w = \text{const}$). If α is also constant, then

$$\varphi(x) = \int_0^x \frac{\alpha u}{Gc_p}\,dx = \frac{\alpha u}{Gc_p}\,x = Ax$$

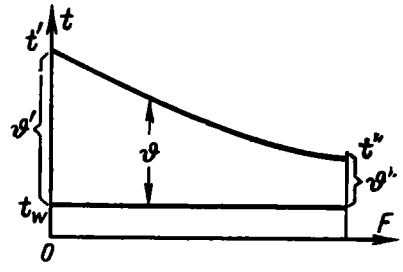

Fig. 6-4. Variation of temperature drop ϑ along a tube with $t_w = \text{const}$ and $\alpha = \text{const}$

The variation of fluid temperature (of the temperature difference), corresponding to the conditions $t_w = \text{const}$ and $\alpha = \text{const}$ is illustrated in Fig. 6-4.

If with given $\alpha(x)$ and $t_w(x)$ the dependence $t(x)$ is found, Newton's law can be used to determine easily the distribution $q_w(x)$. In the special case when $\alpha = \text{const}$ and $t_w = \text{const}$ we have:

$$q_w(x) = \alpha\,[t_w - t(x)] = -\alpha\vartheta = -\alpha\vartheta_0 e^{-Ax}$$

The variation of q_w is similar to the variation of fluid temperature depicted in Fig. 6-4.

If the dependences $\alpha(x)$ and $q_w(x)$ are known or they were determined beforehand, it follows from Eq. (6-10) that

$$dt = \frac{u}{Gc_p}\,q_w(x)\,dx$$

or

$$\int_{t_0}^{t} dt = \frac{u}{Gc_p}\int_0^x q_w(x)\,dx$$

As a result the mass mean fluid temperature, averaged over the cross-section of flow, is described by equation

$$t = t_0 + \frac{u}{Gc_p} \int_0^x q_w(x)\,dx \tag{6-17}$$

For the special case q_w = const it follows from Eq. (6-17) that

$$t = t_0 + \frac{q_w u}{Gc_p} x \tag{6-18}$$

i.e. that the variation of fluid temperature along the tube is linear. If the local heat-transfer coefficient α is also const then from Newton's law it follows that

$$t_w(x) - t(x) = \frac{q_w}{\alpha} = \text{const}$$

i.e. the temperature difference does not vary along the tube.

Since at q_w = const temperature t is a linear function of the coordinate x, the variation of wall temperature t_w is also linear. In a more general case, when $\alpha = \alpha(x)$ and $q_w = q_w(x)$, from Newton's law and Eq. (6-17) we yield:

$$t_w = t(x) + \frac{q_w(x)}{\alpha(x)} = t_0 + \frac{q_w(x)}{\alpha(x)} + \frac{u}{Gc_p} \int_0^x q_w(x)\,dx \tag{6-19}$$

6-5. Averaging Heat-transfer Coefficients and Temperature Difference

Calculation of heat transfer often requires the knowledge of surface-averaged heat-transfer coefficients. The mean heat-transfer coefficient α is determined from Newton's law stating that

$$\bar{\alpha} = \frac{Q_w}{\overline{\Delta t} F} = \frac{\bar{q}_w}{\overline{\Delta t}} \tag{6-20}$$

Calculating the integral mean values of the rate of heat flow $\bar{q}_w$ and temperature difference $\overline{\Delta t}$, formula (6-20) can be presented in the following form

$$\alpha = \frac{\bar{q}_w}{\overline{\Delta t}} = \frac{\frac{1}{F_0}\int_0^{F_0} q_w\,dF}{\frac{1}{F_0}\int_0^{F_0} \Delta t\,dF} = \frac{\int_0^{F_0} \alpha \Delta t\,dF}{\int_0^{F_0} \Delta t\,dF} \tag{6-21}$$

where F_0 denotes the surface over which α and q_w are averaged. If the local heat-transfer coefficient α varies only along one coordinate axis, then

$$\bar{\alpha} = \frac{\int_0^{x_0} \alpha \Delta t\,dx}{\int_0^{x_0} \Delta t\,dx} \tag{6-21'}$$

The integral mean local heat-transfer coefficient is often determined as:

$$\bar{\alpha} = \frac{1}{F_0}\int_0^{F_0} \alpha\,dF \quad \text{or} \quad \bar{\alpha} = \frac{1}{x_0}\int_0^{x_0} \alpha\,dx \tag{6-22}$$

Averaging by formulas (6-21) and (6-22) may yield various results, with the difference reaching in certain cases several tens of per cent.

If $\Delta t = t_w - t_f = \text{const}$, formula (6-21) turns into formula (6-22) and the latter equation can be considered as a special case of Eq. (6-21).

At present both averaging methods are used in heat-transfer calculations when $\Delta t \neq \text{const}$. Preference should be given to the first method, expressed by Eq. (6-21). When $\Delta t \neq \text{const}$ the use of the integral mean heat-transfer coefficient makes it necessary to introduce in the calculation a specially chosen mean temperature difference, and only then a correct value of the heat flux can be obtained.

Below, average values of the heat-transfer coefficient α and of the Nusselt number (and also of other quantities) will be denoted by a horizontal mark placed on top of the letter symbol, $\overline{\alpha}$ and $\overline{\mathrm{Nu}}$.

When the heat-transfer coefficient is averaged over the entire surface considered, $\overline{\alpha}$ will not depend on coordinates, but if the coefficient is averaged for individual sections of the surface, then the average values obtained may vary in general from section to section.

The equations derived in the preceding section permit in principle to calculate the integral mean temperature difference

$$\overline{\Delta t} = \frac{1}{F_0} \int_0^{F_0} \Delta t \, dF \quad \text{or} \quad \overline{\Delta t} = \frac{1}{x_0} \int_0^{x_0} \Delta t \, dx \tag{6-23}$$

which is required to calculate heat transfer by formula (6-21). In general, however, calculation of the integral mean temperature difference may involve considerable difficulties (especially when the mean heat-transfer coefficients are determined by experiment).

The mean local heat-transfer coefficients are therefore often determined by Eq. (6-20), using the *arithmetic mean*

$$\overline{\Delta t}_{ar} = \frac{\Delta t_1 + \Delta t_2}{2} \tag{6-24}$$

or the *logarithmic mean*

$$\overline{\Delta t}_{log} = \frac{\Delta t_1 - \Delta t_2}{\ln \Delta t_1 / \Delta t_2} \tag{6-25}$$

temperature differences (where Δt_1 and Δt_2 are respectively the local temperature difference at the beginning and end of the averaging section). The mean temperature differences $\overline{\Delta t}_{ar}$ and $\overline{\Delta t}_{log}$ are special cases of the integral mean temperature difference, and the use of $\overline{\Delta t}_{ar}$ and $\overline{\Delta t}_{log}$ is generally a convention.

Let us derive formula (6-25). Let $t_w = \text{const}$. The local temperature difference then is determined from Eq. (6-16). As a result

$$\overline{\Delta t} = \overline{\vartheta} = \frac{1}{x} \int_0^x \vartheta \, dx = \frac{1}{x} \int_0^x \vartheta_0 \exp\left[- \int_0^x f(x) \, dx \right] \tag{6-26}$$

where $f(x) = \alpha(x) u / G c_p$. Introducing the integral mean value of the heat-transfer coefficient for the section $0 - x$, we can write

$$\frac{\overline{\alpha} u}{G c_p} x = \overline{f} x = \int_0^x f(x) \, dx$$

Substituting the above expression into Eq. (6-26) and integrating, we get:

$$\bar{\vartheta} = \frac{1}{x}\int_0^x \vartheta_0 e^{-\bar{f}x}\,dx = -\frac{\vartheta_0}{x}\left|\frac{e^{-\bar{f}x}}{\bar{f}}\right|_0^x = -\frac{\vartheta_0}{x\bar{f}}(e^{-\bar{f}x}-1)$$

From equation $\vartheta = \vartheta_0 \exp(-\bar{f}x)$ it follows that

$$\frac{\vartheta}{\vartheta_0} = e^{-\bar{f}x} \quad \text{and} \quad \ln\frac{\vartheta}{\vartheta_0} = -\bar{f}x$$

Substituting these values into the equation for $\bar{\vartheta}$, we get:

$$\bar{\vartheta} = \frac{\vartheta_0}{\ln \vartheta/\vartheta_0}\left[\frac{\vartheta}{\vartheta_0}-1\right] = \frac{\vartheta_0-\vartheta}{\ln \vartheta_0/\vartheta} = \frac{\Delta t_1-\Delta t_2}{\ln \Delta t_1/\Delta t_2} = \overline{\Delta t}_{log}$$

Thus, the logarithmic mean temperature difference corresponds to the integral mean provided $t_w = \text{const}$ and the heat-transfer coefficient is averaged by Eq. (6-22) (or $\alpha = \text{const}$). The other limitations, assumed when deriving formulas (6-10) or (6-16), also hold.

Let us compare $\overline{\Delta t}_{ar}$ and $\overline{\Delta t}_{log}$. The shaded areas on the graph shown in Fig. 6-5 correspond to the exponential and linear laws governing fluid temperature variation along the heat-transfer surface at $t_w = \text{const}$ (curves $t't''$ and tt''). The shaded areas are proportional respectively to the mean temperature difference $\overline{\Delta t}_{ar}$ and $\overline{\Delta t}_{log}$. It follows from the comparison that $\overline{\Delta t}_{ar} > \overline{\Delta t}_{log}$.

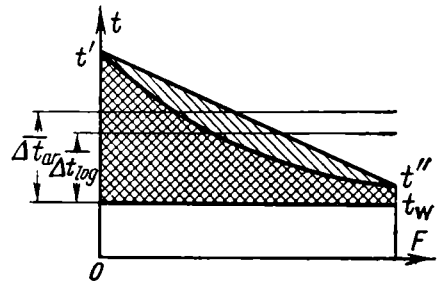

Fig. 6-5. Comparison of the logarithmic and arithmetic mean temperature differences

If $\Delta t_2/\Delta t_1 > 0.5$, then with an accuracy sufficient for most heat-transfer calculations the logarithmic mean temperature difference $\overline{\Delta t}_{log}$ corresponds to the arithmetic mean temperature difference $\overline{\Delta t}_{ar}$ (the difference between them is less than 4 per cent).

As follows from the material treated in this section the numerical values of $\bar{\alpha}$ may depend on the method of determination and, in particular, on the choice of the calculation value for $\overline{\Delta t}$. When obtaining $\bar{\alpha}$, the method of determining it must be indicated. Unfortunately, data of this kind are not always given in published works.

If, for instance, the error admitted in determining α by experiment exceeds the possible inaccuracy, due to the uncertainty of the mean temperature difference $\overline{\Delta t}$, this uncertainty is immaterial.

6-6. Derivation of Empirical Equations

The formulas cited in the preceding sections are used for the primary treatment of heat-transfer data obtained by measurement.

Before experimental results are treated in dimensionless terms, it is necessary to determine the reference or known dimensionless terms on which the unknown value depends. This can be done by means of the method previously described by writing the system of differential equations that describes the process being investigated experimentally, and formulating the conditions of unambiguity.

The mathematical description of the process involved is then expressed in dimensionless form. Let us assume that it follows from the mathematical description that

$$\mathrm{Nu} = f(\mathrm{Re}, \mathrm{Pr})$$

The magnitudes of Re and Pr and the corresponding values of the Nusselt number are calculated from measurement data. The dependence between dimensionless terms is usually represented in power functions, for example:

$$\mathrm{Nu} = c\mathrm{Re}^n\mathrm{Pr}^m$$

where c, n and m are constant dimensionless quantities.

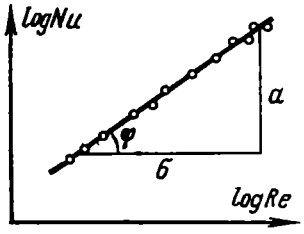

Fig. 6-6. Determining $\mathrm{Nu} = c\mathrm{Re}^n$ function

Relationships of this kind are purely empirical, and are applicable only within the limits of the argument in which they have been substantiated experimentally.

Assume that the Nusselt number depends only on the Reynolds number (or that the Reynolds number of the heat-carrying agent in the experiment is constant and independent of temperature), then

$$\mathrm{Nu} = c\mathrm{Re}^n$$

Taking the logarithm of this equation, we obtain:

$$\log \mathrm{Nu} = \log c + n \log \mathrm{Re}$$

Denoting log Nu by Y, log Re by X and log c by A, we get:

$$Y = A + nX$$

which is the equation of a straight line. The exponent n is the slope of the curve with respect to the x-axis; n consequently can be determined by a graphical representation of test results in coordinates log Nu $= f$ (log Re) (Fig. 6-6). The exponent n is equal to:

$$n = \tan\varphi = \frac{a}{b}$$

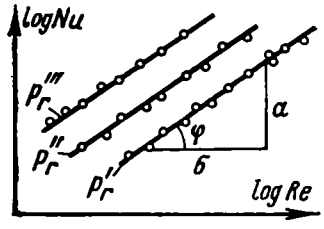

Fig. 6-7. Determining $\mathrm{Nu} = c\mathrm{Re}^n\mathrm{Pr}^m$ function

The constant c is determined from the equation $c = \frac{\mathrm{Nu}}{\mathrm{Re}^n}$ which holds for any point of the straight line.

The power relationship is proved applicable, if a check shows that all points fall on the straight line with logarithmic coordinates. If, however, they fall on a curve, the latter is usually replaced by a broken line, and the values of c and n are different for its individual sections.

When the unknown Nusselt number is a function of two arguments, $\mathrm{Nu} = f(\mathrm{Re}, \mathrm{Pr})$, for instance, the graph is a family of straight lines, and the second argument is taken as a parameter (Fig. 6-7). Further, after the exponent is found from one of the straight lines at one value of Re, the test data are shown

in the graph by the dependence $\log \frac{Nu}{Re^n} = f(\log Pr)$. The power m of the Prandtl number is determined from this graph, and then the factor c is found by equation $c = \frac{Nu}{Re^n Pr^m}$

Experimental data are treated with the aid of electronic computers. Based on mathematical statistics, the constants c, n, m, etc. can be calculated. There exist special standard calculation programs for digital computers, facilitating the work of investigators.

Lately the semi-empirical method has been extensively used in deriving dimensionless equation. The dependence between dimensionless terms is presented as a function obtained preliminarily with an accuracy to within the constants obtained by an analytical solution of the problem. Then the constants are defined more accurately with the aid of experimental results. This way of deriving dimensionless equations should be preferred over the empirical.

Reference dimension. The dimensionless terms of similarity contain the reference dimension l_0. The theory of similarity does not define unambiguously which of the involved dimensions is to be taken as reference, i.e. which dimension is to appear in the dimensionless terms. If several dimensions are given unambiguously, the one which complies with a greater degree with the physical essence of a process is taken as the reference dimension. The other dimensions are usually presented in the equation of similarity in the form of simplexes, $L_1 = l_1/l_0$, $L_2 = l_2/l_0$, etc.

In a number of cases, taken as the reference linear dimension is a combination of unlike physical quantities appearing in the conditions of unambiguity. The dimension of a combination of this kind is a linear quantity and proportional to some linear dimension.

Reference temperature. The dimensionless terms contain physical parameters (properties) of fluid. When composing dimensionless terms the physical properties are often assumed to be constant. Actually, inasmuch as the temperature of fluid varies, the values of its physical properties vary as well. Therefore, when treating experimental data, it is very important to select the so-called reference temperature by which are determined the physical parameters appearing in the dimensionless terms or criteria of similarity.

Experimental and theoretical research indicates that there is no universal reference temperature that will automatically account for the dependence of heat transfer on variation of the physical properties of fluid. Consequently, the predominating view is now that it is proper to take as the reference temperature either the one given or the temperature which can be easily calculated.

In calculations the reference temperature and the linear dimension should be chosen in exactly the same way as when deriving formulae, and if this is ignored, there may be considerable errors.

CHAPTER

7

HEAT TRANSFER FOR A FLAT PLATE IN FORCED LONGITUDINAL FLOW

We assume for the sake of simplicity that the flat plate is exposed to a fluid flow and that velocity and temperature of the fluid which are constant beyond the hydrodynamic and thermal boundary layers and equal respectively to w_0 and t_0.

The flow is directed along the plate, the temperature of the surface of the body does not change in time. There are no inner heat sources in the field, frictional heat is negligibly small.

7-1. Integral Equations of the Boundary Layer

Suggested in Chapter 4 was a simplified form of the differential equation of energy for the thermal boundary layer (4-30). Allowing for $q_y = -\lambda \partial t/\partial y$ and, consequently, $\lambda \partial^2 t/\partial y^2 = -\partial q_y/\partial y$, let us present Eq. (4-30) in the form

$$\rho c_p \left(w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} \right) = -\frac{\partial q_y}{\partial y} \tag{7-1}$$

Let us integrate Eq. (7-1) between the limits from $y = 0$ to $y = \infty$. It will be recalled that beyond the limits of the boundary layer the derivatives appearing in Eq. (7-1) are equal to zero by definition (Sec. 4-4). The increase of the upper limit k to ∞, therefore, does not vary the integral. Upon integration of the right-hand side of the equation we obtain:

$$-\int_0^\infty \frac{\partial q_y}{\partial y}\, dy = q_w \tag{a}$$

where account taken of the fact that $(q_y)_{y=\infty} = -\lambda \left(\frac{\partial t}{\partial y}\right)_{y=\infty} = 0$ (Sec. 4-4).

Before integrating the left-hand side of Eq. (7-1) let us find w_y from the equation of continuity (4-29), according to which

$$dw_y = -\frac{\partial w_x}{\partial x}\, dy$$

taking into account that at $y = 0$ $w_y = 0$ due to wall impermeability, we obtain:

$$w_y = -\int_0^y \frac{\partial w_x}{\partial x}\, dy \tag{7-2}$$

Substituting the found value of w_y into Eq. (7-1) and integrating the left-hand side of this equation, we obtain:

$$\int_0^\infty \left(w_x \frac{\partial t}{\partial x} + w_y \frac{\partial t}{\partial y} \right) dy = \int_0^\infty w_x \frac{\partial t}{\partial x}\, dy - \int_0^\infty \frac{\partial t}{\partial y} \left(\int_0^y \frac{\partial w_x}{\partial x}\, dy \right) dy \tag{b}$$

The second integral in the right-hand side of Eq. (b) can be taken by parts the formula for it being:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Then,

$$\int_0^\infty \frac{\partial t}{\partial y}\left(\int_0^y \frac{\partial w_x}{\partial x}\,dy\right)dy = \int_0^\infty \left(\int_0^y \frac{\partial w_x}{\partial x}\,dy\right)dt =$$

$$= t\left(\int_0^y \frac{\partial w_x}{\partial x}\,dy\right)\Big|_0^\infty - \int_0^\infty t\frac{\partial w_x}{\partial x}\,dy = t_0\int_0^\infty \frac{\partial w_x}{\partial x}\,dy - \int_0^\infty t\frac{\partial w_x}{\partial x}\,dy =$$

$$= \int_0^\infty (t_0 - t)\frac{\partial w_x}{\partial x}\,dy \qquad \text{(c)}$$

Let us substitute expression (c) into equation (b). Inasmuch as the limits of integration do not depend on x, the sequence followed in differentiating in respect to x and integrating in respect to y can be changed. Allowing for this, we obtain:

$$\int_0^\infty w_x\frac{\partial t}{\partial x}\,dy - \int_0^\infty (t_0 - t)\frac{\partial w_x}{\partial x}\,dy = -\int_0^\infty \frac{\partial}{\partial x}\left[w_x(t_0 - t)\right]dy =$$

$$= -\frac{d}{dx}\int_0^\infty w_x(t_0 - t)\,dy \qquad \text{(d)}$$

Equating (a) and (d) and passing from the integration limit ∞ to k, we obtain the following integral-differential equation:

$$\frac{d}{dx}\int_0^k w_x(t_0 - t)\,dy = \frac{-q_w}{\rho c_p} \qquad (7\text{-}3)$$

This equation is called the *integral equation of heat flow for the thermal boundary layer*. The integral and the heat flow rate q_w in the left-hand side of the equation are only functions of x. For approximate calculations the functions $w_x = w_x(y)$ and $t = t(y)$ are usually given, based on experience. It will be noted that the left-hand side of Eq. (7-3) is not very affected by some inaccurate selection of the distribution of $w_x(y)$ and $t(y)$. If the distribution of velocity and temperature is known, then Eq. (7-3) can be used to determine $k = k(x)$. In illustration, a solution of this kind will be exhibited in the following section.

The equation of flow for the case under consideration, expressed in projections on the Ox-axis, was written approaching the boundary layer, Chapter 4, Eq. (4-28). Taking into account that $s = \mu(\partial w_x/\partial y)$, let us represent Eq. (4-28) in the following form:

$$\rho\left(w_x\frac{\partial w_x}{\partial x} + w_y\frac{\partial w_x}{\partial y}\right) = \frac{\partial s}{\partial y} \qquad (7\text{-}4)$$

Comparing Eqs. (7-1) and (7-4), we see that they are similar. In consequence, integrating Eq. (7-4) between the limits from $y = 0$ to $y = \infty$ (or δ) and carrying out similar transformations, we obtain similar results.

The integral equation of momentum for the hydrodynamic boundary layer can be presented in the following form:

$$\frac{d}{dx}\int_0^{\delta} w_x(w_0-w_x)\,dy=\frac{s_w}{\rho} \tag{7-5}$$

where s_w is the tangential stress due to friction at $y = 0$, i.e. on the surface of the wall.

The integral equations of the thermal and hydrodynamic boundary layers, (7-3) and (7-5), are true provided the previously assumed conditions are fulfilled. In the general case the integral equations associated with it become more intricate as well.

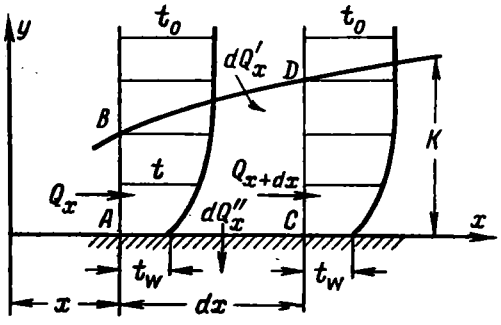

Fig. 7-1. Deriving the integral equation of heat flow

Let us give a physical meaning to the integrals in the left-hand sides of Eqs. (7-3) and (7-5). Consider an infinitely small volume in the thermal boundary layer confined between two sections spaced dx apart (Fig. 7-1). The distance between the two planes confining the considered volume, parallel to the plane of the drawing, is assumed to be unity. A similar control volume will also be considered in the flow for the hydrodynamic boundary layer.

The mass rate of fluid flow through a certain section of the boundary layer and the variation of this rate of flow per unit will respectively be equal to:

$$G(x)=\int_0^{\delta\,(\text{or } k)}\rho w_x\,dy \quad \text{and} \quad \frac{dG}{dx}=\frac{d}{dx}\int_0^{\delta\,(\text{or } k)}\rho w_x\,dy$$

Transported together with the mass of fluid are momentum $J(x)$ and enthalpy $Q(x)$. The variation of J and Q per unit length is determined respectively by equations

$$\frac{dJ}{dx}=\frac{d}{dx}\int_0^{\delta}\rho w_x^2\,dy \quad \text{and} \quad \frac{dQ}{dx}=\frac{d}{dx}\int_0^{k}\rho w_x c_p t\,dy$$

These changes are associated with the transport of momentum $J'(x)$ and enthalpy $Q'(x)$ through the outer border of the boundary layers ($y = \delta$, $y = k$) together with the mass of fluid entrained into the flow in the boundary layer (Fig. 7-1):

$$\frac{dJ'}{dx}=\frac{d}{dx}\int_0^{\delta}\rho w_x w_0\,dy \quad \text{and} \quad \frac{dQ'}{dx}=\frac{d}{dx}\int_0^{k}\rho c_p w_x t_0\,dy$$

In addition, the changes in momentum J and enthalpy Q are due to viscous drag and the heat flow rate on the surface of the wall, s_w and q_w.

Equations (7-3) and (7-5) then can be presented respectively in the following form:

$$\frac{dQ'}{dx}-\frac{dQ}{dx}=-q_w \quad \text{and} \quad \frac{dJ'}{dx}-\frac{dJ}{dx}=s_w$$

The integral equation of heat flow (7-3) was first formulated by G. N. Kruzhilin and the momentum equation (7-5)—by T. Karman. These equations are also suitable for the turbulent boundary layer, if by w_x and t are meant the time-averaged values of velocity and temperature. It will be recalled that on a solid impermeable wall ($y = 0$) the equalities $\lambda_t = 0$ and $\mu_t = 0$ must be satisfied and this was accounted for when deriving the equations involved.

7-2. Heat Transfer in a Laminar Boundary Layer

To calculate heat transfer in a laminar boundary layer with the aid of Eq. (7-3), we must know the velocity distribution in the layer. The distribution of velocity in the laminar boundary layer is nearly parabolic, and it is convenient to describe the velocity distribution curve by the equation of a cubic parabola:

$$w_x = a + by + cy^2 + dy^3 \tag{a}$$

The velocity distribution equation must satisfy the boundary conditions. With $y = 0$ the condition $w_x = 0$ (the "adhesion" condition) is satisfied; we also assume that $(\partial^2 w_x/\partial y^2)_{y=0} = 0$. In addition, on the external border of the boundary layer ($y = \delta$) $w_x = w_0$ and $(\partial w_x/\partial y)_{y=\delta} = 0$.

The condition $(\partial^2 w_x/\partial y^2)_{y=0} = 0$ stems from the differential equation of flow (4-28) should we assume that directly at the wall only the forces of viscosity are of importance (i.e. assuming that frictional forces can be ignored).

Equation (a) will conform to the boundary conditions enumerated, if

$$a = 0,\ b = \frac{3}{2}\frac{w_0}{\delta},\quad c = 0 \quad \text{and} \quad d = -\frac{1}{2}\frac{w_0}{\delta^3}$$

The distribution of velocity then will acquire the following form:

$$\frac{w_x}{w_0} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{b}$$

At velocity distribution according to Eq. (b), it follows from the integral equation of momentum (7-5) that the thickness of the hydrodynamic boundary layer is determined by the expression

$$\delta = \sqrt{\frac{280}{13}}\sqrt{\frac{\nu x}{w_0}} \approx 4.64\sqrt{\frac{\nu x}{w_0}} \tag{7-6}$$

Formula (7-6) shows that δ varies proportionally to the square root of the distance from the front edge of the plate to the given point, and can be expressed in dimensionless terms:

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{w_0 x/\nu}} = \frac{4.64}{\sqrt{\mathrm{Re}_x}} \tag{7-7}$$

Assume that the surface temperature of the solid t_w is independent of x, i.e. $t_w = \text{const}$. For convenience temperature readings for fluid will be taken from t_w. Let us denote:

$$\vartheta = t - t_w \quad \text{and} \quad \vartheta_0 = t_0 - t_w$$

where t_0 is the fluid temperature beyond the thermal boundary layer. The boundary conditions here prove to be similar to those previously assumed for the hydrodynamic layer.

Indeed, at $y = 0$, $\vartheta = 0$. Moreover, $(\partial\vartheta/\partial y)_{y=0} = \text{const}$ and $(\partial^2\vartheta/\partial y^2)_{y=0} = 0$, if we take into account that heat is transported only by conduction in the liquid

closely adjoining the wall. The following conditions hold for the outer border of the thermal layer ($y = k$):

$$\vartheta = \vartheta_0 = \text{const} \quad \text{and} \quad (\partial\vartheta/\partial y)_{y=k} = 0$$

Consequently, it can be assumed that the temperature distribution is described by an equation similar in form to that of velocity distribution:

$$\frac{\vartheta}{\vartheta_0} = 1.5\left(\frac{y}{k}\right) - 0.5\left(\frac{y}{k}\right)^3 \tag{c}$$

From this it follows that

$$\frac{d\vartheta}{dy} = \frac{1.5\vartheta_0}{k} - \frac{1.5\vartheta_0}{k^3}\, y^2$$

$$\left(\frac{d\vartheta}{dy}\right)_{y=0} = \frac{1.5\vartheta_0}{k} \tag{d}$$

We take the integral of heat balance equation (7-3), integrating between the limits of the thermal boundary layer $y = 0$ and $y = k$, assuming $k \leqslant \delta$. Integration between the limits from $y = 0$ to $y = k$ is tantamount here to integration between the limits of both the thermal and the hydrodynamic layers.

If the integration were spread to $\delta < k$, it would mean that in the thermal boundary layer velocity distribution was governed by two laws: with $y < \delta$, in accordance with Eq. (b), and with $\delta \leqslant y \leqslant k$, in accordance with the condition $w_x = w_0 = \text{const}$.

Integration gives:

$$\int_0^k (t_0 - t)\, w_x\, \partial y = \int_0^k (\vartheta_0 - \vartheta)\, w_x\, dy =$$

$$= \vartheta_0 w \int_0^k \left[1 - 1.5\left(\frac{y}{k}\right) + 0.5\left(\frac{y}{k}\right)^3\right]\left[1.5\left(\frac{y}{\delta}\right) - 0.5\left(\frac{y}{\delta}\right)^3\right] dy =$$

$$= \vartheta_0 w_0 \delta\left[\frac{3}{20}\left(\frac{k}{\delta}\right)^2 - \frac{3}{280}\left(\frac{k}{\delta}\right)^4\right]$$

Since $k < \delta$, then $\frac{k}{\delta} < 1$ and, therefore, the second bracketed term in the right side of the equation is small, compared with the first term, and can be ignored.

Substituting the numerical magnitude of the integral, and the expression $(d\vartheta/dy)_{y=0}$ from (d) into (7-3), we obtain

$$\frac{3}{20}\,\vartheta_0 w_0 \frac{d}{dx}(\beta^2\delta) = \frac{3}{2}\, a\, \frac{\vartheta_0}{\beta\delta}$$

or

$$\frac{1}{10}\, w_0 \left(\beta^3\delta \frac{d\delta}{dx} + 2\beta^2\delta^2 \frac{d\beta}{dx}\right) = a$$

where $\beta = \frac{k}{\delta}$.

Proceeding from the analogy between the equations of the heat and dynamic boundary layers and analogy between the adopted velocity and temperature distributions (b) and (c), we may assume that the thicknesses of the heat and dynamic layers k and δ depend in a similar way on x and their ratio is a constant*

* This statement is true, provided not only the hydrodynamic, but also the heat flow develop at the very beginning of the plate ($x = 0$), i.e. there is no unheated section at the beginning of the plate.

which is not a function of x. Then, $d\beta/dx = 0$ and, instead of the preceding equation, we get:

$$\frac{1}{10} w_0 \beta^3 \delta \frac{d\delta}{dx} = a$$

From Eq. (7-6) it follows that

$$\delta \frac{d\delta}{dx} = \frac{140}{13} \frac{\nu}{w_0}$$

Substituting this value in the preceding equation and assuming

$$\sqrt[3]{\frac{130}{140}} \approx 0.98 \approx 1,$$

we find that

$$\frac{k}{\delta} = \frac{1}{\sqrt[3]{\mathrm{Pr}}} \tag{7-8}$$

More exact solutions give a similar result.

Substituting δ from Eq. (7-7) in Eq. (7-8), we get:

$$k = \frac{4.64x}{\sqrt{\mathrm{Re}_x}\sqrt[3]{\mathrm{Pr}}} \tag{7-9}$$

where

$$\mathrm{Re}_x = \frac{w_0 x}{\nu}$$

For dropping liquids $\mathrm{Pr} \geqslant 1$ as a rule and, consequently, $k \leqslant \delta$, i.e. the condition assumed in integrating the equation of heat flow is satisfied. The Prandtl number of gases varies between 0.6 and 1.0; for air, for example, $\mathrm{Pr} \approx \approx 0.7$ over a wider range of temperature. Although $k > \delta$, the difference in thickness of the thermal and hydrodynamic boundary layers is small; at a Prandtl number of 0.6, for instance, $k = 1.18\delta$. Experiments show that this difference between k and δ can in practice be ignored.

The results obtained are unsuitable for liquid metals for which $k \gg \delta$.

Determine the heat-transfer coefficient omitting the minus sign in Eqs. (4-22) and (d), we get

$$\alpha = \frac{\lambda}{\vartheta_0} \left(\frac{\partial \vartheta}{\partial y}\right)_{y=0} = \frac{3}{2} \frac{\lambda}{k} \tag{7-10}$$

The heat-transfer coefficient, consequently, is inversely proportional to the thickness of the thermal boundary layer.

Equation (7-10) may be expressed in dimensionless terms. For this purpose we multiply both sides of the equation by x/λ and substitute the value of k from Eq. (7-9). This gives:

$$\mathrm{Nu}_x = 0.33 \sqrt{\mathrm{Re}_x} \sqrt[3]{\mathrm{Pr}} \tag{7-11}$$

where

$$\mathrm{Nu}_x = \frac{\alpha x}{\lambda} = \frac{\alpha l}{\lambda} \frac{x}{l} = \mathrm{Nu}_l X$$

$$\mathrm{Re}_x = \frac{w_0 x}{\nu} = \frac{w_0 l}{\nu} \frac{x}{l} = \mathrm{Re}_l X$$

$\mathrm{Pr} = \frac{\nu}{a}$; l = length of plate along the flow. Eq. (7-11) can be presented as follows:

$$\mathrm{Nu}_l = 0.33 X^{-0.5} \mathrm{Re}_l^{0.5} \mathrm{Pr}^{1/3} \tag{7-12}$$

hence

$$\mathrm{Nu}_l = aX^{-0.5} \quad \text{or} \quad \alpha = cx^{-0.5} \tag{e}$$

where $a = 0.33\mathrm{Re}_l^{0.5}\mathrm{Pr}^{1/3}$ and $c = a\lambda l^{-0.5}$ incorporating the proportionality factor 0.33, velocity w_0, length of plate l, and the physical parameters λ, ν, and a, are independent of x.

In accordance with Eq. (e), the heat-transfer coefficient α is infinitely large at $x = 0$, and becomes of a finite and gradually decreasing magnitude with increasing x (Fig. 7-2). This mode of variation of α is due to the fact that the temperature difference $\vartheta_0 = t_0 - t_w$ does not change along the plate, whereas the wall temperature gradient diminishes continuously with rising x [see Eqs. (d) and (7-9)].

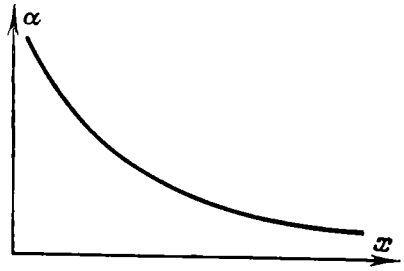

Fig. 7-2. Variation of the heat-transfer coefficient along a plate with a laminar boundary layer

Formula (7-11) is derived on the assumption that the surface temperature of the plate is constant, the physical parameters of fluid independent of temperature, and that there is no unheated section at the beginning of the plate.

Both experience and theory indicate that disregard of these factors leads to substantial errors.

Dependence of heat transfer on variation of physical parameters of fluid. Equation (7-9) was derived on the assumption that all physical parameters are constant and consequently do not change over the cross-section of the layer, in spite of temperature variation. Actually, the physical parameters depend on temperature.

The physical parameters in Eqs. (7-11) and (7-12), including the Prandtl number, are usually decided based on the temperature of the free-stream flow t_0. The physical parameters of various fluids depend on temperature in different ways. The heat-transfer coefficient of liquids consequently depends on the kind of liquid, its temperature, the direction of heat flow, and the magnitude of the temperature difference.

The effect of these factors on heat transfer is the result of temperature variation in the thermal boundary layer and the corresponding variation of the physical parameters that are functions of temperature. The effect caused by variation in viscosity is particularly important.

Numerical calculations of the velocity and temperature fields, taking into consideration variation of viscosity, show that variation of the viscosity of liquid affects both velocity and temperature distribution. Velocity distribution differs, depending on the direction of heat flow, at any temperature difference ϑ_0. The distribution of dimensionless velocities $W_x = \frac{w_x}{w_0}$ and temperatures $\Theta = \frac{t - t_w}{t_0 - t_w}$ over a certain cross-section of the boundary layer at equal Reynolds and Prandtl numbers of the external flow is illustrated in Fig. 7-3.

The temperature of a cooled fluid at the wall is lower than that of a heated fluid and its viscosity is consequently larger. The increase in viscosity results in deceleration of flow, and similarity between temperature and velocity fields is disturbed.

Similar calculations for gas (air), taking the variation of all physical parameters into account, indicate that the temperature and velocity fields change only slightly, but at high wall temperatures and large temperature differences the results are different. Velocity distribution in a heated gas is then qualitatively similar to curve *2* (Fig. 7-3*a*), since the viscosity coefficients of liquids and gases depend on temperature in different ways (Figs. 4-1 and 4-2).

In order to take variation of the physical parameters into account, the system of differential equations describing convection must be altered. In deriving them, the variables of the physical parameters are not to be removed from the sign of the derivative. Equations of the following kind describing the variation of physical parameters with temperature must also be added to the basic system of differential equations:

$$\lambda = f_1(t),\ \mu = f_2(t),\ c_p = f_3(t),$$
$$\text{and } \rho = f_4(t)$$

The physical parameters of gases may be assumed, with sufficient accuracy, to vary along the lines of simple power equations like $\eta = \eta_1 (T/T)^n$, where η_1 is the magnitude of a parameter at a temperature T_1; n is a constant in the temperature range investigated.

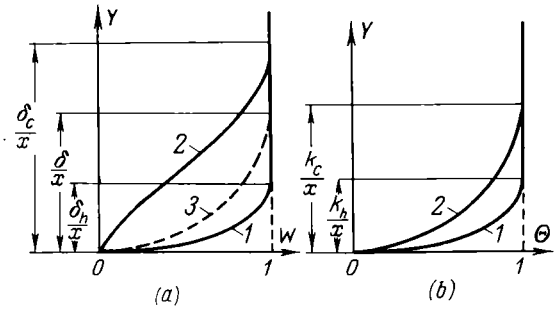

Fig. 7-3. Variation of velocity (*a*) and temperature (*b*) for heated and cooled liquids: *1*—heating; *2*—cooling; *3*—isothermal process

Variation of the physical parameters can then be taken into account by introducing an additional argument $\Theta_w \equiv T_w/T_0$ in the dimensionless equation, where T_0, K, is the temperature of the gas at a distance from the wall, or the mean gas temperature in the passage, and T_w, K, is the surface temperature at the wall. The ratio T_w/T_0 is known as the *temperature factor*.

It has been found by experiment that heat transfer in a gas cooled or heated at small temperature differences is practically independent of the temperature factor, provided the physical parameters are chosen for the temperature of the external flow. At temperature differences of the order of hundreds of degrees, heat transfer in a heated gas depends considerably on the temperature factor.

The dependence of the physical parameters of liquids on temperature is more complex and different in nature. No general and uniform method that takes account of the effect of variation of physical parameters on heat transfer in liquids has yet been elaborated theoretically.

It has been established experimentally that the dependence of heat transfer on the direction of heat flow and the temperature difference can be accounted for by introducing an additional factor $(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25}$ in the dimensionless equation, the subscripts f and w indicating that the corresponding Prandtl numbers are being selected based on both the temperature of the liquid at a distance from the solid and the wall temperature. This correction accounts first of all for the effect of variation of fluid viscosity on heat transfer.

The factor has been suggested by M.A. Mikheev. It was shown later that for some practical conditions the exponent n at the ratio $\mathrm{Pr}_f/\mathrm{Pr}_w$ must be variable, however, the correction $(\mathrm{Pr}_f/\mathrm{Pr}_w)^n$ does not differ materially from that suggested by M.A. Mikheev.

For heated liquids $\frac{\mathrm{Pr}_f}{\mathrm{Pr}_w} > 1$, and for cooled $\frac{\mathrm{Pr}_f}{\mathrm{Pr}_w} < 1$. The $\mathrm{Pr}_f/\mathrm{Pr}_w$ ratio of

a definite liquid in flow differs more widely from unity, the greater the temperature difference.

If $\vartheta_0 \to 0$, then $(\mathrm{Pr}_f/\mathrm{Pr}_w) \to 1$. With q_w given, the equation $q_w = \alpha (t_0 - t)$ shows the temperature difference to be very small, when α is very large. It can then be assumed that $\frac{\mathrm{Pr}_f}{\mathrm{Pr}_w} \approx 1$.

The correction $(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25}$ does not apply to gases.

The process of heat transfer in fluids that are in a state close to critical develops in a special way, then the correction $(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25}$ is not applicable.

Several authors have accounted for variation of physical parameters by introducing into the dimensionless equation the simplexes $\frac{\lambda_f}{\lambda_w}$, $\frac{\mu_f}{\mu_w}$, and $\frac{c_{pf}}{c_{pw}}$, where the subscripts f and w indicate that the corresponding parameters are selected either for the temperature of the fluid at a distance from the solid, or for the temperature of the wall.

Dependence of heat transfer on variation of surface temperature along the plate. A variation in t_w along the plate may seriously affect the process of heat transfer. Variation in wall temperature results in variation in temperature distribution in the thermal boundary layer, in changes in the thickness of the layer, and in the temperature gradient in the fluid at the surface. The heat-transfer coefficient for any point on the plate depends on the development of the boundary layer in the preceding section, and on the variation in wall temperature along these sections. With variation of the physical parameters of fluid this dependence becomes even more complicated.

The variation in surface temperature or in the temperature difference may often be described by the power law:

$$\vartheta_w (x) = Ax^m \tag{7-13}$$

where $\vartheta_w (x) = t_w (x) - t_0$; $t_0 = \text{const}$;
$t_w (x) =$ local surface temperature;
A and $m =$ constants independent of x.

At $m = 0$, $\vartheta_w = A = t_w - t_0 = \text{const}$, which corresponds to the problem previously considered at $t_w = \text{const}$.

A number of researchers [7.1-7.3, etc.] have studied the heat transfer with a non-isothermal plate. Analysis of the results obtained indicates that the thickness of the thermal boundary layer diminishes with increasing m, and that the rate of heat transfer rises.

The effect of the longitudinal surface temperature gradient can be accounted for by the ratio of the rate of heat transfer for a plate with a variable surface temperature ($m \neq 0$) and a constant one ($m = 0$); let us denote this ratio by ε:

$$\varepsilon = \frac{\mathrm{Nu}_{x(m\neq 0)}}{\mathrm{Nu}_{x(m=0)}} \quad \text{or} \quad \mathrm{Nu}_{x(m\neq 0)} = \varepsilon \mathrm{Nu}_{x(m=0)}$$

Values of ε have been determined analytically, and checked experimentally for particalur values of m, and are given in Table 7-1 [7.1].

TABLE 7-1

Dependence of $\varepsilon = f(m)$ at $\mathrm{Pr} > 1$

m	-0.25	0 ($t_w = \text{const}$)	0.1	0.2	0.3	0.4	0.5 ($q_w = \text{const}$)	0.8	1.0	2.0
ε	0.655	1	1.09	1.17	1.25	1.30	1.36	1.52	1.60	1.98

Effect of an unheated initial section. In the presence of an unheated initial section the hydrodynamic and the thermal boundary layers do not develop simultaneously which effects the coefficient of heat transfer. The presence of a surface taking no part in the process of heat transfer corresponds to the special case when the surface temperature of a plate changes along its length.

Extensive experimental studies of the effect of the unheated initial section on heat transfer have been carried out by I.I. Zhyugzhda and A.A. Zhukauskas

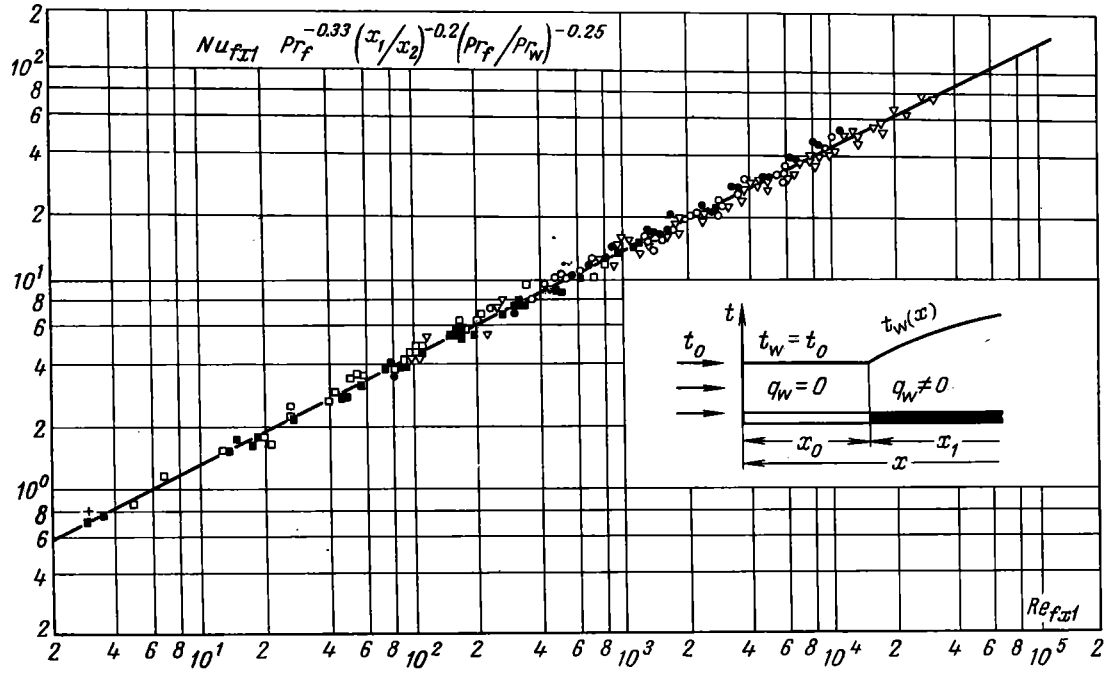

Fig. 7-4. Local heat transfer with a laminar boundary layer and a non-heated initial section, $m = 0.4$ ($\varepsilon = 1.30$)

[7.1]. In the tests conducted the ratio of the length of the initial unheated section x_0 to the total length l was changed from 0.425 to 0.86, with Prandtl numbers changing from 0.7 to 510 (for air, water and transformer oil) and Reynolds numbers Re_{fx1} varying from 3 to 3×10^4 (Fig. 7-4).

The equation derived for calculating the local heat-transfer coefficients for the plate with a laminar boundary layer and an unheated initial section is as follows

$$\mathrm{Nu}_{fx1} = 0.33\varepsilon\, \mathrm{Re}_{fx1}^{0.5}\, \mathrm{Pr}_f^{0.33}\, (x_1/x)^{0.2}\, (\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25} \qquad (7\text{-}14)$$

where the coordinate $x_1 = x - x_0$ substituted into the dimensionless numbers is measured from the beginning of the heated section. The physical parameters are chosen by the temperature of the free-stream flow t_0 which is indicated by the subscript "f" (an exception is the Prandtl number, Pr_w, selected by the wall temperature in the given section). We determine the mean heat-transfer coefficient at $x_0 = 0$:

$$\bar{\alpha} = \frac{\bar{q}}{\Delta t} = \frac{\int_0^l \alpha\vartheta\, dx}{\int_0^l \vartheta\, dx} = \frac{\int_0^l cx^{-0.5} Ax^m\, dx}{\int_0^l Ax^m\, dx} = \frac{2c}{\sqrt{l}} \frac{m+1}{2m+1} = 2\alpha_{x=l} \frac{m+1}{2m+1}$$

With $m = 0$ ($t_w = \text{const}$) we obtain $\bar{\alpha} = 2\alpha$ (α being taken at $x = l$). With $q_w = \text{const}$, $m = 0.5$, and $\bar{\alpha} = 1.5\alpha$. When calculating the mean rate of heat transfer, the value of Pr_w must be taken for the mean wall temperature.

With linear variation of the temperature difference expressed by a relationship of the type of

$$\vartheta_w(x) = \vartheta_w(0)\left(1 + b\frac{x}{l}\right)$$

the value of ε proves to be dependent on x. The relationship $\alpha \sim x^{-0.5}$ becomes disturbed in this case. Fig. 7-5 gives values of $b = +1$ and $b = -0.25$ suggested by D.A. Labuntsov [7.1, 7.2]; here l is the total length of the plate, $b = 0$ corresponds to an isothermal surface of the wall. Curves *1* represent the variation of the local heat-transfer coefficients, while curves *2* and *3* show the variation of the mean heat-transfer coefficients averaged by formulae (6-22) and (6-21). Higher values of the heat-transfer coefficient $\bar{\alpha}$ correspond to a rise in temperature difference along the plate ($b > 0$), and lower values of $\bar{\alpha}$ to a drop in the temperature difference.

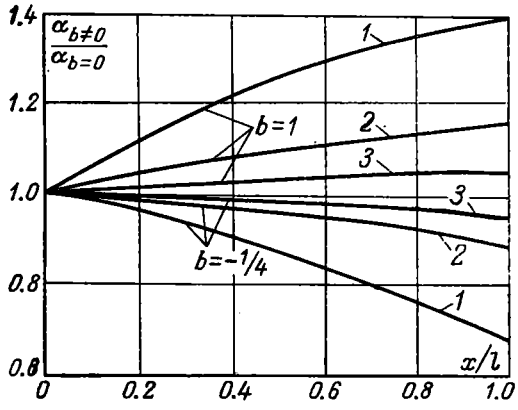

Fig. 7-5. Heat transfer from a non-isothermal plate with linear variation in temperature difference (drop):
1—local heat-transfer coefficients; *2*—heat transfer coefficient averaged by formula (6-21); *3*—averaged by formula (6-22)

The numerical value of $\bar{\alpha}$ averaged by formula (6-21) depends only slightly on the variation of wall temperature, and is close to the value of the mean heat-transfer coefficient at $t_w = \text{const}$. This conclusion holds both for the linear and exponential variation in wall temperature (temperature drop).

7-3. Change from Laminar to Turbulent Flow

The change from the laminar pattern of flow to turbulent takes place over some section of flow (Fig. 7-6). Flow in this section is unstable and is referred to as transition flow.

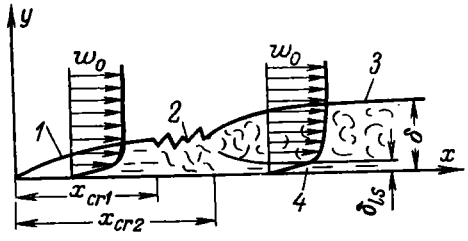

Fig. 7-6. Boundary layer schematic:
1—laminar boundary layer; *2*—transition region; *3*—turbulent boundary layer; *4*—viscous (laminar) sublayer

The laws governing heat transfer in fluid in laminar and turbulent flow are different, therefore the determination of the boundaries of the two patterns of flow is of great importance.

The pattern of flow is judged by the critical Reynolds numbers

$$\text{Re}_{cr1} = \frac{w_0 x_{cr1}}{\nu} \quad \text{and} \quad \text{Re}_{cr2} = \frac{w_0 x_{cr2}}{\nu}$$

where x is the coordinate (distance) along the plate, measured from the front edge of the surface exposed to flow. Knowing Re_{cr1} and Re_{cr2}, we can calculate the values of x_{cr1} and x_{cr2}, determining respectively the points at which deterioration of the laminar layer begins and stabilized turbulent flow sets in. Experimental studies show that a change from laminar to turbulent flow may occur

at $\mathrm{Re}_{cr} = w_0 x_{cr}/\nu$ ranging from 10^4 to 4×10^6. The coordinates x_{cr1} and x_{cr2} depend on a number of factors.

The change from laminar to turbulent flow is affected by such characteristics of the external flow as the degree (intensity) of turbulence, the scale of turbulence, pulsation frequency. When flow is accelerated ($\partial p/\partial x < 0$, flow in the convergent zone of the boundary layer) the change is delayed, and when flow is decelerated ($\partial p/\partial x > 0$, flow in the divergent zone of the boundary layer) the change sets in at smaller values of x (or Re_x).

In addition to external flow parameters, a change from the laminar pattern of flow to turbulent is affected by parameters which are associated in some degree with the body exposed to the flow. The critical Reynolds numbers Re_{cr1} and Re_{cr2} depend on the intensity of heat transfer, waviness, roughness of the surface exposed to flow, aerodynamical shape of the front edge of the plate involved, vibration of the plate. Some of the factors are interrelated.

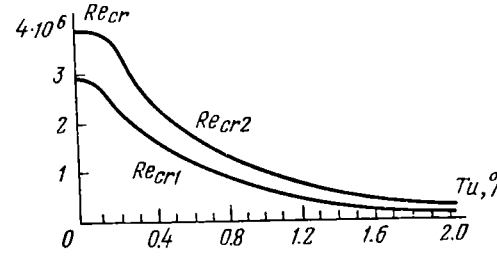

Fig. 7-7. The values of Re_{cr1} and Re_{cr2} depending on the degree of turbulence of free-stream flow

Shown in Fig. 7-7 is the dependence of the Reynolds critical numbers on the measure of turbulence of the free-stream flow, determined by the expression

$$\mathrm{Tu} = \frac{\sqrt{\frac{1}{3}\left(\overline{w_x'^2} + \overline{w_y'^2} + \overline{w_z'^2}\right)}}{w_0}$$

where $\overline{w_x'^2}$, $\overline{w_y'^2}$, $\overline{w_z'^2}$ = time-averaged squared components of velocity fluctuation;

w_0 = external flow velocity.

With comparatively small values of Tu the change of the pattern of flow does not depend on the degree of turbulence of the external flow, and is determined by the characteristics of the laminar layer proper (its stability). An increase in Tu leads to a smaller Re_{cr}.

In practice, the section in which the change takes place can be determined, in particular, by variation of the time-averaged velocity $\overline{w}_x(y)$. With turbulent flow the velocity $\overline{w}_x$ increases abruptly near the wall, and the velocity $\overline{w}_x(y)$ evens out at a distance from it. The evening-out of velocity is explained by the turbulent transport of momentum.

Data on critical Reynolds numbers were obtained conducting tests mainly with air as a fluid. When $\mathrm{Tu} < 0.1$ per cent the value of the lower critical Reynolds number is independent of the turbulence of the main stream and is equal to 3.1×10^6 [7.4, 7.5] for isothermal flow. According to the data proposed by L.M. Zysina-Molozhen for a plate exposed to longitudinal gradientless air flow the dependence of Re_{cr1} on Tu and the temperature factor T_w/T_0 can be described by equation

$$\mathrm{Re}_{cr1} = 3.1 \times 10^6 \varphi(\mathrm{Tu})\, \psi(T_w/T_0)$$

where $\varphi(\mathrm{Tu}) = 1$ at $\mathrm{Tu} < 0.12$ per cent; $\varphi = 0.23\mathrm{Tu}^{-0.7}$, if $\mathrm{Tu} = 0.12$-1.0 per cent; with $\mathrm{Tu} > 1.0$ per cent $\varphi = 0.23\mathrm{Tu}^{-1.76}$. The function ψ is deter-

mined by equation $\psi = (T_w/T_0)^{-2.3}$, where T_w and T_0 are respectively wall temperature and mainstream temperature.

The essential effect of the temperature factor is due to gas viscosity rising with increasing temperature and, as a result, deceleration of the flow at the wall with increasing (T_w/T_0), as shown in Fig. 7-3. The deceleration of the flow near the wall with the velocity of flow remaining constant at a distance from it contributes to a drop in flow stability, the appearance of additional (secondary) motion directed across the main flow along the plate.

According to available data [7.5] $\mathrm{Re}_{cr2} \approx 1.4\mathrm{Re}_{cr1}$ at $\mathrm{Tu} < 0.1$ per cent and $\mathrm{Re}_{cr2} \approx 1.6\mathrm{Re}_{cr1}$ at $\mathrm{Tu} > 0.6$ per cent (isothermal, gradientless flow along a plate).

The flow in the transition region is not stable. Turbulence appears in some part of the boundary layer, then the turbulently flowing fluid is entrained by the flow. The change from the laminar to the turbulent state of flow occurs in irregular time intervals. Such alternating flow is characterized by the *factor* ω which indicates for what fraction of some time interval the fluid is in turbulent flow at a definite spot in the fluid. Hence, when $\omega = 1$ the flow is turbulent all the time and a coefficient $\omega = 0$ indicates that the flow is laminar all the time. In this way, the boundary values of x_{cr1} and x_{cr2} acquire the property of time-averaged values.

The presence of a large number of influencing factors and the absence of data on the value of Tu in industrial installations make it difficult to determine exactly the section where the change from laminar into turbulent flow occurs. In calculations, therefore, the section $\Delta x = x_{cr2} - x_{cr1}$ is often replaced with a point and the critical Reynolds number is set approximately on the basis of test data. With the front edge of a plate being sufficiently streamlined, we may assume that

$$\mathrm{Re}_{cr1} \approx \mathrm{Re}_{cr2} \approx \mathrm{Re}_{cr} \approx 10^5$$

7-4. Heat Transfer in a Turbulent Boundary Layer

The transport of heat and momentum across a turbulent boundary layer can be described by Eqs. (4-42) and (4-43):

$$q = -(\lambda + \lambda_t)\frac{\partial \bar{t}}{\partial y} = -(\lambda + \rho c_p \varepsilon_q)\frac{\partial \bar{t}}{\partial y}$$

$$s = (\mu + \mu_t)\frac{\partial \bar{w}_x}{\partial y} = (\mu + \rho\varepsilon_s)\frac{\partial \bar{w}_x}{\partial y}$$

Let us present the equations in the following form:

$$q = -\lambda\left(1 + \frac{\mathrm{Pr}}{\mathrm{Pr}_t}\frac{\varepsilon_s}{\nu}\right)\frac{\partial \bar{t}}{\partial y} \tag{7-15}$$

$$s = \rho\nu\left(1 + \frac{\varepsilon_s}{\nu}\right)\frac{\partial \bar{w}_x}{\partial y} \tag{7-16}$$

where Pr_t denotes the ratio $\varepsilon_s/\varepsilon_q$.

The quantity Pr_t is called the Prandtl number for turbulent flow. As was shown in Sec. 4-5, the kinematic coefficients of turbulent transport of heat and momentum, ε_q and ε_s, depend on the parameters of the process of turbulent flow. In consequence, in the general case, the Prandtl number for turbulent flow can also be a parameter of a process. Allowing for Eqs. (7-15) and

(7-16), the differential equations of energy (4-44) and flow (4-45) for a turbulent boundary layer take the appearance:

$$\bar{w}_x \frac{\partial \bar{\vartheta}}{\partial x} + \bar{w}_y \frac{\partial \bar{\vartheta}}{\partial y} = a \frac{\partial}{\partial y}\left[\left(1 + \frac{\mathrm{Pr}}{\mathrm{Pr}_t}\frac{\varepsilon_s}{\nu}\right)\frac{\partial \bar{\vartheta}}{\partial y}\right] \tag{7-17}$$

$$\bar{w}_x \frac{\partial \bar{w}_x}{\partial x} + \bar{w}_y \frac{\partial \bar{w}_x}{\partial y} = \nu \frac{\partial}{\partial y}\left[\left(1 + \frac{\varepsilon_s}{\nu}\right)\frac{\partial \bar{w}_x}{\partial y}\right] \tag{7-18}$$

When $\mathrm{Pr} = 1$ ($a = \nu$) and $\mathrm{Pr}_t = 1$, equations (7-17) and (7-18) become identical. Then, with identical boundary conditions the fields of temperature $\bar{\vartheta}$ and velocity $\bar{w}_x$ will be similar.

To integrate equations (7-17) and (7-18) there should be available data on the coefficients of turbulent transport of heat and momentum. Use can also be made of the integral-differential Eqs. (7-3) and (7-5), but for this purpose we must know, in particular the distribution of velocity and temperature in the turbulent flow.

To create perfect calculation formulae, it is necessary to combine theoretical and experimental studies, permitting an understanding of the mechanism of turbulent transport of heat and momentum under different conditions of flow.

Let us determine the profile (distribution) of averaged velocity by means of Eqs. (4-47) and (4-50):

$$s_t = \rho l^2 \left(\frac{d\bar{w}_x}{dy}\right)^2 = \rho (\varkappa y)^2 \left(\frac{d\bar{w}_x}{dy}\right)^2$$

whence

$$\sqrt{\frac{s_t}{\rho}} = l \frac{d\bar{w}_x}{dy} = \varkappa y \frac{d\bar{w}_x}{dy}$$

with individual terms of the above equation having the dimension of velocity.

Let us assume that the tangential stress of turbulent flow does not change with y, i.e. that $\sqrt{s_t/\rho} = \sqrt{s_w/\rho} = \text{const}$. Denote $\sqrt{s_w/\rho}$ by w_* and give it the name of dynamic velocity. Then

$$w_* = \varkappa y \frac{d\bar{w}_x}{dy}, \quad d\bar{w}_x = \frac{w_*}{\varkappa}\frac{dy}{y}$$

and

$$\bar{w}_x = \frac{w_*}{\varkappa} \ln y + c \tag{7-19}$$

Equation (7-19) expresses the so-called *logarithmic distribution of the mean velocity of turbulent flow in the region adjoining a wall.* Let us determine the constant c from the condition $\bar{w}_x(0) = 0$. It follows from Eqs. (7-19) that with $y \to 0$ $\bar{w}_x = -\infty$, i.e. an absurd, incongruous result is obtained.

Account should be taken of the viscous forces which must be great directly at the wall. The layer of fluid adjoining the wall, in which viscosity forces predominate and which is a component part of the turbulent boundary layer, is referred to as the *viscous sublayer* (or laminar sublayer). Accounting only for the viscous forces, the equation of motion can be presented in the form $d^2\bar{w}_x/dy^2 = 0$, whence it follows that $dw_x/dy = \text{const} = c_1$ and $\bar{w}_x = c_1 y + c_2$, i.e. in the viscous sublayer the variation of velocity is linear. Thus, in the given

case $s = s_w = \mu d\overline{w}_x/dy = \text{const}$. Then:

$$s_w = \mu \frac{w_b}{\delta_s} = \rho w_*^2 \qquad (7\text{-}20)$$

where δ_s = thickness of viscous sublayer;

$w_b = w_x(\delta_s)$ = velocity on the outer boundary of the viscous sublayer.

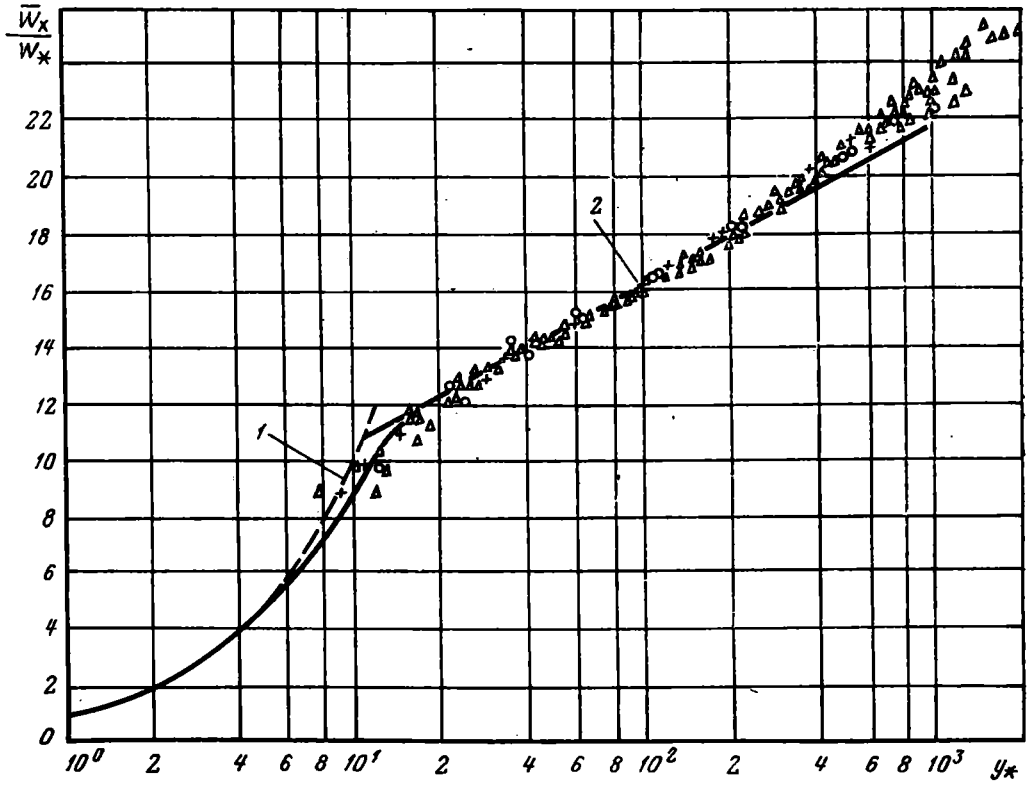

Fig. 7-8. Distribution of dimensionless velocity over the thickness of a turbulent boundary layer:
1—velocity variation in accordance with Eq. (7-22); *2*—velocity variation in accordance with Eq. (7-23); +—tests with air, Δ—with water; ○—with transformer oil

It follows from Eq. (7-20) that

$$\delta_s = \nu w_b / w_*^2$$

Let us determine the integration constant c in Eq. (7-19) from the condition that at $y = \delta_s = \nu w_b/w_*^2$ $w_x = \overline{w}_x(\delta_s) = w_b$. We obtain:

$$c = \frac{w_b}{w_*} - \frac{1}{\varkappa} \ln \delta_s = \frac{w_b}{w_*} - \frac{1}{\varkappa} \ln \frac{\nu w_b}{w_*^2}$$

Substituting the value of c into Eq. (7-19) and upon some transformations (account is taken that the difference of logarithms is equal to the logarithm of the quotient):

$$\overline{W}^+ = \frac{\overline{w}_x}{w_*} = \frac{1}{\varkappa} \ln y_* + \eta \qquad (7\text{-}21)$$

Formula (7-21) is called the *universal logarithmic distribution* of mean velocity in the wall-adjoining regions of turbulent flow. Here

$$y_* = \frac{w_* y}{\nu} \quad \text{and} \quad \eta = \frac{w_b}{w_*} - \frac{1}{\varkappa} \ln \frac{w_b}{w_*}$$

Formula (7-21) has been repeatedly correlated with test data at various values of y_* (excepting very small values of y_* inside the viscous sublayer). Correlation results can be expressed, in particular, by the graph shown in Fig. 7-8.

Curve *1* corresponds to linear variation of velocity in the viscous sublayer:

$$w_x = w_* y_* = \frac{w_*^2}{\nu} y = \frac{s_w}{\mu} y \qquad (7\text{-}22)$$

Curve *2* reflects logarithmic distribution of mean velocity in wall-adjoining turbulent part of the boundary layer. In this region

$$\frac{\overline{w_x}}{w_*} = 5.6 \log y_* + 4.9 \qquad (7\text{-}23)$$

The point of intersection of curves *1* and *2* corresponds to $y_* = w_* y/\nu$, equal about to 12. Now we can evaluate the calculation thickness of the viscous sublayer:

$$\delta_s = 12 \frac{\nu}{w_*} = 12\nu \sqrt{\frac{\rho}{s_w}} \qquad (7\text{-}24)$$

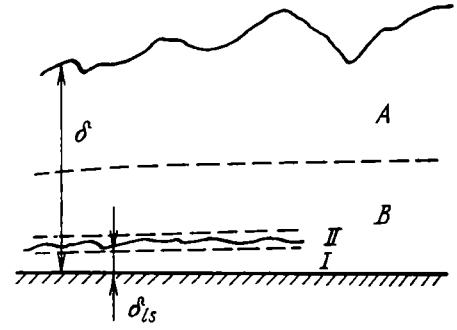

Fig. 7-9. Structure of a turbulent boundary layer:
A—external region; *B*—wall-adjoining region (*I*—viscous sublayer, *II*—intermediate layer); in the illustration the scales of thickness are shifted

At great values of y_* velocity distribution deviates from logarithmic.

Experimental studies have shown that fluid is in intricate motion in the turbulent layer, as illustrated in Fig. 7-9. There is no strictly laminar flow in the viscous sublayer along the wall. Fluctuations, especially broad-scale (low-frequency ones), penetrate the viscous layer, where their flow is regulated by the viscous forces. Generally speaking, the motion in the viscous sublayer is not of the steady-state kind, and the boundary of the sublayer is not defined clearly.

The outer border of the viscous sublayer is a high-power generator of pulsating flow. Turbulence of a higher intensity is observed in the wall-adjoining turbulent region. If for instance, the degree of turbulence in the free-stream flow may amount to fractions of one per cent, in the wall-adjoining region it may reach several tens of per cent. The wall-adjoining region constitutes about 20 per cent of the boundary-layer thickness (the thickness of the viscous sublayer is one-two order less). The flow in the outer region of the boundary layer, constituting about 80 per cent of its thickness, depends, in particular, on the pattern of flow in the external stream.

The outer border of the turbulent boundary layer pulsates continuously, which is traced to the periodic penetration of the fluid from the outer flow, where the degree of turbulence may be low, into the outer region of the boundary layer. Such interaction between the boundary layer with the external stream leads to the formation of the region of intermittent flow.

By analogy with the viscous sublayer, a *thermal sublayer* can be singled out directly at the wall. It is characterized by the predominance of heat transport by conduction over turbulent transport.

The thicknesses of the viscous sublayer δ_s and thermal k_s coincide at $\mathrm{Pr} = 1$. With $\mathrm{Pr} > 1$ $k_s < \delta_s$. The latter inequality is equivalent to the assertion that in the part of the viscous sublayer between $y = k_s$ and $y = \delta_s$ heat is being transported not only by conduction, but also by pulsations. The pulsations, penetrating the viscous sublayer prove to be essential for the transport of heat, but they do not contribute materially to the transport of momentum, compared with the molecular viscous transport. This pattern of flow must manifest itself especially with very viscous fluids ($\mathrm{Pr} \gg 1$).

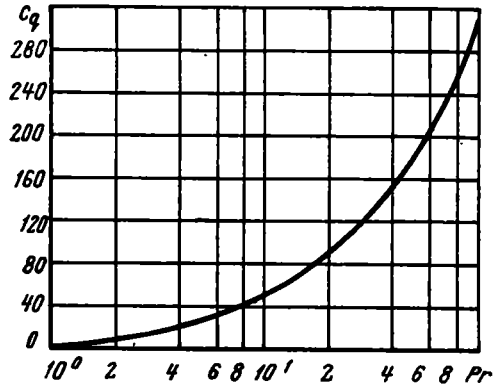

Fig. 7-10. Dependence of c_q in Eq. (7-26) on Prandtl number

In the limiting case when $\mathrm{Pr} \ll 1$ the reverse pattern should take place.

For very viscous media characterized by a low thermal conductivity, such as liquids with large Prandtl numbers, $\mathrm{Pr} = \mu c_p/\lambda$, the thermal sublayer offers the main thermal resistance to heat flow.

In view of the intensive turbulent transport the thicknesses of the thermal and dynamic boundary layers, k and δ, practically coincide. With turbulent flow the thickness of the boundary layer δ is greater than with laminar flow, due to the effect of turbulent viscosity.

Inasmuch as in the thermal sublayer heat transport is mainly due to conduction, temperature variation over its thickness is described by the equation of a straight line (as for a plane wall, Sec. 2-1).

Temperature distribution in the sublayer can be presented in the following manner:

$$\Theta = \mathrm{Pr} y_* \tag{7-25}$$

where $\Theta = \vartheta/\vartheta_*$; $\vartheta_* = q_w/\rho c_p w_*$.

Temperature distribution in the zone of logarithmic distribution of velocity obeys the logarithmic law:

$$\Theta = \frac{\mathrm{Pr}_t}{\varkappa} \ln y_* + c_q(\mathrm{Pr}) \tag{7-26}$$

The quantity c_q is a function of the Prandtl number (Fig. 7-10) and it accounts for temperature variation due to the different thickness of the sublayers, $k_s \neq \delta_s$.

Knowledge of the distribution of velocity and temperature permits calculation of heat transfer with the aid of the integral equations of heat flow and momentum, derived in Sec. 7-1. To avoid cumbersome calculations, associated with the use of integral equations, let us make use of a simplified conclusion, assuming, in doing so, that $\mathrm{Pr} \geqslant 1$, but it differs but little from unity.

Proceeding from the linear distribution of velocity and temperature, we can write for the viscous and thermal sublayers:

$$s_w = \mu \frac{w_b}{\delta_s} \quad \text{and} \quad q_w = \lambda \frac{\vartheta_b}{k_s}$$

The values of s_w and q_w do not change over the thicknesses δ_s and k_s. It follows from the above equations that

$$q_w = s_w \frac{\lambda}{\mu} \frac{\vartheta_b}{w_b} \frac{\delta_s}{k_s} \tag{7-27}$$

where $\vartheta_b = t_b - t_w$;
t_b = temperature at $y = k_s$, i.e. on the outer boundary of the thermal sublayer; and, respectively,
w_b = velocity at $y = \delta_s$;
t_w = fixed temperature of wall surface.

Molecular transport of heat and momentum can be ignored for the turbulent part of the boundary layer. Let us also assume that here $\mathrm{Pr}_t = 1$ ($\varepsilon_s = \varepsilon_q$). The distribution of mean velocity and temperature then will be identical. Then it follows from Eqs. (7-15) and (7-16) that in the turbulent part of the turbulent layer

$$q_t = s_t c_p \frac{\partial \bar{t}/\partial y}{\partial \bar{w}_x/\partial y}$$

Inasmuch as $\delta_s \ll \delta$, $k_s \ll k$ and $\delta \approx k$, the above equation becomes

$$q_t = s_t c_p \frac{t_0 - t_b}{w_0 - w_b} \tag{7-28}$$

On the border of the thermal sublayer $y = k_b$ there is no discontinuity in the magnitude of the thermal flow, therefore, the values of the heat flow rate q, expressed in accordance with Eqs. (7-27) and (7-28), can be equated. In doing so, we ignore the possible difference in tangential stress due to friction, s, in Eqs. (7-27) and (7-28), the difference being due to the fact that in the general case Pr_t is not unity near the wall (since $k_s \neq \delta_s$).

Let us solve Eqs. (7-27) and (7-28) in respect to the temperature differences:

$$t_b - t_w = \frac{q_w}{s_w} \frac{\mu}{\lambda} w_b \frac{k_s}{\delta_s} \quad \text{and} \quad t_0 - t_b = \frac{q_w}{s_w c_p} w_0 \left(1 - \frac{w_b}{w_0}\right)$$

Summing up these equations, we obtain:

$$t_0 - t_w = \frac{q_w w_0}{s_w c_p} \left[1 + \frac{\mu c_p}{\lambda} \frac{w_b}{w_0} \frac{k_s}{\delta_s} - \frac{w_b}{w_0}\right] \tag{7-29}$$

According to Eq. (7-24) $\delta_s \approx 12\, \nu/w_*$, whence

$$w_b = \delta_s \frac{w_*^2}{\nu} = 12 w_* = 12 \sqrt{\frac{s_w}{\rho}} \tag{7-30}$$

Let us assume that the ratio of the thicknesses of the thermal and viscous sublayers is described by Eq. (7-8) previously derived for the ratio of the thickness of the thermal boundary layer to that of the hydrodynamic for fluid in laminar flow:

$$\frac{k_s}{\delta_s} = \frac{1}{\sqrt[3]{\mathrm{Pr}}} \tag{7-31}$$

Substituting into Eq. (7-29) the values of w_b and k_s/δ_s defined by Eqs. (7-30) and (7-31), and solving Eq. (7-29) in respect to q_w, we get:

$$q_w = \frac{s_w c_p (t_0 - t_w)}{w_0 \left[1 + \frac{12}{w_0} \sqrt{\frac{s_w}{\rho}} (\mathrm{Pr}^{2/3} - 1)\right]} \tag{7-32}$$

The tangential stress due to skin friction, s_w, is characterized by the friction factor c_f, which by definition is

$$c_f = \frac{2 s_w}{\rho w_0^2} \tag{7-33}$$

Substituting into Eq. (7-32) the value of $s_w = c_f \rho w_0^2/2$ and dividing the left- and right-hand sides of Eq. (7-32) by $\rho c_p w_0 (t_0 - t_w)$, we obtain:

$$\mathrm{St} \equiv \frac{\alpha}{\rho c_p w_0} = \frac{c_f/2}{1 + 12\sqrt{\frac{c_f}{2}}(\mathrm{Pr}^{2/3} - 1)} \tag{7-34}$$

The complex $\alpha/\rho c_p w_0$ is dimensionless and referred to as the *Stanton* number, denoted by the symbol St. It can be expressed in terms of the Nusselt, Reynolds

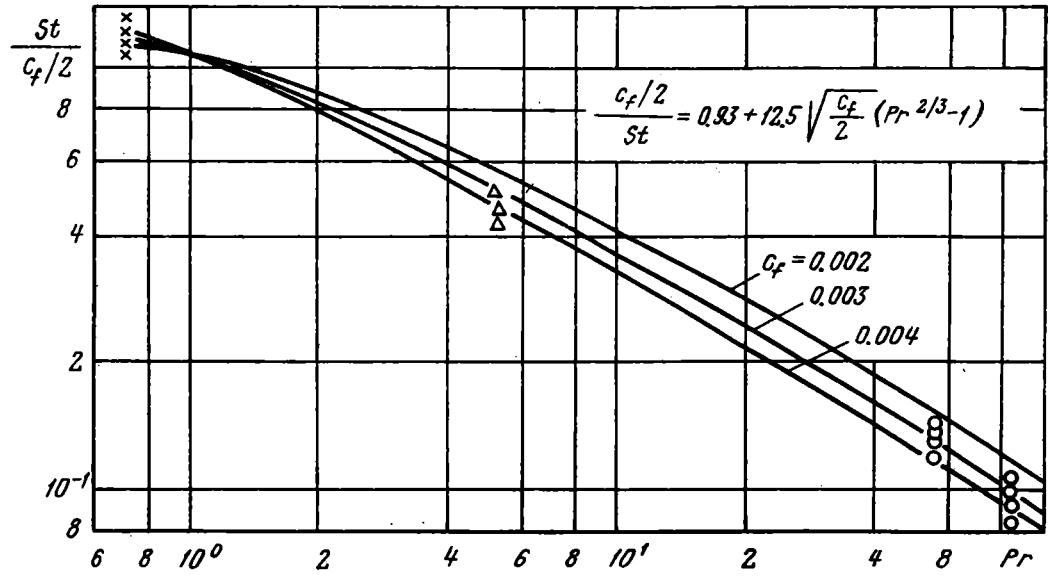

Fig. 7-11. Heat transfer for a plate with a turbulent boundary layer:
×—air; Δ—water; O—transformer oil

and Prandtl numbers, Nu, Re and Pr:

$$\mathrm{St} = \frac{\mathrm{Nu}}{\mathrm{Re\,Pr}} = \frac{\alpha}{\rho c_p w_0} \tag{7-35}$$

With Pr = 1 Eq. (7-34) simplifies and reduces into

$$\mathrm{St} = \frac{c_f}{2} \tag{7-36}$$

Equation (7-36) is the mathematical expression of the analogy between the transport of heat and momentum at Pr = 1 and $\mathrm{Pr}_t = 1$. This analogy was first shown by O. Reynolds in 1874. Formula (7-36) describes sufficiently well heat transfer in gases at small temperature differences.

The quantity Pr_t varies over the thickness of the boundary layer. According to data available [7.6] in the region for which the logarithmic laws governing the distribution of velocity and temperature are valid the turbulent Prandtl number is equal about to 0.8 (tests conducted with air, water and transformer oil). With this taken into account we arrive at the following formula:

$$\mathrm{St} = \frac{c_f/2}{0.93 + 12.5\sqrt{c_f/2}\,(\mathrm{Pr}^{2/3} - 1)} \tag{7-37}$$

Compared with formula (7-34), equation (7-37) contains some changed constants.

In Fig. 7-11 formula (7-37) is correlated with test data obtained with different Prandtl numbers.

When formula (7-37) is used to calculate heat transfer in liquids, it is advisable to multiply the obtained value of the Stanton number by the correction $(\mathrm{Pr}_f/\mathrm{Pr}_w)^n$, where $n = 0.25$ approximately. More correct values of the exponent n can be taken from Fig. 7-12 [7.6]. With air flow the correction $(T_0/T_w)^m$ is introduced, where $m \approx 0.25$ when heating of a gas is involved ($T_w > T_0$).

Formula (7-36)

$$\mathrm{St} = \frac{c_f}{2}$$

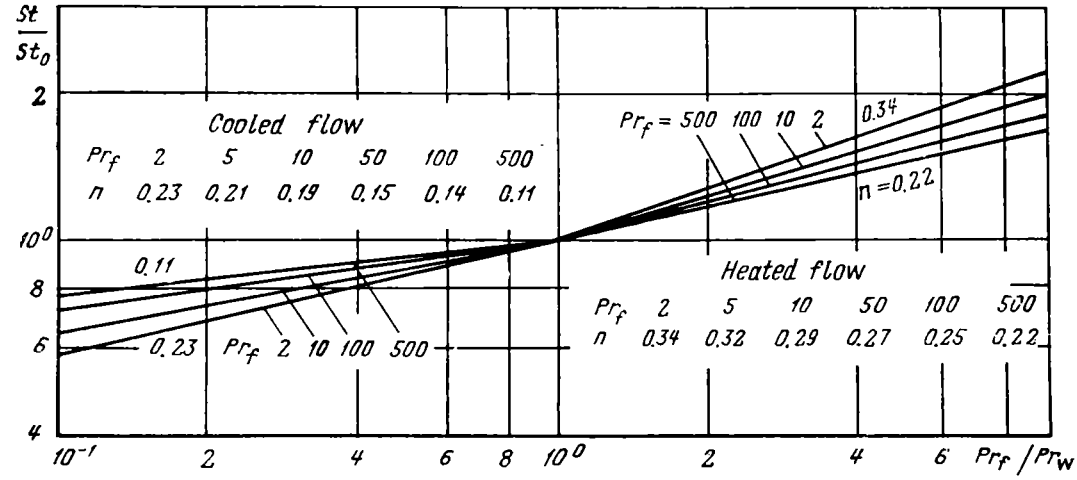

Fig. 7-12. Effect of the variation of physical properties of a liquid on heat transfer in the presence of a turbulent boundary layer, St_0 is determined by formula (7-37)

valid at $\mathrm{Pr} = 1$, can be applied to the case $\mathrm{Pr} > 1$ with the aid of the experimentally determined function $f(\mathrm{Pr}) = \mathrm{Pr}^{0.43}$, introduced into Eq. (7-36) as a multiplier.

Making use of Prandtl's formula

$$c_f = \frac{0.0592}{\mathrm{Re}_{fx}^{0.2}} \tag{7-38}$$

and introducing the correction $(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25}$, we obtain the calculation formula widely used in practice

$$\mathrm{Nu}_{fx} = 0.0296\,\mathrm{Re}_{fx}^{0.8}\,\mathrm{Pr}_f^{0.43}\,(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25} \tag{7-39}$$

The temperature of fluid at a distance from the solid t_0 (excepting Pr_w, chosen on the basis of $\bar{t}_w$) is taken as the reference. The reference dimension is the coordinate x, measured from the beginning of the section where heat transfer takes place. These recommendations refer both to formula (7-39) and formula (7-37).

According to formula (7-39) $\alpha = cx^{-0.2}$. Then, the integral mean local heat-transfer coefficient α is equal to $\bar{\alpha} = 1.25\alpha_{x=l}$.

If the entire plate is exposed to the turbulent layer (as is the case with a high degree of turbulence of the free-stream flow, bluff front edge, etc.), the local heat-transfer coefficient varies along the plate as illustrated in Fig. 7-13 by curve *1*. In the presence of a laminar boundary layer on the front part of the

plate the heat-transfer coefficient changes following a more intricate law (Fig. 7-13, curve *2*). The mean rate of heat transfer then must be calculated for individual sections with different patterns of flow.

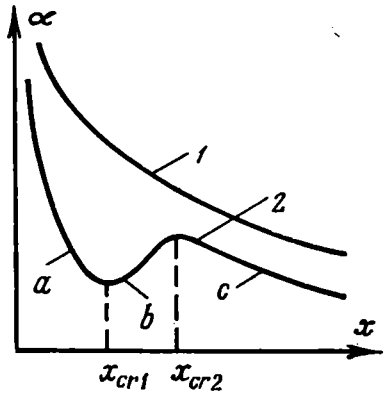

Fig. 7-13. Variation of the heat-transfer coefficient along the plate:
1—fully-developed turbulent flow in the boundary region; *2*—combined flow (*a*—laminar flow, *b*—transition flow, *c*—turbulent flow)

The region of transition flow $\Delta x = x_{cr1} - x_{cr2}$ cannot always be determined with sufficient accuracy. In calculations, therefore, the change from laminar to turbulent flow is often assumed to take place at a certain value of x, i.e. the section Δx is replaced with a point.

With developed forced turbulent flow the rate of heat transfer is usually independent of the Grashof number (with the near-critical region being a possible exception).

The formulae determining heat transfer for a plate in forced longitudinal flow may also be used to calculate heat transfer for a single cylinder exposed to fluid flow, if its diameter exceeds materially the thickness of the boundary layer. A more thorough knowledge of the theory of heat transfer in a turbulent boundary layer can be gained from special literature [7.6-7.11].

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CHAPTER

8

HEAT TRANSFER WITH FORCED FLOW OF FLUID IN TUBES

8-1. Peculiarities of Flow and Heat Transfer in Tubes

The process of heat transfer in a fluid flowing in tubes is more complicated than heat transfer from a surface exposed to infinite flow.

At a distance from the plate the fluid is not affected by the processes at the wall. The tube is finite in the cross-section. This results in the fluid being subjected to the decelerating effect of viscosity at a certain distance from the tube inlet. Since the tube is finite, the temperature of the fluid varies both over the cross-section and along the passage, which affects heat transfer.

Below, main attention will be paid to fluid flow and heat transfer in smooth straight tubes with a lengthwise invariable round cross-section, and, just as before, the dissipation of mechanical energy will be ignored. There are no inner heat sources in the fluid.

Flow in a tube may be laminar or turbulent. The pattern of flow is judged by the magnitude of

$$\mathrm{Re} = \frac{\bar{w}d}{\nu}$$

where $\bar{w}$ is the mean fluid velocity, and d is the inside diameter of the tube.

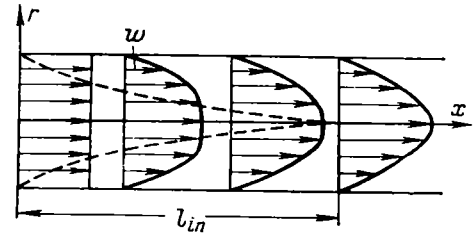

Fig. 8-1. Stabilization of velocity distribution in fluid flow inside a tube

Flow is said to be laminar, if $\mathrm{Re} < \mathrm{Re}_{cr} \approx 2000$.

The critical lower magnitude of the Reynolds number is 2000. With a Reynolds number greater than 2000 the flow once disturbed does not settle back again to the laminar pattern.

In industrial tubes turbulent flow is fully developed at $\mathrm{Re} > \mathrm{Re}_{cr2} \approx 10^4$.

The flow of fluid with $\mathrm{Re} = 2 \times 10^3$-$10^4$ is referred to as transition flow; heat transfer in flow of this kind is said to be transient.

If the fluid flows into the tube from a large volume and the tube edges are somewhat rounded at the inlet, inlet velocity distribution will be uniform (Fig. 8-1). While the fluid is in flow a hydrodynamic boundary layer of gradually increasing thickness forms at the wall. In sufficiently long tubes the boundary layer fills the entire cross-section of the tube at a certain distance from the inlet. In fluids with constant physical properties, after the turbulent layer fills the tube, a constant velocity distribution sets in, characteristic of the given pattern of flow.

The distance at which the constant velocity distribution sets in, measured from the tube inlet, is called the *length of the hydrodynamic initial section*, or section of hydrodynamic stabilization.

Stabilized flow ($x \geqslant l_{in}$) is independent of inlet velocity distribution ($x = 0$), but the distribution of velocity, both at $x < l_{in}$ and at $x \geqslant l_{in}$ may depend on the process of heat transfer.

An initial hydrodynamic section is observed with both laminar and turbulent flow of fluid in tubes. However, at $Re > Re_{cr1}$ fluid flow along the initial section develops in a peculiar way. In the front section of the tube the fluid may be in laminar flow. Upon reaching critical thickness, the forming laminar boundary layer may change into a turbulent layer. The turbulent layer increases in thickness until it fills the entire cross-section of the tube. At the point where laminar flow changes to turbulent the zone of the initial section is characterized by alternating flow. The change in the pattern of flow may take place beyond the initial hydrodynamic section.

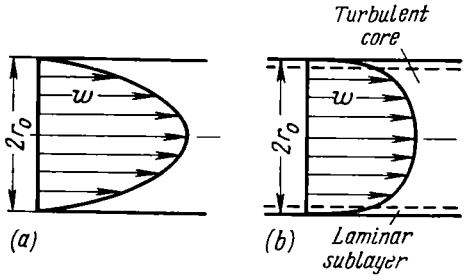

Fig. 8-2. Velocity distribution over the cross-section of a tube with stabilized fluid flow: *a*—laminar; *b*—turbulent

With $Re \gtrsim 5 \times 10^4$ a turbulent boundary layer develops practically at the very beginning of the tube. If the fluid comes from a large volume into a sharp-edged tube, eddies form at the tube inlet and quickly break down the laminar boundary layer.

The length of the initial hydrodynamic section and of its fractions exposed respectively to laminar and turbulent boundary layers depends on the Reynolds number and on a number of other factors.

Many factors are interrelated.

With hydrodynamically stabilized flow ($x > l_h$) the distribution of velocity in isothermal laminar flow is parabolic, as illustrated in Fig. 8-2*a*:

$$w_x = w_{max}[1 - (r/r_0)^2]$$

where r_0 = tube radius;
w_{max} = velocity at the tube axis (at $r = 0$).

The mean velocity then is half the maximum: $\bar{w}_x = 0.5 w_{max}$.

With turbulent flow almost the entire section of the tube is filled with the liquid in turbulent flow, with a viscous sublayer forming at the wall. At large Reynolds numbers the thickness of the viscous sublayer is a negligible fraction of the tube diameter. In spite of this the viscous sublayer represents the main thermal resistance, since heat is transferred through it by conduction.

The distribution of velocity over the cross-section of stabilized turbulent flow in a tube follows a truncated parabola (Fig. 8-2*b*). It attains its maximum value at the tube axis as before, but varies most abruptly at the wall.

The distribution of velocity in the turbulent part of flow (see Sec. 7-4) can be described with the aid of the universal logarithmic law expressed by Eq. (7-21)

$$\frac{\bar{w}_x}{w_*} = \frac{1}{\chi} \ln y_* + \eta$$

where $w_* = \sqrt{s_w/\rho}$; $y_* = w_* y/\nu$; $y = r_0 - r$ (Fig. 8-2). According to numerous experimental data for the turbulent core, characterized by $y^* \gtrsim 30$, $\varkappa = 0.4$ and $\eta = 5.5$; for the intermediate region arranged between the turbulent core and viscous sublayer for which y_* ranges between 5 and 30 $1/\varkappa = 5.0$ and $\eta = 3.05$. Within the limits of the viscous sublayer ($y_* \cong 0$ to 5) the linear variation in speed is assumed to be

$$\frac{\bar{w}_x}{w_*} = y_* \quad \text{or} \quad \bar{w}_x = \frac{s_w}{\mu} y$$

The stress of friction on the wall is a function of the Reynolds number, predetermining the dependence of velocity distribution over the section on Re. The greater the Reynolds number the more pronounced is the change in velocity near the wall, and less pronounced—in the central part of the flow, i.e. the velocity profile becomes better filled (Fig. 8-3). In consequence, the ratio of velocity averaged over the cross-section of the tube to maximum velocity ($r = 0$) will depend on the Reynolds number. This quantity has been shown by experiment to change slightly and be equal to 0.8-0.9.

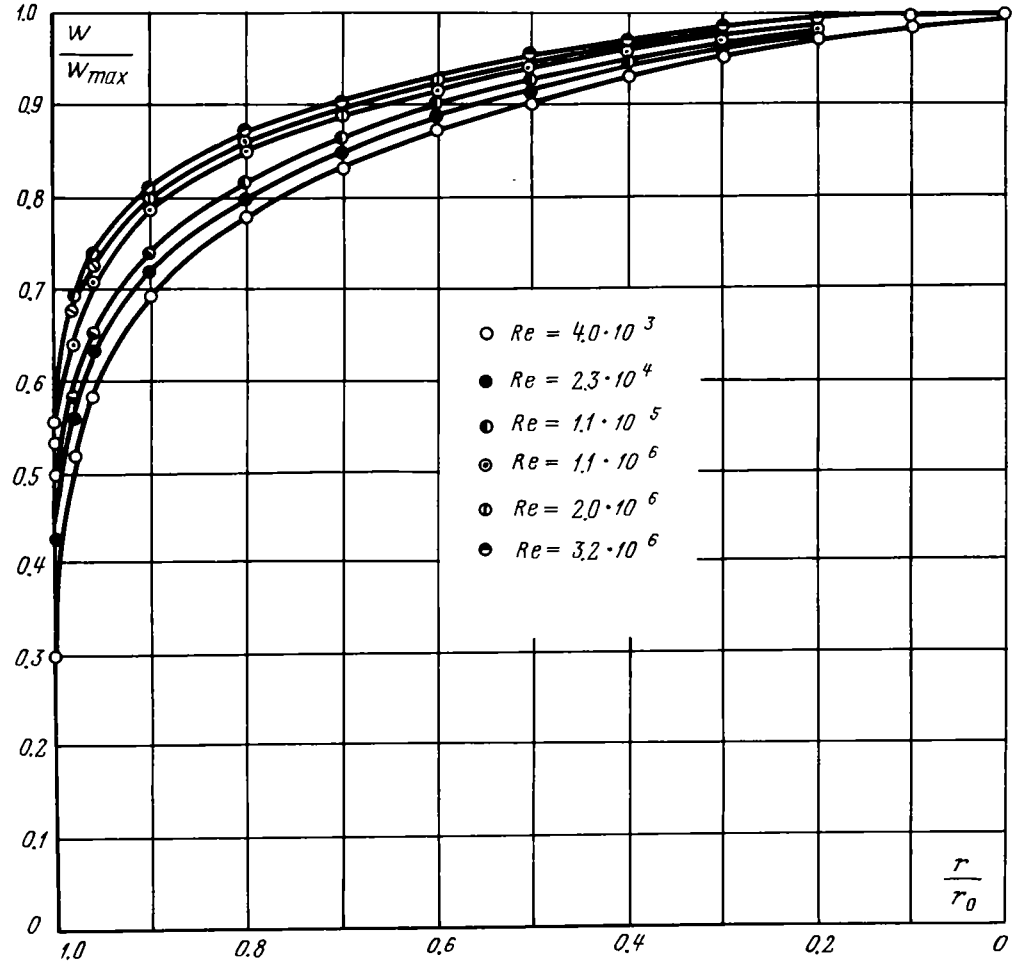

Fig. 8-3. Velocity distribution in a circular tube at different Reynolds numbers

The given data concerning the distribution of velocity in a turbulent flow pertain above all to isothermal flows or to flows involving fluids with practically invariable physical properties.

As the fluid flows along the tube heating or cooling of the layers adjoining the tube walls is observed, when fluid temperature differs from that of the tube. At the beginning of the tube the central core of the fluid is yet at a temperature, equal to the inlet temperature, this core does not take part in heat transfer, temperature variation concentrates in the wall-adjoining layer. Thus, a thermal boundary layer forms at the surface of the tube along its initial section, and the thickness of this layer increases with distance from the inlet. At a cer-

tain distance from the inlet, equal to $l_{in.th}$ the thermal boundary layer fills the entire cross-section of the tube; from here on all the fluid participates in heat transfer, with the intensity of heat transfer no more depending on inlet velocity and temperature distribution. The section of the tube $l_{in.th}$ long is called the *initial thermal section* or *section of thermal stabilization.*

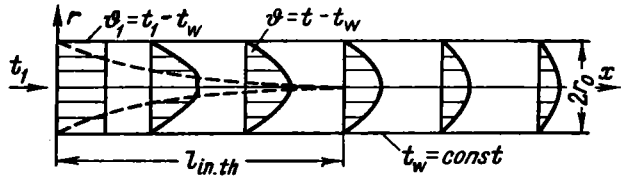

Fig. 8-4. Variation of temperature distribution in fluid flow in a tube

If at $x > l_{in.th}$ the law prescribing the boundary conditions on the wall does not change, heat transfer of this kind is referred to as *stabilized.* Unlike the velocity profiles, at $x > l_{in.th}$ the temperature profiles do not remain invariable, even in cases of constant physical parameters of the fluid involved, as illustrated in Fig. 8-4. An important change in the boundary conditions may lead to an effect similar to that involving the formation of a new thermal boundary layer [for instance, in the event of a sharp increase in the heat flow rate (the heat load), disturbance of the flow by some local obstacle].

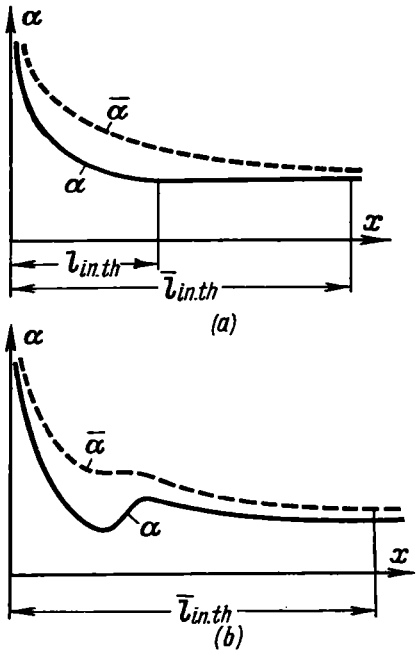

Fig. 8-5. Variation of local and mean heat-transfer coefficients along a tube:
a—invariable pattern of flow; *b*—mixed flow

For a fluid with constant physical properties and with simplest boundary conditions (for instance, t_w = const, q_w = const) the local coefficient of stabilized heat transfer is constant (Fig. 8-5). At t_w = const the derivative $(\partial t/\partial r)_{r=r_0}$ and the temperature drop $\vartheta = \bar{t} - \bar{t}_w$, where $\bar{t}$ is the bulk temperature over the cross-section of flow, diminish along the tube at an equal rate if $x > l_{in.th}$ (or remain constant at q_w = const).

Along the initial section the derivative $(\partial t/\partial r)_{r=r_0}$ diminishes at a considerably greater rate than the temperature drop. Thus, as follows from the heat-transfer equation,

$$\alpha = -\frac{\lambda}{\vartheta}\left(\frac{\partial t}{\partial r}\right)_{r=r_0}$$

and falls sharply along the section of thermal stabilization until it becomes constant for the fluid in stabilized flow (Fig. 8-5*a*).

The heat-transfer coefficient will change in a different manner along a tube, if the pattern of flow changes in the initial section, for example, in Figure 8-5*b*. The heat-transfer coefficient diminishes along the section of laminar flow and increases as the laminar section deteriorates. Then, heat transfer stabilizes when turbulent flow sets in.

The length of the thermal-stabilization section depends on many factors, for instance, on the thermal conductivity of the fluid concerned, the presence of hydrodynamic stabilization, the Reynolds number, temperature distribution on the inlet, etc.

Theory shows that with a liquid of constant physical parameters and a homogeneous inlet temperature in laminar flow, at $t_w = \text{const}$

$$\frac{l_{in.th}}{d} = 0.055\,\text{Pe}$$

and with $q_w = \text{const}$

$$\frac{l_{in.th}}{d} = 0.07\,\text{Pe}$$

These equations correspond to preliminarily hydraulically stabilized flow.

The Reynolds number may be equal to about 2000 for laminar flow. The length of the initial thermal section for gases ($\text{Pr} \approx 1$) may be equal to about hundred tube diameters, if account is taken of the fact that $\text{Pe} = \text{RePr}$. The magnitude of $l_{in.th}$ for every viscous fluid ($\text{Pr} \gg 1$) may vary from several hundreds to several tens of thousand tube diameters. In the latter case heat transfer is practically present only within the initial section.

According to numerous experimental data, for turbulent flow $l_{in.th} =$ (from 10 to 15) d.

Let us determine the mean heat-transfer coefficient for the tube if $l > l_{in.th}$, where l is the length of the tube. Let on the section $0 \leqslant x < l_{in.th}$ $\alpha = \alpha(x) = \alpha_{in.th}$, and at $x \geqslant l_{in.th}$ $\alpha = \alpha_\infty = \text{const}$. Then

$$\bar{\alpha} = \frac{\bar{q}_w}{\overline{\Delta t}} = \frac{\int_0^l \alpha\,\Delta t\,dx}{\int_0^l \Delta t\,dx} = \frac{\int_0^{l_{in.th}} \alpha_{in.th}\,\Delta t\,dx + \int_{l_{in.th}}^{l} \alpha_\infty\,\Delta t\,dx}{\int_0^{l_{in.th}} \Delta t\,dx + \int_{l_{in.th}}^{l} \Delta t\,dx} \qquad \text{(a)}$$

The integrals between the limits 0 and $l_{in.th}$ can be presented as follows:

$$\int_0^{l_{in.th}} \alpha_{in.th}\,\Delta t\,dx = \bar{q}_{in.th} l_{in.th} =$$

$$= \bar{\alpha}_{in.th}\,\overline{\Delta t}_{in.th} l_{in.th} \quad \text{and} \quad \int_0^{l_{in.th}} \Delta t_{in.th}\,dx = \overline{\Delta t}_{in.th} l_{in.th}$$

Substituting the values of the integrals into Eq. (a), we get:

$$\bar{\alpha} = \frac{\bar{\alpha}_{in.th} + \dfrac{\alpha_\infty}{\overline{\Delta t}_{in.th} l_{in.th}} \int_{l_{in.th}}^{l} \Delta t\,dx}{1 + \dfrac{1}{\overline{\Delta t}_{in.th} l_{in.th}} \int_{l_{in.th}}^{l} \Delta t\,dx}$$

or

$$\frac{\bar{\alpha}}{\alpha_\infty} = \frac{\dfrac{\bar{\alpha}_{in.th}}{\alpha_\infty} + \varphi(l)}{1 + \varphi(l)} \qquad \text{(8-1)}$$

where

$$\varphi(l)=\frac{1}{\overline{\Delta t}_{in.th}l_{in.th}}\int_{l_{in.th}}^{l}\Delta t\,dx=\frac{\overline{\Delta t}_{\infty}}{\overline{\Delta t}_{in.th}}\frac{l-l_{in.th}}{l_{in.th}}$$

$\overline{\Delta t}_{\infty}$, $\overline{\Delta t}_{in.th}$ are respectively the mean temperature differences on the sections $(l_{in.th}, l)$ and $(0, l_{in.th})$.

If $\overline{\Delta t}_{\infty} \approx \overline{\Delta t}_{in.th}$, then $\varphi(l) \approx (l - l_{in.th})/l_{in.th}$.

Substituting into Eq. (8-1) this value of the function we obtain:

$$\frac{\bar{\alpha}}{\alpha_{\infty}}=1+\frac{l_{in.th}}{l}\left(\frac{\bar{\alpha}_{in.th}}{\alpha_{\infty}}-1\right) \qquad (8\text{-}2)$$

It follows from Eq. (8-2) that in long tubes ($l \gg l_{in.th}$) $\bar{\alpha} \to \alpha_{\infty}$, i.e. with large l the values of $\bar{\alpha}$ and α_{∞} practically coincide. For instance, if $\bar{\alpha}_{in.th}/\alpha_{\infty} = 1.3$, with an accuracy of 3 per cent the mean heat-transfer coefficient $\bar{\alpha}$ will be equal to the local α_{∞} at $l = 10\,l_{in.th} = \bar{l}_{in.th}$.

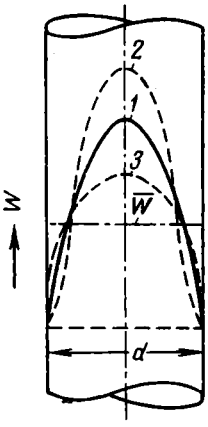

Fig. 8-6. Velocity distribution over the cross-section for viscous liquids flowing in tubes: *1*—isothermal flow; *2*—cooled liquid; *3*—heated liquid

The length of tube $\bar{l}_{in.th}$ at which it may be assumed with sufficient accuracy that the mean heat-transfer coefficient $\bar{\alpha}$ is equal to the coefficient of stabilized heat transfer α_{∞} is usually used in practical calculation of the mean rate of heat transfer. It is clear that $l_{in.th}$ is a fictitious calculation quantity, whose numerical value depends either on the accuracy of analytical calculations or on the accuracy of experimental data.

Owing to the variation in physical parameters, there may be two patterns of non-isothermal laminar flow (Re < 2000): *viscous* and *viscous-gravitational*. Heat transfer is governed by different laws in the two patterns of flow.

For the viscous-gravitational pattern of flow the viscosity and buoyancy forces are commensurable.

Viscous non-isothermal flow occurs when gravitational forces predominate over buoyancy forces, i.e. it is characteristic of the flow of viscous fluids unaffected by natural convection.

Compared to viscous-gravitational, the viscous pattern of flow is more probable to occur the smaller the tube diameter, the larger the fluid viscosity, and the smaller the temperature difference.

With viscous flow the distribution of velocity over the cross-section of a tube is not parabolic, because the viscosity also changes owing to temperature variation over the cross-section. Velocity distribution also depends on whether the fluid undergoes heating or cooling (Fig. 8-6). With an identical cross-section mean temperature, the temperature of the fluid at the wall is higher with heating than with cooling. The higher the temperature of a dropping liquid, the lower is its viscosity, therefore the velocity of a heated liquid is higher at the wall of a conduit than that of a cooled one, and the rate of heat transfer is augmented.

We are familiar with a similar phenomenon from consideration of heat transfer from a flat plate exposed to liquid flow. The heat-transfer coefficient of a heated liquid is larger than that of a cooled one; the difference between the two coefficients increases with a rise in temperature difference.

Apart from the effect of viscosity variation, velocity distribution in viscous-gravitational flow is greatly influenced by the intensity and direction of free convection, arising from difference between the densities of the hotter and cooler particles of the fluid. In the absence of forced flow, and at a definite variation the velocity distribution for natural convection is as illustrated in Fig. 8-6.

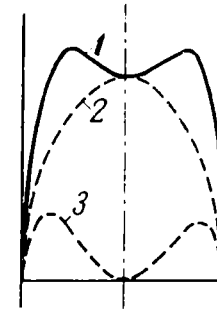

Fig. 8-7. Velocity distribution over the cross-section of a tube with coincident direction of forced and free convection:
1—summary curve; *2*—at the expense of forced convection; *3*—at the expense of free convection

According to the relative direction of the forced and free convection, three cases can be distinguished:

(a) the natural and forced convection flows are in the same direction;

(b) the two flows are mutually perpendicular;

(c) the natural and forced convection flows are opposite in direction.

The first case occurs in vertical tubes with a heated fluid flowing upward or a cooled fluid flowing downward. The free convection brings about an increase in the velocity of fluid at the wall (Fig. 8-7), and there may be two maxima on the velocity distribution diagram.

The second case is observed in horizontal tubes. Transverse circulation of the fluid originates over the tube cross-section under the effect of free convection.

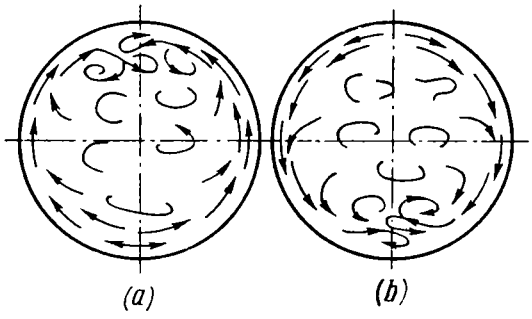

Fig. 8-8. Transverse circulation in a horizontal tube resulting from forced and free convection:
a—with heating of fluid; *b*—with cooling of fluid

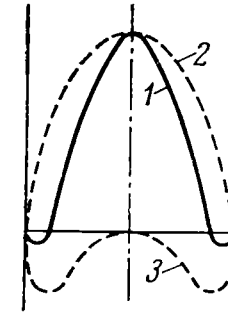

Fig. 8-9. Velocity distribution over the cross-section of a tube with forced and free convection in opposite directions:
1—summary curve; *2*—forced convection; *3*—free convection

Ascending currents of free convection appear in a heated fluid at the wall, and descending, at the centre of the tube; the opposite is the case with cooled fluids (Fig. 8-8). The fluid consequently moves as though along a spiral. The rate of heat transfer increases, owing to the better mixing of the fluid. Other things being equal, the intensity of heat transfer is larger than with coincident directions of forced and free convection.

The third case occurs when a heated fluid is in downward flow in a vertical tube, or a cooled fluid is in upward flow. The velocity of the fluid at the wall

diminishes here owing to the oppositely directed currents of free convection. Eddies may form in the fluid at the wall (Fig. 8-9). The heat-transfer coefficients then are practically equal to these determined from the equations derived for fluid in turbulent flow [8.1].

Flow has its peculiarities if heat transfer is not uniform over the periphery of a conduit or occurs on only one side of it. So, for instance, if a flat (slit-like) conduit is horizontally arranged and only its bottom side is heated, the flow will be considerably disturbed due to free convection, and slightly disturbed, if top heating of the conduit is practised.

Non-isothermal conditions may thus exclude strict laminar flow, i.e. parallel-stream flow with parabolic velocity distribution.

The intricacy and diversity of processes involving fluid flow and heat transfer in tubes permit us to single out an enormous number of practical problems differing in the initial differential equations describing them and in the conditions of unambiguity. Many of these problems have been solved. Most intricately formulated problems cannot be solved with sufficient accuracy, or their solution is not feasible. The use of electronic computers makes it possible to bring the solution of problems to obtaining numerical values for the unknown variables. However, even then the regions in which the obtained values can find practical application sometimes remain undetermined. For instance, the calculation of viscous-gravitational flow with the aid of a computer may fail to show the conditions under which the flow changes into turbulent (for the critical Reynolds number may change somewhat).

In consequence, experimental studies are presently of great importance when investigating convective heat transfer. In conducting experimental work attempts to interrelate individual variables involve an intricate problem which in the general case cannot be solved with sufficient accuracy without the aid of theory (if only limited). An organic combination of calculation-analytical and experimental studies, therefore, yields presently the most reliable universal results.

8-2. Integral Equation for Developed Heat Transfer

Let us now consider an approximate method suitable for finding the heat-transfer coefficients of a fluid in hydrodynamically and thermally stabilized flow in a straight round tube.

We assume the fluid to be incompressible, its physical parameters to be constant, and the heat of friction to be negligible and that there are no inner heat sources.

The energy equation for axis-symmetrical steady-state flow takes the following form:

$$\rho c_p \left(w_r \frac{\partial t}{\partial r} + w_x \frac{\partial t}{\partial x} \right) = \lambda \left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r} \frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial x^2} \right)$$

being written in cylindrical coordinates, with r the radius, and x the coordinate along the axis in the direction of flow.

We also assume that much more heat is transported in the radial direction than in the axial. The term $\frac{\partial^2 t}{\partial x^2}$ may then be ignored. In addition, $w_r = 0$.

Note that in turbulent flow heat is transported not only by conduction, but also by the way of turbulent fluctuation or diffusion. The equation of energy then can be presented in the following form:

$$\frac{\partial}{\partial r} \left[(\lambda + \lambda_T) r \frac{\partial t}{\partial r} \right] = \rho c_p w_x r \frac{\partial t}{\partial x}$$

where $\lambda_T = \rho c_p \varepsilon_q$ is the turbulent diffusivity—an analogue of thermal conductivity, and t and w_x are respectively the local (point) time-average temperature and velocity of turbulent flow.

Let us assume $q_w = \text{const}$. As was shown in Chapter 6, at $q_w = \text{const}$

$$\frac{d\bar{t}_f}{dF} = \frac{q_w}{Gc_p} = \text{const}$$

For a round tube ($dF = 2\pi r_0 dx$)

$$\frac{d\bar{t}_f}{dx} = \frac{2q_w}{\rho c_p \bar{w}_x r_0} = \frac{2\alpha (t_w - \bar{t}_f)}{\rho c_p \bar{w}_x r_0}$$

Here $\bar{t}_f$ is the calorimetric (enthalpy) mean temperature of the fluid in the given section; $\bar{w}_x$ is the mean velocity in the same section; and r_0 is the tube radius.

In these conditions the mean temperature of fluid will be a linear function of x. With $\alpha = \text{const}$ (stabilized flow) both the variation of $\bar{t}_f$, and of the wall temperature is linear:

$$\frac{q_w}{\alpha} = t_w - \bar{t}_f = \text{const}$$

With constant physical parameters the variation of local fluid temperature along the tube is also linear. Hence,

$$\frac{\partial t}{\partial x} = \frac{2q_w}{\rho c_p \bar{w}_x r_0} = \text{const}$$

Substituting the value of $\partial t/\partial x$ in the energy equation, we obtain:

$$\frac{d}{dr}\left[(\lambda - \lambda_T) r \frac{dt}{dr}\right] = 2q_w \frac{w_x}{\bar{w}_x} \frac{r}{r_0}$$

or

$$\frac{d}{dR}\left[(\lambda + \lambda_T) R \frac{dt}{dR}\right] = 2q_w r_0 W_x R$$

where $W_x = \frac{w_x}{\bar{w}_x}$ and $R = \frac{r}{r_0}$ are respectively the dimensionless velocity and radius.

Separating the variables and integrating from 0 to R and from 0 to $(\lambda + \lambda_T) R \frac{dt}{dR}$, we obtain:

$$(\lambda + \lambda_T) R \frac{dt}{dR} = 2q_w r_0 \int_0^R W_x R \, dR$$

From this it follows that

$$\frac{dt}{dR} = \frac{2q_w r_0}{(\lambda + \lambda_T) R} \int_0^R W_x R \, dR \qquad \text{(a)}$$

The calorimetric mean temperature of a fluid at constant c_p and ρ is determined by the following equation:

$$\bar{t}_f = \frac{1}{f_0 \bar{w}_x} \int_0^{f_0} w_x t \, df$$

For a round tube

$$f = \pi r^2 \text{ and } df = d(\pi r^2) = 2\pi r \, dr$$

Then,

$$\bar{t}_f = \frac{2}{r_0^2 \bar{w}_x} \int_0^{r_0} w_x t r \, dr = 2 \int_0^1 t W_x R \, dR$$

We integrate this equation by parts, following the formula

$$\int_a^b u \, dv = uv \Big|_a^b - \int_0^b v \, du$$

Then denoting $t = u$ and $dv = W_x R \, dR$ or $v = \int_0^R W_x R \, dR$. Then

$$\bar{t}_f = 2\left[t \int_0^R W_x R \, dR \Big|_0^1 - \int_0^1 \left(\int_0^R W_x R \, dR\right) dt\right] =$$

$$= 2\left[t_w \int_0^1 W_x R \, dR - \int_0^1 \left(\int_0^R W_x R \, dR\right) dt\right] \qquad \text{(b)}$$

The integral $\int_0^1 W_x R \, dR$ can be rearranged as follows:

$$\int_0^1 W_x R \, dR = \frac{2\pi}{2\bar{w}_x \pi r_0^2} \int_0^{r_0} w_x r \, dr = \frac{\int_0^{r_0} w_x 2\pi r \, dr}{2\bar{w}_x \pi r_0^2} = \frac{\bar{w}_x \pi r_0^2}{2\bar{w}_x \pi r_0^2} = \frac{1}{2}$$

Then, substituting the obtained value of the integral in (b), we have:

$$\bar{t}_f = t_w - 2 \int_0^1 \left(\int_0^R W_x R \, dR\right) dt$$

Substituting dt in this expression we can write in accordance with equation (a):

$$\bar{t}_f = t_w - \frac{4 q_w r_0}{\lambda} \int_0^1 \frac{\left(\int_0^R W_x R \, dR\right)^2}{\left(1 + \frac{\lambda_T}{\lambda}\right) R} \, dR$$

From this it follows that

$$\frac{\lambda(t_w - \bar{t}_f)}{2 q_w r_0} = 2 \int_0^1 \frac{\left(\int_0^R W_x R \, dR\right)^2 dR}{\left(1 + \frac{\lambda_T}{\lambda}\right) R} = 2 \int_0^1 \frac{\left(\int_0^R W_x R \, dR\right)^2 dR}{\left(1 + \frac{\mathrm{Pr}}{\mathrm{Pr}_T} \frac{\varepsilon_s}{\nu}\right) R}$$

where $\mathrm{Pr}_t = \varepsilon_s / \varepsilon_q$ is the turbulent Prandtl number. By definition

$$\frac{\lambda (t_w - \bar{t}_f)}{2 q_w r_0} = \frac{\lambda}{\alpha d} = \frac{1}{\mathrm{Nu}_d}$$

Using this last notation, we can write the following integral equation for developed heat transfer:

$$\frac{1}{\mathrm{Nu}_d} = 2 \int_0^1 \frac{\left(\int_0^R W_x R \, dR\right)^2 dR}{\left(1 + \frac{\mathrm{Pr}}{\mathrm{Pr}_T} \frac{\varepsilon_s}{\nu}\right) R} \qquad \text{(8-3)}$$

Equation (8-3) was derived by Lyon. His equation is universal, for it is suitable for both the turbulent and laminar patterns of flow. If the distribution of velocities $w_x(r)$ is known the heat-transfer coefficient can be calculated from Eq. (8-3).

For laminar flow $\lambda_T = 0$, and Lyon's integral (8-3) becomes simplified:

$$\frac{1}{\mathrm{Nu}_d} = 2\int_0^1 \frac{dR}{R}\left(\int_0^R W_x R\,dR\right)^2 \qquad (8\text{-}3')$$

Analytical calculation methods for heat transfer in fluids flowing in tubes, including fluids with variable properties, are considered in recommended literature [8.1-8.3].

8-3. Heat Transfer in Fluid Flow in Plain Tubes of Circular Cross-section

(a) *Heat Transfer in Laminar Flow*

Heat transfer in hydrodynamically and thermally stabilized flow can be calculated by formula (8-3'). For liquid with invariable physical properties in hydrodynamically developed laminar flow

$$w_x = 2\bar{w}_x[1-(r/r_0)^2] \quad \text{or} \quad W_x = 2(1-R^2)$$

where

$$W_x = \frac{w_x}{\bar{w}_x}, \quad \text{and} \quad R = \frac{r}{r_0}$$

Substituting this value of W_x in Eq. (8-3) and integrating, we obtain:

$$\frac{1}{\mathrm{Nu}_d} = 2\int_0^1 \frac{dR}{R}\left[\int_0^R 2(1-R^2)\,R\,dR\right]^2 = \frac{11}{48}$$

From this it follows that

$$\mathrm{Nu}_d = \frac{48}{11} \approx 4.36$$

Thus, for the process of steady-state heat transfer the Nusselt number is constant and equal to 4.36.

This value is based on the condition that $q_w = \text{const}$. At $t_w = \text{const}$ theory states that $\mathrm{Nu}_d = 3.66$. These values are for a parabolic distribution of velocities, a distribution that occurs with vanishing temperature differences or invariable fluid parameters. The obtained result therefore may sometimes diverge considerably from experimental data. In addition, the theory considered above does not account for heat transfer in the initial section of the tube.

The processes of flow and heat transfer develop identically with a plate in longitudinal flow (Chapter 7) and in the inlet section of a tube, since at the beginning of the tube the thicknesses of the boundary layers are small compared with the cross-sectional dimensions of the channel. Accordingly, heat transfer at the inlet of a tube can be described with sufficient accuracy by the equations derived for a plate in longitudinal flow. Owing to the considerable constraint of the flow, the regularities of the process vary with distance from the inlet.

The necessity to account for the variation of physical parameters and other affecting factors makes analytical calculations very intricate and labour-con-

suming. Empirical formulae are, therefore, preferred at present for practical calculations. Let us consider some of the experimental results.

For the case of $q_w = \text{const}$ [8.4] in the research carried out at the Krzhizhanovsky Power Institute in Moscow, the following empirical formula was suggested for the local heat-transfer coefficients of viscous flow:

$$\mathrm{Nu}_{f(x),\,x} = 0.33\mathrm{Re}^{0.5}_{f(x),\,x}\mathrm{Pr}^{0.43}_{f(x)}\left(\mathrm{Pr}_{f(x)}/\mathrm{Pr}_{w(x)}\right)^{0.25}(x/d)^{0.1} \qquad (8\text{-}4)$$

The reference dimension used here is the distance of the investigated cross-section from the inlet of the tube; the sectional mean temperature of the fluid is taken as the reference temperature (the quantity $\mathrm{Pr}_{w(x)}$ is taken for the local wall temperature).

According to formula (8-4) $\alpha = cx^{-0.4}$, where c is independent of x. Averaging the heat-transfer coefficient by formula (6-21), we find that $\bar{\alpha} = 1.4\alpha_{x=l}$.

In experimental studies [8.4] heat transfer took place from the beginning of the tube (the rate of heat transfer was measured from the point at which $x/d = 2$), the relative length of the tube was $l/d \leqslant 216$, where l = length of tube, and d = inside diameter. Equation (8-4) is close to that derived for a plate in longitudinal flow. The term $(x/d)^{0.1}$ is assumed to account for conduit curvature and constraint of the flow by the wall of the tube.

If the tube is longer than the initial thermal section and heat transfer initiates at the tube inlet, the mean heat transfer coefficients for a viscous flow can be determined by the following equation [8-1]:

$$\overline{\mathrm{Nu}} = 1.55\left(\mathrm{Pe}\frac{d}{l}\right)^{1/3}(\mu_w/\mu_f)^{-0.14}\varepsilon_l \qquad (8\text{-}5)$$

Here the mean heat-transfer coefficient is reduced to the logarithmic mean temperature difference. The physical properties of the fluid present in Nu and Pe, and also the value of μ_f are taken by the temperature $t = t_w - 0.5\,\Delta t_{log}$ (the value of μ_w being based on the mean wall temperature). The reference dimension in $\overline{\mathrm{Nu}}$ and Pe is the inside tube diameter.

The factor ε_l is a correction for the hydrodynamic initial section which is forming together with the initial thermal section, and it can be calculated by formula

$$\varepsilon_l = 0.10\left(\frac{1}{\mathrm{Re}}\frac{l}{d}\right)^{-1.7}\Big/\left(1 + 2.5\,\frac{1}{\mathrm{Re}}\frac{l}{d}\right)$$

which holds at $\frac{1}{\mathrm{Re}}\frac{l}{d} < 0.1$, or taken from a graph (Fig. 8-10). The reference values are similar to these of the $\overline{\mathrm{Nu}}$ and Pe numbers.

If there is an unheated section in the beginning of the tube of a length $l_0 < l_{in.th}$, approximate results may be obtained by formula (8-5), substituting in the expression $\frac{1}{\mathrm{Re}}\frac{l}{d}$, the sum $l_0 + l$ for l. When $l_0 \geqslant l_{in.th}$ ε_l should be assumed equal to unity.

Equation (8-5) has been derived based on $\frac{1}{\mathrm{Pe}}\frac{l}{d} < 0.01$ and $0.07 \leqslant \frac{\mu_w}{\mu_f} \leqslant 1500$. The accounting for the effect of viscosity with the aid of ratio $\left(\frac{\mu_w}{\mu_f}\right)^{-0.14}$ is true for dropping liquids and is unsuitable for gases.

Formula (8-5) can be used with a constant or slightly lengthwise varying wall temperature.

According to Petukhov [8.1], at $GrPr > 8 \times 10^5$ the viscous-gravitational pattern of flow takes place (here $Gr = q\beta\Delta t d^3/\nu^2$; $\Delta t = |(t_w - t_0)|$; t_0 = fluid temperature at the tube inlet; the physical parameters present in GrPr are taken by the temperature $t = 0.5\ (t_0 + t_w)$.

With viscous-gravitational flow the heat-transfer coefficients are greater than these determined by formulae (8-4) and (8-5). Natural convection may increase the heat-transfer coefficient 5 times under certain conditions.

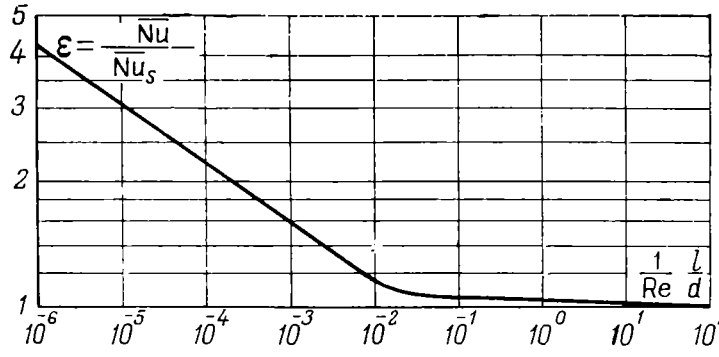

Fig. 8-10. Heat transfer for hydrodynamic initial section of a round tube at laminar flow and at t_w = const

It is rather difficult to account for the effect of natural convection with the tube in different positions, combined with different cooling and heating conditions. A comparatively slight difference in boundary conditions affects essentially experimental results, which hampers deriving of generalized relationships, true for all cases of the viscous-gravitational pattern of flow.

The mean heat-transfer coefficient for the viscous-gravitational pattern of flow can be estimated approximately by formula [8.5]

$$\overline{Nu}_{f,\,d} = 0.15 Re_{f,\,d}^{0.32} Pr_f^{0.33} (Gr_{f,\,d} Pr_f)^{0.1} (Pr_f/Pr_w)^{0.25} \bar{\varepsilon}_l \qquad (8\text{-}6)$$

The reference temperature used here is the mean fluid temperature in the tube. The reference dimension is the tube diameter. The factor $\bar{\varepsilon}_l$ accounts for the variation of the mean heat-transfer coefficient along the tube. If $l/d \geqslant 50$, then $\bar{\varepsilon}_l = 1$. With $l/d < 50$, the correction $\bar{\varepsilon}_l$ can be taken from Table 8-1 [8.6].

TABLE 8-1

Values of $\bar{\varepsilon}_l$ for Laminar Flow

l/d	1	2	5	10	15	20	30	40	50
$\bar{\varepsilon}_l$	1.90	1.70	1.44	1.28	1.18	1.13	1.05	1.02	1

Petukhov, Krasnoshchekov, Nolde and others have made extensive research work in heat transfer in both viscous and viscous-gravitational flow [8.7-8.10]. Experiments with water at q_w = const have shown [8.10] that free convection may result in a considerable variation in wall temperature over the circumference of a horizontal tube; when the fluid is heated, the temperature at the upper generating line is considerably larger than that at the lower. Whenever it is necessary to carry out thorough calculations of the rate of heat transfer for viscous-gravitational flow, one should refer to the literature cited above.

(b) Heat Transfer in Turbulent Flow

In considering the turbulent boundary layer earlier, we found [Eq. (7-32)] that

$$q_w = \frac{s_w c_p (t_0 - t_w)}{w_0 \left[1 + \frac{12}{w_0}\sqrt{\frac{s_w}{\rho}}(\mathrm{Pr}^{2/3} - 1)\right]}$$

For fluid flow in tubes we assume that $w_0 = \bar{w}$ and $t_0 = \bar{t}$, where $\bar{w}$ and $\bar{t}$ are the mean velocity and temperature in the tube. With non-separating flow, where the drop in pressure or hydraulic resistance is determined by friction, s_w can be determined, if the friction ξ for stabilized flow is known.

In Fig. 8-11 the drop in pressure $\Delta p = p_1 - p_2$ across sections *1* and *2* of the tube is spent in stabilized flow to overcome wall friction (and also for rearrangement of flow in the initial section of the tube). Then,

$$\Delta p f = s_w F$$

where f = cross-sectional area of the tube;
F = surface area of the tube between sections *1* and *2*.

According to the Darcy law

$$\Delta p = \xi \frac{l}{d} \frac{\rho \bar{w}_2}{2}$$

Then

$$s_w = \Delta p \frac{f}{F} = \xi \frac{l}{d} \frac{\rho \bar{w}^2}{2} \frac{f}{F}$$

For a circular tube

$$\frac{l}{d} \frac{f}{F} = \frac{1}{4}$$

from which

$$s_w = \frac{\xi}{8} \rho \bar{w}^2 \tag{8-7}$$

Substituting relationship (8-7) in Eq. (7-32) and dividing both sides of this equation by $\rho c_p \bar{w} (\bar{t} - t_w)$, we get:

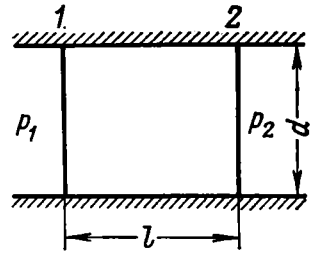

Fig. 8-11. Deriving Eq. (8-7)

$$\mathrm{St} \equiv \frac{\alpha}{\rho c_p \bar{w}} = \frac{\xi/8}{1 + 12\sqrt{\xi/8}(\mathrm{Pr}^{2/3} - 1)} \tag{8-8}$$

It will be recalled that the Stanton number, St, can be presented in the following form:

$$\mathrm{St} = \mathrm{Nu}/\mathrm{RePr}$$

If Pr = 1, then instead of Eq. (8-8) we have:

$$\mathrm{St} = \frac{\xi}{8} \quad \text{or} \quad \mathrm{Nu} = \frac{\xi}{8}\mathrm{RePr} \tag{8-9}$$

B.S. Petukhov and V.V. Kirillov [8.11] suggested the formula

$$\mathrm{Nu}_{fd} = \frac{\xi/8\, \mathrm{Re}_{fd}\mathrm{Pr}_f}{1.07 + 12.7\sqrt{\xi/8}(\mathrm{Pr}^{2/3} - 1)} \varepsilon_t \tag{8-10}$$

the constants in which are more exact. In Eq. (8-10) $\varepsilon_t = (\mu_f/\mu_w)^n$; $n = 0.11$ for a liquid undergoing heating and $n = 0.25$ for a cooled liquid*.

* When calculating heat transfer by means of formula (8-10) it is advisable to determine the friction factor ξ by the equation suggested by G. K. Filonenko: $\xi = 1/(1.82 \log \mathrm{Re}_{fd} - 1.64)^2$.

Equation (8-10) can be used to calculate the heat-transfer coefficients for stabilized heat-transfer processes. The reference temperature used here is either that averaged over the cross-section (when calculating local heat-transfer coefficients) or the mean temperature of the fluid in the tube (when calculating mean heat-transfer coefficients). The only exception is the coefficient of dynamic viscosity μ_w, chosen by wall temperature. Taken as reference is the inside diameter of the tube. Eq. (8-10) can be used to calculate heat transfer in various liquids at $Pr \geqslant 0.7$.

Equation (8-9) can be used to derive the calculation formula for $Pr \geqslant 1$, by introducing into it the experimentally determined function $f(Pr) = 0.91Pr^{0.43}$. The hydraulic friction factor ξ will be determined with the aid of formula

$$\xi = 0.184Re_d^{-0.2}$$

Then, introducing an additional correction for the variation of physical properties of dropping liquids, $\varepsilon_t = (Pr_f/Pr_w)^{0.25}$, we obtain the formula suggested by M.A. Mikheev [8.5]:

$$\overline{Nu}_{fd} = 0.021Re_{fd}^{0.8}Pr_f^{0.43}(Pr_f/Pr_w)^{0.25} \qquad (8\text{-}11)$$

Formula (8-11) describes the mean rate of heat transfer in straight plain tubes with $(l/d) > 50$. The reference temperature used here is the mean temperature of the fluid (liquid) in the tube, and the reference dimension is the inside diameter. The Prandtl number Pr_w is chosen by the mean wall temperature, $\bar{t}_w$.

A.S. Sukomel et al. [8.12] have worked out the following formula for calculating local heat-transfer coefficients for a gas in turbulent flow in a straight plain tube

$$Nu_{f(x)d} = 0.022Re_{f(x)}^{0.8}Pr_{f(x)}^{0.43}\varepsilon_l \qquad (8\text{-}12)$$

The reference temperature in the above formula is the section-mean temperature of the gas and the reference dimension—the inside diameter of the tube. The quantity ε_l is the correction accounting for the change of the heat-transfer coefficient in the initial thermal section. With $(x/d) \geqslant 15$ the correction $\varepsilon_l \approx 1$. At $(x/d) < 15$ and with turbulent flow originating at the very beginning of the tube investigations [8.12] have shown that the correction factor ε_l can be determined from the formula

$$\varepsilon_l = 1.38(x/d)^{-0.12} \qquad (8\text{-}13)$$

It follows from Eq. (8-13) that along the initial section the heat-transfer coefficient diminishes with increasing x.

When formulas (8-10) and (8-11) are used to calculate the mean rate of heat transfer for short tubes $(l/d \leqslant 50)$ the obtained values of $\overline{Nu}$ must be multiplied by the correction $\bar{\varepsilon}_l = \bar{\alpha}/\alpha_\infty$, where α_∞ is the heat-transfer coefficient at $(l/d) \to \infty$ [practically, $(l/d) > 50$].

As was mentioned in Sec. 8-1, the lengths of the initial hydrodynamic and thermal sections depend on a number of factors, for example, on the Reynolds number, degree of flow turbulence at the inlet, initial velocity distribution, thermal boundary conditions, etc. The same factors affect the correction coefficients ε_l and $\bar{\varepsilon}_l$. The values of the correction factors used presently in practical calculations, therefore, are not universal and reflect the nature of the research work that has been carried out to determine these factors. The smaller the ratio l/d (or x/d), the larger may be the difference between the correction factors and the greater the calculation error.

The correction factor $\bar{\varepsilon}_l = \bar{\alpha}/\alpha_\infty$ can be determined from Eq. (8-1). If no sufficient information on the conditions in which a process is developing is available, use can be made of the simpler formula (8-2). Evaluating $\bar{\varepsilon}_l$ with the aid of Eq. (8-13), we can come to the following formula:

$$\bar{\varepsilon}_l \approx 1 + \frac{2}{l/d} \tag{8-14}$$

where l is the length of the averaging section, measured from the inlet cross-section of the tube.

The local heat-transfer coefficient may depend on the variation in wall temperature along the tube. For turbulent flow the non-isothermal surface of the wall tells but slightly on heat transfer.

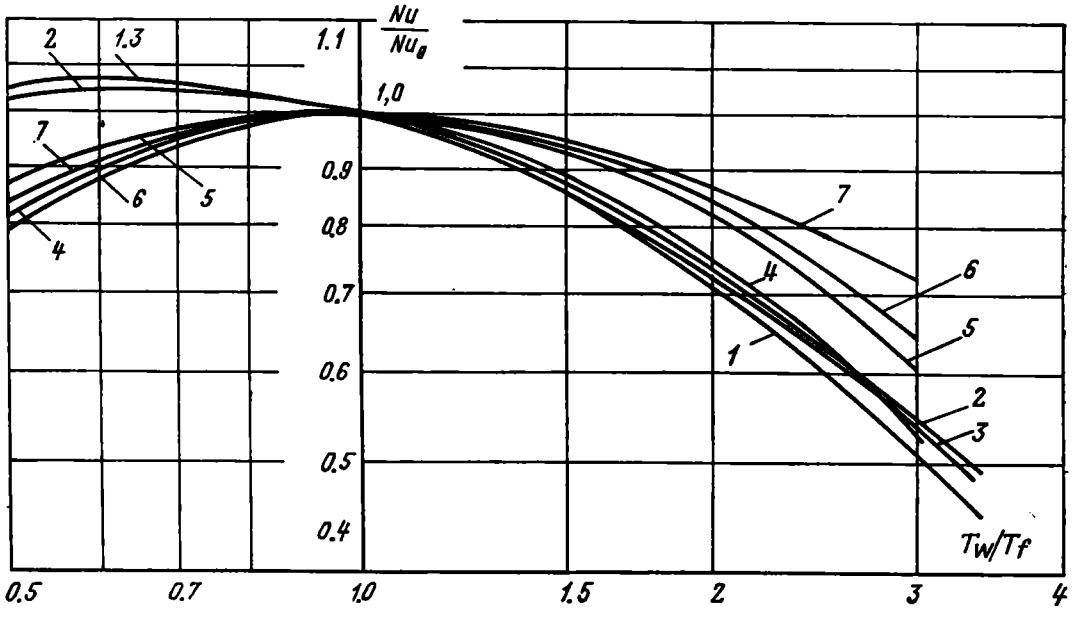

Fig. 8-12. The influence of temperature on the local rate of heat transfer at a turbulent flow of different gases in a tube ($x/d = 70$):
1—argon; *2*—air; *3*—helium; *4*—steam; *5*—carbon dioxide; *6*—ammonia; *7*—methane

When heat transfer takes place in a gas at great temperature differences the heat-transfer coefficients may differ from these calculated by means of Eqs. (8-10) through (8-12) [corrections of the type $(\mathrm{Pr}_f/\mathrm{Pr}_w)^n$ and $(\mu_f/\mu_w)^n$ are not applicable to gases]. The variation of the rate of heat transfer is usually accounted for by introducing into the right-hand side of Eqs. (8-7) through (8-9) the function $f(\Theta_w)$, where $\Theta_w \equiv T_w/T_f$; T_w is the mean or local wall temperature K, depending on whether the mean or local heat-transfer coefficient is being calculated, and T_f = respectively the mass mean temperature in the tube of the gas or in a given cross section of the tube, K.

Shown in Fig. 8-12 are the results obtained in measuring the local rate of heat transfer in a heated ($\Theta_w > 1$) and cooled ($\Theta_w < 1$) gas. When monoatomic and diatomic gases are cooled heat transfer is practically independent of the temperature factor, if the physical parameters are chosen by T_f. According to data available [8.13, 8.14] this independence occurs up to $\Theta_w = 0.08$. The rate of heat transfer from cooled polyatomic gases diminishes somewhat with increasing temperature difference.

Heat transfer in a heated gas depends materially on Θ_w (Fig. 8-12).

It will be noted in conclusion that it follows from Eqs. (8-11) and (8-12) that $\alpha \sim \bar{w}^{0.8}$, i.e. with turbulent flow the local heat-transfer coefficient depends on velocity to a greater degree than in the event of laminar flow. It also follows from Eq. (8-12) that at $(x/d) > 15$ $\alpha \sim d^{-0.2}$, i.e. that the smaller the diameter of the tube the greater the heat-transfer coefficient.

(c) *Heat Transfer in Transition Flow*

At great Reynolds numbers, ranging about from 2×10^3 to 10^4 heat transfer depends on a great number of factors which are very difficult to be taken into account.

The transition pattern of flow is characterized by the appearance of intermittent, or alternating, flow (see Sec. 7-3). In Fig. 8-13 the alternating factor ω,

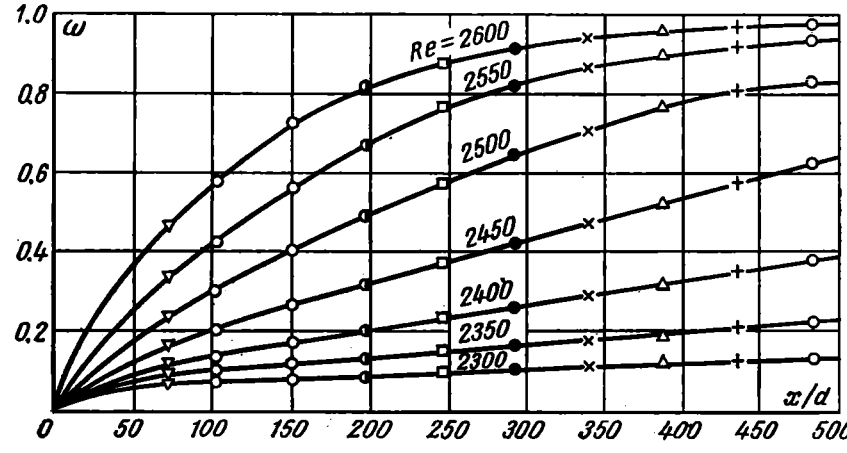

Fig. 8-13. Dependence of the alternation factor ω on the relative distance x/d and Reynolds number

characterizing a practical intermittent flow, is plotted for different Reynolds numbers against the relative distance from the tube inlet. At a constant Reynolds number the alternation factor increases with distance from the tube inlet, and it also increases with increasing Reynolds numbers. Thus, the greater the Reynolds number, the smaller the length of the tube along which the laminar pattern of flow may predominate.

In the general case a boundary layer can be considered in the initial section of the tube in which laminar, transition and turbulent patterns of flow are observed. The change from laminar to turbulent flow may occur in the core of the stream and in the boundary layer not simultaneously. Experimental studies have shown that in the presence of laminar flow in the boundary layer pronounced turbulent flow may exist in the core of the stream. The greater the degree of turbulence at the tube inlet, the shorter the laminar boundary layer [8.15].

The presence of free convection together with forced convection may change materially the development of the process. The intricate pattern of flow characteristic of the transition region of Reynolds numbers hampers a quantitative description of the process of heat transfer. No generalized calculation methods exist for heat transfer in the transition zone. The maximum and minimum heat-transfer coefficients can be evaluated approximately with the aid of the formulae derived for turbulent flow and viscous flow respectively.

8-4. Heat Transfer in Fluid Flow in Non-circular Conduits and Curved Rough Tubes

(a) *Heat Transfer in Tubes of Non-circular Cross-section*

Heat transfer in round tubes is best studied at present. Its calculation in conduits of non-circular cross-section is usually reduced to the determination of its value in an equivalent tube of circular cross-section and of a diameter equal to

$$d_{eq} = \frac{4V}{F} = \frac{4f}{P} \tag{8-15}$$

where f is the clear cross-section of the tube; and P is the full (wetted) perimeter of cross-section. The equivalent or hydraulic diameter d_{eq} is thus the quadruple ratio of the volume of the fluid V filling the tube to the surface F. With round tubes $d_{eq} = d$.

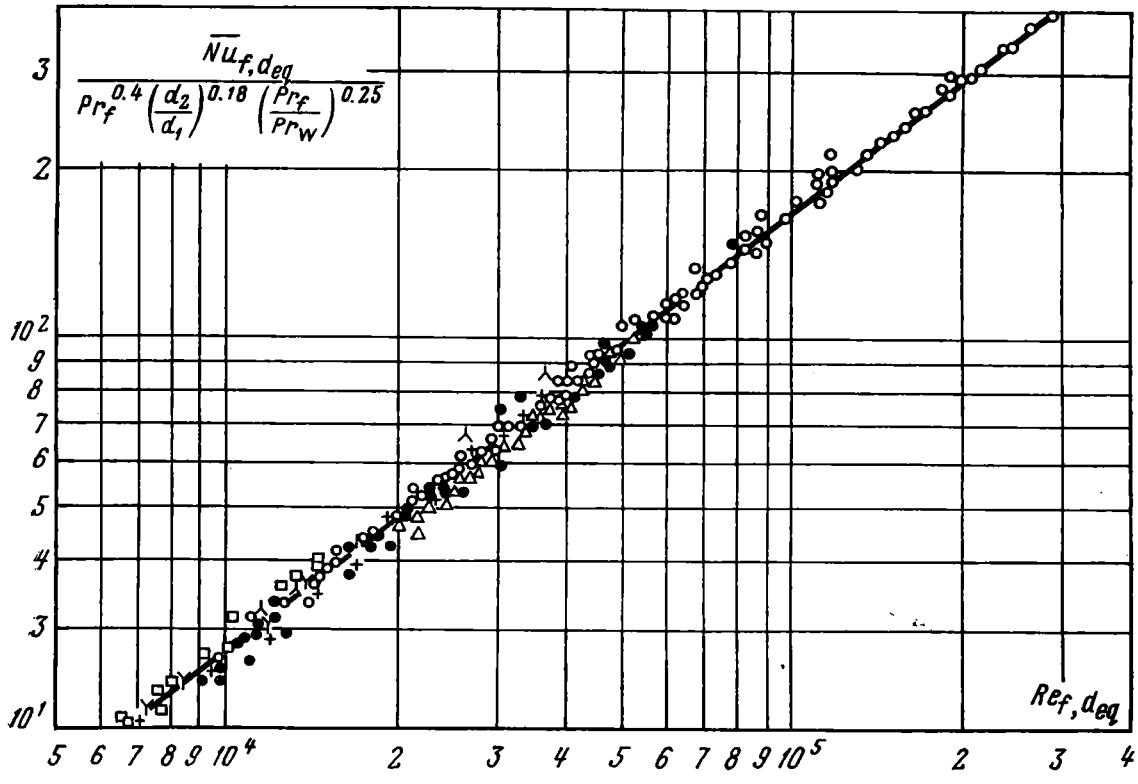

Fig. 8-14. Heat transfer with turbulent flow in annular conduits of air, water, and transformer oil

This method of calculation with the aid of equivalent diameter d_{eq} is an approximate one only. Its exact range of application has not been established. Experimental research, however, indicates that the approximate method often yields satisfactory results. As recommended by Mikheev [8.6], for fluid in turbulent flow in conduits of rectangular (the ratio of sides a/b = 1 to 40) and triangular cross-section, and for banks of tubes in parallel flow heat transfer can be calculated with the aid of the hydraulic equivalent diameter.

According to [8.16] the method is unsuitable for fluids in laminar flow, and for liquid metals.

According to [8.17] the mean heat-transfer coefficients at the inner wall for gases and liquids in turbulent flow in conduits of annular cross-section may be calculated by the equation

$$\overline{Nu}_{fd_{eq}} = 0.017 Re^{0.8}_{fd_{eq}} Pr_f^{0.4} (Pr_f/Pr_w)^{0.25} (d_2/d_1)^{0.18} \quad (8\text{-}16)$$

The reference temperature in this equation is the mean temperature of the fluid in the tube (excepting Pr_w), and the reference dimension $d_{eq} = d_2 - d_1$. The peculiarities of heat transfer developing in annular conduits are accounted for by the factor $(d_2/d_1)^{0.18}$, where d_1 is the inner diameter of the conduit, and d_2 is the outer diameter.

Formula (8-16) is correlated graphically with experimental data in the chart in Fig. 8-14. It holds at the following values: $\frac{d_2}{d_1} = 1.2$ to 1.4, $l/d = 50$ to 460 and $Pr_f = 0.7$ to 100.

(*b*) *Heat Transfer in Curved Tubes*

Heat exchangers in which one of the heat-carrying agents flows inside a curved conduit are often encountered in engineering applications. The centrifugal forces appearing in a fluid flowing along a curved conduit create circulation currents in the cross-section, so-called secondary circulation (Fig. 8-15).

As a result the fluid begins to flow along a helical line, i.e. a complex pattern of flow sets in. The centrifugal effect diminishes with increasing radius R, and it disappears in a straight tube for which $R = \infty$. Secondary circulation can be observed in both turbulent and laminar flow. With the latter there is an ordered flow of the fluid made up of several streams with complex paths which do not intermingle.

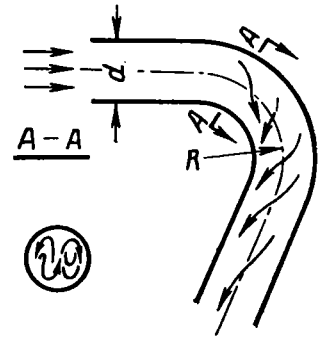

Fig. 8-15. Flow in a bend

Secondary circulation has been shown experimentally to appear only at Reynolds numbers exceeding certain critical values of Re'_{cr}, with $Re'_{cr} < Re_{cr1} \approx$ $\approx$ 2000 (for straight tubes).

The following formula has been proposed [8.18] for determining the magnitude of Re'_{cr} for fluid flowing through helical coils:

$$Re'_{cr} = \frac{16.4}{\sqrt{d/R}} \quad (8\text{-}17)$$

where d is the inner diameter of the tube, and R is the coil radius. Formula (8-17) holds at $\frac{d}{R} \geqslant 8 \times 10^{-4}$.

Further increase in the Reynolds number may bring about fully developed turbulent flow. The critical Reynolds number Re''_{cr} for fluid flow in curved tubes (helical coils) is larger than the critical Reynolds number Re_{cr1} for straight tubes. The transition to turbulent flow is also smoother than in straight tubes.

The magnitude of Re''_{cr} with a ratio of $\frac{d}{R} \geqslant 8 \times 10^{-4}$ can be determined from the following formula [8.18]:

$$Re''_{cr} = 18\,500 \left(\frac{d}{2R}\right)^{0.28} \tag{8-18}$$

Equations (8-17) and (8-18) are illustrated graphically in Fig. 8-16. The charts representing Re'_{cr} and Re''_{cr} delineate three zones. With $Re < Re'_{cr}$ the flow is laminar with no secondary circulation (region *1*); at $Re'_{cr} < Re < Re''_{cr}$ the flow is laminar with a second circulation (region *2*); at $Re > Re''_{cr}$ the flow is turbulent and there is secondary circulation (region *3*).

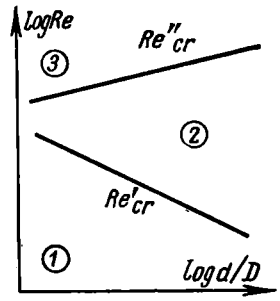

Fig. 8-16. Dependence of Re'_{cr} and Re''_{cr} on d/D in curved tubes (coils)

According to [8.18] heat-transfer coefficient at $Re'_{cr} < Re < Re''_{cr}$ can be calculated by equation (8-11).

When $Re > Re''_{cr}$ the same formula must be used, but the value of the heat-transfer coefficient obtained must be multiplied by the factor ε_{cur}, which is determined for coiled tubes by the following equation:

$$\varepsilon_{cur} = 1 + 1.8 \frac{d}{R} \tag{8-19}$$

In coils the centrifugal effect spreads over the entire length of the tube; the effect occurring in elbows and bends of straight tubes is only local, but it does influence the flow downstream of the bend. Consequently, the rate of heat transfer downstream of the bend must be a little greater than the rate upstream of the bend, and it then diminishes to magnitudes characteristic of heat transfer in straight tubes. No data are yet available to account for this effect.

(*c*) *Heat Transfer in Rough Tubes*

Turbulent flow in rough tubes is accompanied by important hydrodynamic transformations linked with the height of the surface irregularities δ and the thickness of the laminar sublayer δ_{ls}. To simplify, two principal cases of the phenomenon may be considered: surface irregularities deeply submerged in the laminar sublayer ($\delta \ll \delta_{ls}$), and surface irregularities protruding beyond this sublayer ($\delta \gg \delta_{ls}$).

In the first case the surface irregularities do not disturb the laminar pattern of fluid flow in the sublayer, and the fluid flows over them without separation or eddy formation. There is then no difference between a rough and plain tube. Such a pattern of flow is the more probable, the smaller the Reynolds number and relative roughness δ/d (d = tube diameter), since the thickness of the sublayer increases with diminishing Reynolds number. The idea of relative roughness acquires here a purely hydrodynamical meaning.

If $\delta \gg \delta_{ls}$ flow becomes disturbed in the laminar sublayer, and the fluid flow takes on a separated, eddy pattern. The turbulent pulsations increase at the wall, especially at the top of the surface irregularities. Since with fluid in turbulent flow the thermal resistance to heat transfer is concentrated in the sublayer, a change in the pattern of flow causes an increase in the rate of heat transfer. With laminar flow heat transfer coefficient and the pressure drop are independent of the degree of relative roughness. In this case heat transfer can increase

due to the fact that heat-exchange surface of a rough tube is greater than that of a plain one (finning effect).

For fluid in turbulent flow roughness begins to affect heat transfer and pressure drop at various Reynolds numbers. The smaller the ratio δ/d, the larger the limiting Re_{max} at which a change in pattern of heat transfer occurs. Increase in the heat-transfer coefficient is accompanied here by an increase in the pressure drop Δp.

In experimental work roughness was created by mechanical treatment (knurling, threading). Compared with plain tubes, it was shown by experiment that heat transfer in rough tubes depends, in addition, on the shape of the surface irregularities, the magnitude of the relative roughness δ/d and the distance between surface irregularities.

Experimental data obtained in the Moscow Power Institute [8.19] are presented grafically in Fig. 8-17. The experiments used water as the fluid; roughness was provided by triangular thread of $\frac{l}{d} = 104$, where d is the inner diameter of a round tube reduced to surface irregularities. The heat-transfer coefficient was taken for a fictitious tube surface πdl.

It follows from the chart that heat transfer in a rough tube may be tripled under certain conditions, compared with a plain tube. This permits roughness to be used to intensify heat transfer.

If the roughness is created in an irrational way, the heat-transfer coefficient may even be lower than for a plain tube. There may also be a reduction in the heat-transfer coefficient, if the surface irregularities are high, and a stagnation zone may form between them and the wall surface. The effect shows itself in a different way, depending on the Reynolds number, shape of the surface irregularities, distance between them, etc.

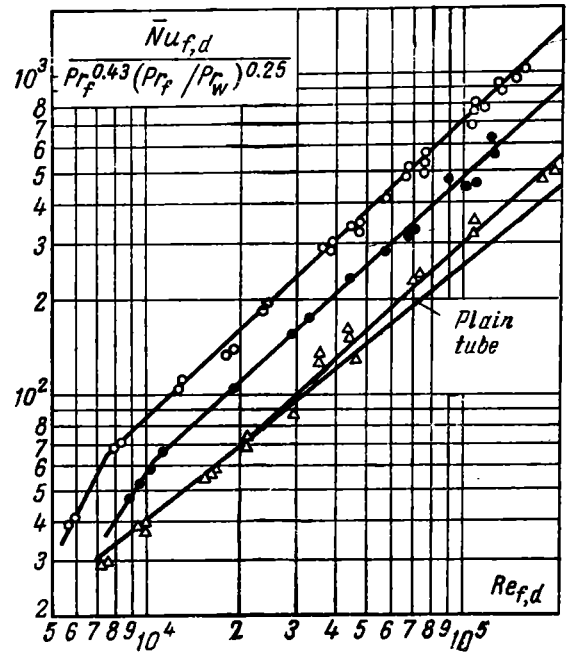

Fig. 8-17. Heat transfer with turbulent fluid flow in rough circular tubes (d = 16.7 mm, water; roughness simulated by triangular thread):
Δ—δ/d = 0.0063; ●—δ/d = 0.011; ○—δ/d = 0.038

With one and the same relative height δ/d heat transfer can be both intensified and impaired. Therefore, unfounded extrapolation of experimental results for appliance to an unexplored region of reference parameters is especially dangerous in the case being considered.

For this purpose, it is advisable to create roughness with relative pitches $(s/\delta)_{opt}$ = 12-14 (s being the distance between adjacent irregularities) [8.20]. At $s/\delta \geqslant 8$ the mean heat-transfer coefficient may be calculated by the formula suggested by Gomelauri [8.20], who has generalized the results of his tests with water and transformer oil, the tests of Nunner (air), and Brauer and Fedynsky (water):

$$\overline{Nu}_{fd_{eq}} = 0.022 Re^{0}_{fd_{eq}} Pr_f^{0.47} (Pr_f/Pr_w)^{0.25} \varepsilon_r \qquad (8\text{-}20)$$

Here

$$\varepsilon_r = \exp\left[0.85\frac{(s/\delta)_{opt}}{s/\delta}\right] \quad \text{at} \quad \frac{s}{\delta} > \left(\frac{s}{\delta}\right)_{opt}$$

$$\varepsilon_r = \exp\left[0.85\frac{s/\delta}{(s/\delta)_{opt}}\right] \quad \text{at} \quad \frac{s}{\delta} < \left(\frac{s}{\delta}\right)_{opt}$$

This formula was derived from the results of tests on heat transfer with heat-carrying agents in turbulent flow in tubes and annular conduits. It may be taken, when using formula (8-16), that the optimum relative pitch $(s/\delta)_{opt}$ is 13 at any Prandtl number between 1 and 80. The reference temperature and linear dimension are chosen as for formula (8-16).

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CHAPTER

9

HEAT TRANSFER FOR TUBES AND BANKS OF TUBES IN FORCED CROSS FLOW

9-1. Heat Transfer for a Single Round Tube in Cross Flow

The process of heat transfer for a single tube in infinite cross flow is characterized by a number of peculiarities. Smooth unseparated flow of fluid past the cylinder, as illustrated in Fig. 9-1, takes place only at Reynolds numbers of about $\mathrm{Re} = \frac{w_0 d}{\nu} \leqslant 5$ (w_0 = velocity of free-stream or overtaking flow, d = outside diameter). With $\mathrm{Re} > 5$ the cylinder in cross flow is not a streamlined body. The boundary layer forming at the front half of the tube separates

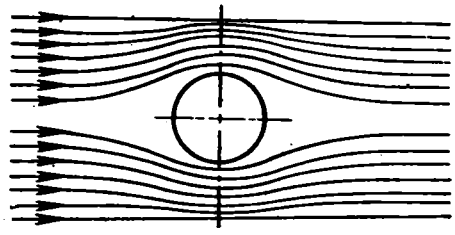

Fig. 9-1. Cylinder in non-separated cross flow

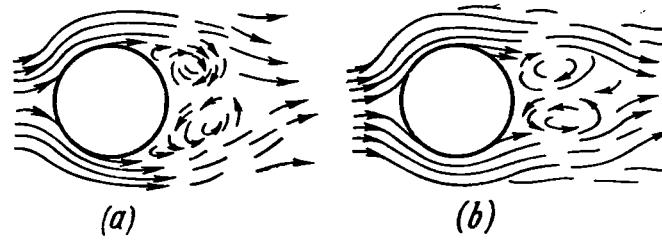

Fig. 9-2. Cylinder in cross flow with separation of laminar layer (*a*) and turbulent layer (*b*)

from its surface at the rear half, and two symmetrical eddies form behind the cylinder. These eddies straighten out at a distance from the cylinder with further increase of Reynolds number. Then the eddies separate periodically from the tube and are swept downstream by the fluid flow, forming an eddy path behind the cylinder (Fig. 9-2).

Up to $\mathrm{Re} \approx 10^3$ the frequency of eddy separation increases, then in the region where Reynolds numbers range from 10^3 to 2×10^5 it becomes practically constant and is characterized by the Strouhal number $\mathrm{Sh} = fd/w_0 = 0.2$, where f is frequency.

The separation of the boundary layer results from an increase in pressure along the flow and deceleration of the fluid at the solid wall. As the fluid flows past the front half of the cylinder the cross section of flow diminishes and the velocity of fluid increases, resulting in that the statical pressure drops at the wall. On the contrary in the rear part of the cylinder static pressure rises, since velocity diminishes here.

Velocity and, consequently, kinetic energy of the fluid are small at the surface of the cylinder, owing to the effect of viscosity. The increase in velocity observed along the flow results in stagnation of the fluid, which leads to appearance of a return flow. The return flow pushes the laminary layer away from the surface of the body with the result that flow is separated and eddies form (Fig. 9-3).

The phenomena of flow separation and formation of eddies are the main peculiarities of a tube in cross flow.

Separation of the laminar boundary layer is observed at comparatively low Reynolds numbers and a slight turbulence of the free-stream flow upstream of the cylinder, occurring at an angle φ around 82 degrees (the angle φ is measured from the rear point of the tube) and takes place downstream of the point of minimum pressure.

At large Reynolds numbers the stagnation of flow due to increase of pressure does not lead to separation, but to a change to turbulent flow in the layer. The turbulent boundary layer possesses high kinetic energy since it acquires additional energy from the outer flow due to turbulent fluctuations. The result is that the point of flow separation shifts downstream. The turbulent boundary layer separates from the wall of the tube at an angle $\varphi \approx 140$ degrees. The shifting of the point of separation results in a smaller eddy zone downstream of the cylinder (Fig. 9-2), and an improvement of flow over the cylinder.

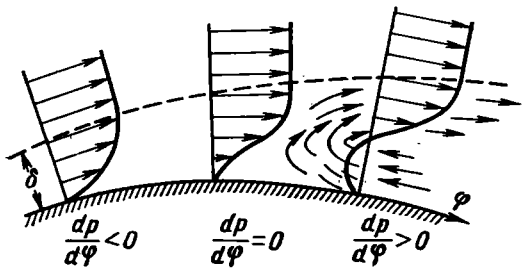

Fig. 9-3. Velocity distribution at the surface of a cylinder and the formation of reversed flow

Flow becomes turbulent in the boundary layer at quite large Reynolds numbers. Various authors have found the turbulent boundary layer to appear at a critical Reynolds number of $Re_{cr} = 1 \times 10^5$ to 4×10^5.

The critical Reynods number is essentially affected by the turbulence of the free-stream flow and other factors (see Sec. 7-3). The greater the initial turbulence, the smaller the Reynolds number at which the turbulent boundary layer sets in. The intensity of flow turbulence may depend on the design of the pilot plant and vary with flow velocity. The critical Reynolds number Re_{cr} may be assumed equal to 2×10^5.

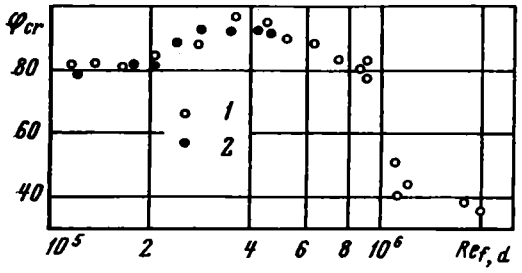

Fig. 9-4. Dependence of φ_{cr} corresponding to the transition of a laminar boundary layer into a turbulent one on Reynolds number:

○—d = 50 mm; ●—d = 30.7 mm; water; q_w = const

According to A.A. Zhukauskas and P.M. Dauetas at $Re > 2 \times 10^5$ the angle φ_{cr}, corresponding to the change from laminar flow in the boundary layer to turbulent flow, depends on the Reynolds number (Fig. 9-4).

The peculiarities of fluid flow over a tube also influence the process of heat transfer. The variation of the heat-transfer coefficient around a cylinder is illustrated in Fig. 9-5. Curve *1* corresponds to heat transfer with separation of the laminar boundary layer, and curve *2* to heat transfer with separation of the turbulent boundary layer.

The drop in the heat-transfer coefficient behind the tube is explained by an increase in thickness of the laminar boundary layer. From curve *1* it is seen that minimum heat transfer is at the point where separation of the boundary layer takes place; the rear of the cylinder is exposed to fluid in complex eddy flow. Heat transfer at the rear of the cylinder is small at low Reynolds numbers, increases with rising Reynolds numbers, and may become equal to the intensity of heat transfer at the front of the tube.

Curve *2* has two minima. The first corresponds to the change from laminar to turbulent flow, and is accompanied by a sharp increase in the heat-transfer coefficient: the increase at large Reynolds numbers may be of an order of two or three times. The second minimum occurs where separation of the turbulent boundary layer takes place. The drop in the rate of heat transfer upstream of this point may be explained by stagnation of the boundary layer. Downstream of it the tube is exposed to eddies of a complex nature, and heat transfer increases somewhat there.

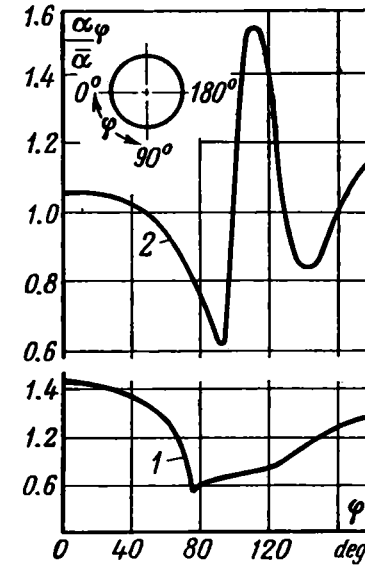

Fig. 9-5. Variation of local heat-transfer coefficient α around a cylinder in normal gas flow (cylinder in cross flow):
1—Re = 70 800; *2*—Re = 219 000; $\bar{\alpha}$—circumferential mean local heat-transfer coefficient· q_w = const

At a rate of heat flow q_w = const the intensity of heat transfer in the laminar boundary layer is on the average 15 to 20 per cent greater than with t_w = const. It is clear from the foregoing that the heat transfer from a cylinder is closely linked with the nature of the flow in which the cylinder is placed. Because of the complex nature of flow over a single cylinder the character of variation in heat transfer is complicated. Hence any theoretical solution of the problem is difficult for various fields of flow. Detailed experimental studies of the circumferential mean rate of heat transfer for a cylinder have been carried out by A.A. Zhukauskas [9.1, 9.2], using as well the experimental results obtained by other investigators. Generalization of the available data indicated that the circumferential mean heat-transfer coefficient for a cylinder may be described by the following equations:

at $5 < \mathrm{Re} < 10^3$

$$\overline{\mathrm{Nu}}_{fd} = 0.5\mathrm{Re}_{fd}^{0.5}\mathrm{Pr}_f^{0.38}(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25} \tag{9-1}$$

at $10^3 < \mathrm{Re} < 2 \times 10^5$

$$\overline{\mathrm{Nu}}_{fd} = 0.25\mathrm{Re}_{fd}^{0.6}\mathrm{Pr}_f^{0.38}(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25} \tag{9-2}$$

at $\mathrm{Re} = 3 \times 10^5$ to 2×10^6

$$\overline{\mathrm{Nu}}_{fd} = 0.023\mathrm{Re}_{fd}^{0.8}\mathrm{Pr}_f^{0.37}(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25} \tag{9-3}$$

Here the linear dimension is the outside diameter of the tube, and the velocity of flow is reduced to the narrowest cross-section of the conduit where the cylinder is positioned. As reference temperature is taken the mean temperature of the fluid involved; the Prandtl number for the wall, Pr_w, is selected, by way of exception, for the mean temperature at the tube surface.

According to formulae (9-1) and (9-2) the pattern of heat transfer changes at $\mathrm{Re} = 10^3$. In the apparatus used by A.A. Zhukauskas turbulence of flow was slight. If the overtaking flow is artificially turbulized, the heat-transfer coefficient will be larger than follows from Eq. (9-2), and a change in the pattern of heat transfer will then occur at Reynolds numbers lower than 10^3.

The flow can be made artificially turbulent by means of bluff grids; flow is especially disturbed on the discharge side of a fan or pump. If a cylinder is

placed, for example, directly behind a fan or pump, the rate of heat transfer may be doubled.*

Fig. 9-6 shows the effect of the degree of turbulence of free stream flow on mean heat transfer for a cylinder. Tests conducted with air indicate that in the interval $100 \leqslant \mathrm{Re\,Tu} \leqslant 10^4$ and at Tu equal to or less than 14 per cent mean heat transfer can be described by equation [9.3]

$$\overline{\mathrm{Nu}} = \overline{\mathrm{Nu}}_0 [1 + 0.09\,(\mathrm{Re\,Tu})^{0.2}]$$

where Nu_0 is calculated by formula (9-2).

In a fluid at high temperature, the temperature of the tube may be close to the temperature limit of its material. Calculations based on mean heat-transfer

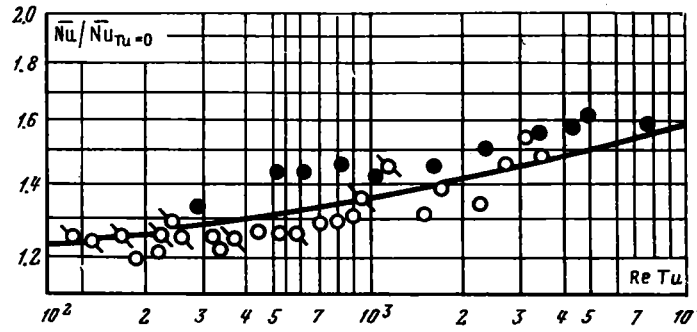

Fig. 9-6. Mean heat transfer of a cylinder in cross flow of artificially turbulized air: ○—Tu = 2 to 4%; ○—Tu = 6 to 8%; ●—Tu = 11 to 14%

coefficients give the mean temperature of the wall, but the local temperatures may be higher or lower than the mean temperature. The temperature of a thin-walled tube made of a material of low thermal conductivity may vary considerably around its circumference, if the coefficients of heat transfer of the outer surface exceed those of a fluid coolant flowing inside the tube. The highest local heat-transfer coefficient on the outside will have a corresponding maximum local wall temperature. With Reynolds numbers $\mathrm{Re} < 2 \times 10^5$ the maximum values of α and t_w will be encountered on the front of the cylinder. From Kruzhilin's theoretical solution [9.4], the process of heat transfer at the front of the cylinder ($\varphi = 0$) is described by equation

$$\mathrm{Nu}_d = 1.04\mathrm{Re}_d^{0.5}\mathrm{Pr}^{1/3}$$

This equation has been quite well corroborated by tests with air.

Local heat transfer for tubes placed in cross flow of various fluids was studied under various boundary conditions by A.A. Zhukauskas, I.I. Zhyugzhda, V.I. Katinas et al. Some calculation formulas for the local and mean rates of heat transfer at individual sections can be found in literature [9.5].

Formulae (9-1) and (9-2) are true when the angle ψ formed by the direction of flow and tube axis (the angle of attack) is a right angle. If $\psi < 90°$ the rate of heat transfer diminishes, and at $\psi = 30$ to 90° this decrease in heat transfer can be evaluated with the aid of the following approximate relationship

$$\alpha_\psi = \bar{\alpha}_{\psi=90°}\,(1 - 0.54\,\cos^2\psi)$$

* Fine-mesh grids are sometimes used to reduce turbulences; the fine eddies formed by them attenuate rapidly.

where $\bar{\alpha}_\psi$, $\bar{\alpha}_{\psi=90°}$ are heat-transfer coefficients respectively at $\psi < 90°$ and $\psi = 90°$.

The angle of attack $\psi = 0$ corresponds to the longitudinal fluid flow of a tube. All other things being equal cross-sectional fluid flow is associated with higher rates of heat transfer. It will be noted that the natures of cross and longitudinal flow past a single tube differ essentially.

The hydrodynamic theory of heat transfer relates heat transfer to the drop in pressure due to the resistance offered by friction. The total resistance of a cylinder in cross flow is the sum of the resistance of friction and resistance of shape. Shape resistance is due to separation of flow and subsequent formation of eddies. Friction resistance is only a small fraction of the total resistance. Total resistance of a cylinder is usually measured. Therefore the hydrodynamic theory of heat transfer is not applicable to cylinders in vortex flow.

9-2. Heat Transfer in Flow Across Banks of Tubes

Heat exchangers seldom consist of a single tube placed in cross flow, because the heat transfer surface is small here. Usually many tubes are assembled in

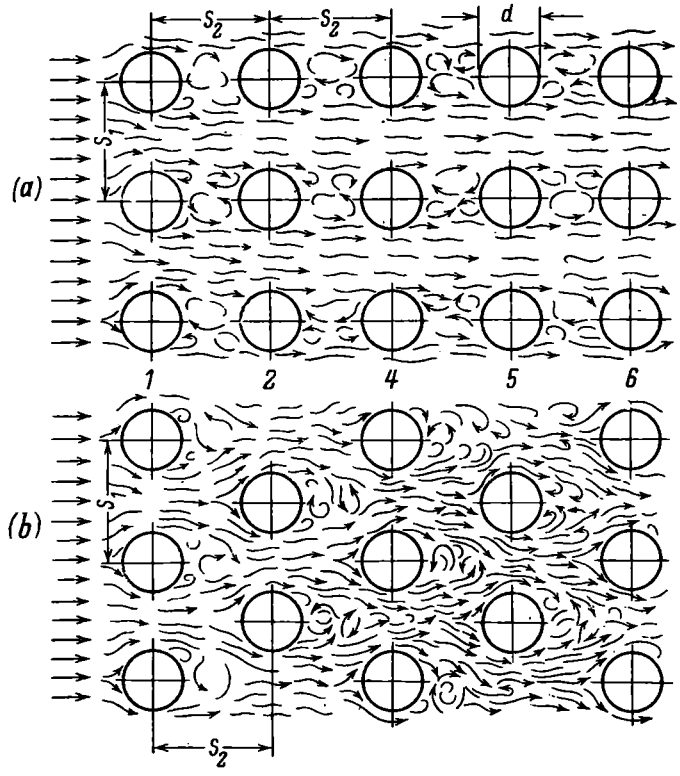

Fig. 9-7. Geometry of in-line (*a*) and staggered (*b*) arrangement of tubes in banks, and the pattern of fluid flow across these banks of tubes

banks. The arrangement of tubes in a bank may be various, but those most frequently encountered in engineering practice are banks of staggered or of in-line tubes, as illustrated in Fig. 9-7.

A bank of tubes is characterized by its transverse pitch s_1 (the distance between the tube axes in the direction transverse to the flow) and longitudinal pitch s_2 (the distance between the axes of two adjacent rows of tubes, arranged

one after another in the direction of fluid flow). Beside s_1 and s_2 the tubes in banks are also characterized by the outer diameter of the tubes and by the number of rows in the direction of fluid flow (each bank shown in Fig. 9-7 consists of five rows). The pitches s_1 and s_2 and the tube diameter d are usually constant for any bank of tubes, and do not vary either in the direction of or across flow. The flow of fluid across a bank of tubes is rather complex. Adjacent tubes of the bank affect the flow past the neighbouring tubes. As a result the heat transfer for tubes of a bank differs from that characteristic of a single tube. A bank is usually placed inside a channel or conduit. The flow across it can therefore be linked with the flow in the conduit.

Two basic patterns of fluid flow are known: laminar and turbulent. These patterns can take place at flow across a bank. The pattern of the flow across a bank depends in many ways on the nature of the flow in the channel upstream of it. If at a given rate of fluid flow and temperature variation along the channel the pattern of flow would have been turbulent in the absence of a bank of tubes, it will necessarily be turbulent across a bank of tubes, since the latter is an excellent turbulizer. However, if a bank of tubes is placed in a channel where flow would have been laminar without it, then both patterns of flow may be ensured, depending on the Reynolds number. The smaller the Reynolds number, the more stable the laminar flow will be, and the larger the Reynolds number, the easier to make the flow turbulent. At low Reynolds numbers flow may remain laminar, the passages between the tubes forming individual narrow channels of varying cross-section (the limiting case where the distances between tubes are very large is an exception).

The turbulent pattern of flow in banks of tubes is most often encountered in engineering practice. For instance, the tubular heating surfaces of boilers are exposed to turbulent cross flow of gases.

Heat transfer in turbulent flow, however, may be governed by different laws, due to the different nature of flow on the walls of tubes. The law of heat transfer changes with the appearance of a turbulent boundary layer at the surface of tubes. Experiments with individual tubes have shown this boundary layer to appear at $\mathrm{Re} > 2 \times 10^5$. In banks of tubes it may appear at smaller Reynolds numbers. For banks of tubes the critical Reynolds number Re_{cr} may be taken as 1×10^5. In this case the velocity present in the Re number is calculated by the narrowest cross-section of the bank, and the reference dimension is the external diameter of the tube.

With $\mathrm{Re} < 1 \times 10^5$ the front of a tube is exposed to a laminar boundary layer, and the back to disordered eddies. Thus, whereas the fluid is in turbulent flow in the space between the tubes (annulus), a layer of fluid may be in laminar flow on the front part of the tube—a *mixed* fluid flow may take place.

Change in the nature of fluid flow affects heat transfer. Three basic patterns of flow and heat transfer can be distinguished for banks of tubes in cross flow, which we will denote as laminar, mixed, and turbulent.

The pattern of mixed flow is that best understood. It is often encountered in engineering applications, especially in boilers. The Reynolds numbers of mixed flow range from 1×10^3 to 1×10^5. Let us consider the basic peculiarities of this flow pattern.

The flow of fluid across the first row of tubes in either staggered or in-line arrangement is similar to the flow around a single cylinder. The nature of flow across the remaining tubes of the bank depends largely on the type of bank (Fig. 9-7). The tubes of the second and subsequent rows of an in-line bank are positioned in the vortex region of the upstream tubes, and circulation of the

fluid in the vortex region is slight, because it flows mainly through the longitudinal passages between the tubes. Hence both the fronts and backs of the tubes in an in-line bank are affected much less by the flow than the same parts of a single tube or than the fronts of the tubes in the first row. In a staggered bank the nature of the flow across deeply arranged tubes differs only little from that across the tubes of the first row.

The distribution of the local heat-transfer coefficients around the tubes in different rows of a bank corresponds to the described nature of the flow across

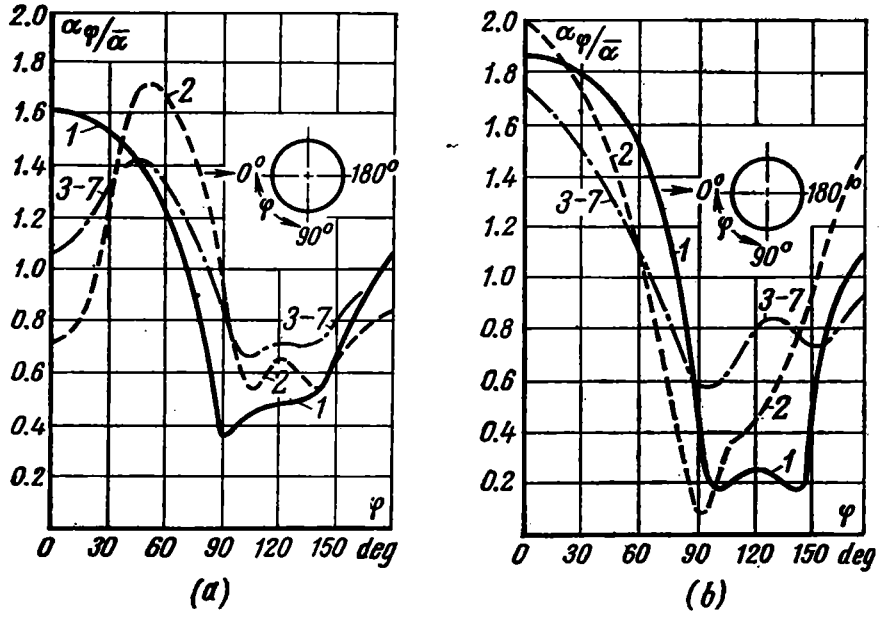

Fig. 9-8. Variation of heat-transfer coefficients around the tubes of different rows of in-line (*a*) and staggered (*b*) banks; Re = 14 × 10³, air

the bank. The distribution of the local heat-transfer coefficients α at a definite Reynolds number is illustrated graphically in Fig. 9-8, where φ is the angle measured from the front point of a tube, the digits indicate the numbers of the rows.

Examination of the curves shows that the variation in the local heat-transfer coefficients α around the tubes in the first row of both types of banks corresponds to the distribution of α characteristic of a single tube in cross flow. The nature of the curves changes for the second and subsequent rows of an in-line bank: maximum heat transfer is observed not at the front point, but at $\varphi \approx 50°$. There are two such maxima located exactly in the sections of the tubes subject to impact of the overtaking streams.

The front of the tubes, however, is not directly affected by the flow, therefore heat transfer is not large here. In staggered banks maximum intensity of heat transfer is observed at the front of the tubes in all the rows (with possible exceptions only at large Reynolds numbers or small ratios of s_2/d).

The mean rate of heat transfer also changes in the first rows of banks. Numerous investigations (N.V. Kuznetsov, V.M. Antufiev) made it possible to formulate a number of general conclusions: (a) mean heat-transfer rate varies in the first row and is determined by the initial turbulence of flow; (b) mean heat-

transfer rate becomes steady from about the third row, because the turbulence in the deeper rows depends on the arrangement of the bank, which is essentially a system of eddy formers.

With overtaking flow of small turbulence the heat transfer in the first row of a staggered bank is about 60 per cent of that in the third and subsequent rows, while heat transfer in the second row is about 70 per cent. The intensity of heat transfer in the first row of an in-line bank is also about 60 per cent of the intensity in the third and subsequent rows, but about 90 per cent in the second row. The variation of heat transfer in the rows of a bank is presented diagrammatically in Fig. 9-9, in this graph the ratio ε_i of the mean heat-transfer coefficient $\bar{\alpha}_i$ of any row to that of the third row $\bar{\alpha}_{i=3}$ is plotted along the ordinate, and the numbers of the rows—along the abscissa.

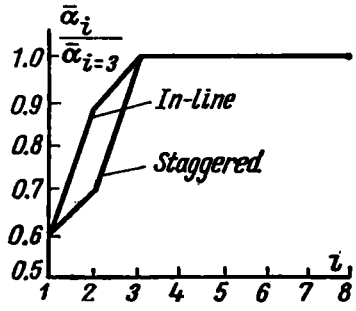

Fig. 9-9. Diagram of variation of heat-transfer coefficients for the individual rows of in-line and staggered banks of tubes

As has been mentioned, the increase in the intensity of heat transfer across the rows is due to additional turbulization of flow in a bank. If the flow is artificially made more turbulent upstream of the bank (e.g. by means of various eddy formers—sudden expansion, passage through a fan or pump), heat transfer in the first rows may be equal to or even larger than in the more deeply arranged rows. The pattern of flow and intensity of heat transfer in these rows are determined by the layout of the bank and are independent of the initial turbulence.

Thus, with a highly turbulent overtaking flow the bank of tubes can even act as a deturbulizing device. No reliable data, however, are available on this point, which could be used to determine the mean coefficient of heat transfer $\bar{\alpha}$ for the first two rows. Calculations can be based on the assumption that $\varepsilon_i = 1$ for all rows of the bank.

With multiple-row banks of tubes the fraction of the heat transferred in the first rows is insignificant compared with the quantity of heat transferred in the whole bank, and inaccuracy in determining ε_i will not lead to serious errors in calculating $\bar{\alpha}$.

Heat transfer in a bank of tubes also depends on the distance between the individual tubes. It is customary to express this distance by the dimensionless characteristics s_1/d and s_2/d, known respectively as the *relative transverse* and *longitudinal* pitches.

With mixed flow ($\mathrm{Re}_{fd} \approx 10^3$ to 10^5) the mean heat-transfer coefficient of any row of a bank can be determined by the following equation [9.6, 9.7]

$$\mathrm{Nu}_{fd} = c\mathrm{Re}_{fd}^{n}\mathrm{Pr}_f^{0.33}\,(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25}\,\varepsilon_i\,\varepsilon_s \qquad (9\text{-}4)$$

where $c = 0.41$ and $n = 0.60$ for staggered banks, and $c = 0.26$ and $n = 0.65$ for in-line banks. The reference dimension in formula (9-4) is the outside diameter of tubes. The fluid velocity in the dimensionless term Re_{fd} is calculated for the narrowest cross-section of the bank. The mean temperature of the fluid is taken as reference temperature (Pr_w, chosen for the wall temperature, is an exception).

The correction factor ε_s accounts for the effect of the relative pitches. For the deeper rows in an in-line bank [9.7]

$$\varepsilon_s = (s_2/d)^{-0.15}$$

and in a staggered bank [9.8]

$$\text{at } \frac{s_1}{s_2} < 2 \quad \varepsilon_s = (s_1/s_2)^{1/6}$$

$$\text{at } \frac{s_1}{s_2} \geqslant 2 \quad \varepsilon_s = 1.12$$

In experimental studies the relative pitches were varied from 1.24 to 4.04 [9.7]; s_1/d = 1.3 to 2.6; s_2/d = 0.61 to 3.9 and $\frac{s_1/d}{s_2/d}$ was varied about from 0.33 to 3.4 [9.8].

Such a variation of the relative pitches resulted in that when tests were conducted at a w_{nar} = const in the deeply arranged rows of an in-line bank of tubes heat transfer varied by about 20 per cent and of a staggered bank of tubes—by 30 per cent.

To determine the mean heat-transfer coefficient for the whole bank, α, it is necessary to average the mean values of $\bar{\alpha}$ obtained for its individual rows:

$$\bar{\bar{\alpha}} = \frac{\sum_{i=1}^{i=n} \bar{\alpha}_i F_i}{\sum_{i=1}^{i=n} F_i}$$

where $\bar{\alpha}_i$ = mean heat-transfer coefficient for the ith row;
F_i = total heat-transfer surface of tubes of the ith row;
n = number of rows in the bank.

The formula becomes simplified, if $F_1 = F_2 = \ldots = F_n$:

$$\bar{\bar{\alpha}} = \frac{\bar{\alpha}_1 + \bar{\alpha}_2 + (n-2)\bar{\alpha}_3}{n}$$

where

$$\bar{\alpha}_1 = \varepsilon_1 \bar{\alpha}_3 \quad \text{and} \quad \bar{\alpha}_2 = \varepsilon_2 \bar{\alpha}_3$$

The correction factor ε_i accounts for the change in the intensity of heat transfer in the first rows of tubes. When $\frac{s_2}{d} \leqslant 4$ and turbulence of the overtaking flow is not high ε_i may be determined from the diagram shown in Fig. 9-9.

The processes of fluid flow and heat transfer are still not as well understood in the laminar and turbulent regions as in the mixed pattern. But the data currently available, however, permit us to conclude that the intensity of heat transfer in the first and second rows with turbulent flow is also less than that in the deeper rows of the bank. The process becomes stabilized from the third row.

Bergelin *et al.* have studied the mean rate of heat transfer in close-set ten-row banks of staggered and in-line tubes (s_1/d and s_2/d being equal to or less than 1.25). Heat transfer is then described by the equation

$$\overline{\mathrm{Nu}}_{fd} = c\mathrm{Re}_{fd}^{1/3}\mathrm{Pr}_f^{1/3}(\mathrm{Pr}_f/\mathrm{Pr}_w)^{1/4} \qquad (9\text{-}5)$$

where c = 1.8 for a staggered bank of tubes and c = 1.2 for an in-line one. All reference values are chosen as for formula (9-4). Formula (9-5) is true at Re_{fd}

between 10 and 200 for staggered banks and between 10 and 150 for in-line banks. It does not, however, account for the possible effect of natural convection (Eq. 9-5). Other things being equal heat transfer in the laminar region of flow is 50 per cent larger in staggered banks than in in-line banks. This difference in the intensity of heat transfer diminishes in the region of mixed flow, and becomes zero at the limit with $Re \approx 10^5$. In the turbulent region of flow in staggered and in-line banks heat transfer differs relatively little.

At $Re > 2 \times 10^5$ the heat transfer in inner rows of staggered and in-line banks of tubes may be calculated by the formula suggested by Zhukauskas [9.2]

$$\overline{Nu}_{fd} = 0.021 Re_{fd}^{0.84} Pr_f^{0.36} (Pr_f/Pr_w)^{0.25} \quad (9\text{-}6)$$

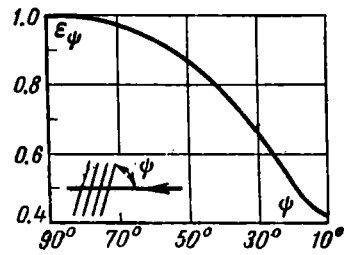

Fig. 9-10. Dependence of heat transfer in banks of tubes on the angle of attack ψ

The change from one pattern of flow to the other has been shown experimentally to take place not at a definite Reynolds number, but within a relatively small range. Therefore when we speak of Reynolds numbers of 150, or 200, or of $Re = 10^5$ these quantities should be considered as averaged values. The magnitudes of the Reynolds numbers also depend on the relative pitches of the banks. For example, the character of heat transfer changes in straight-forward in-line banks $\left(\frac{s_1}{d} \gg \frac{s_2}{d}\right)$ at higher Reynolds numbers than in "lattice" types of in-line banks $\left(\frac{s_1}{d} \ll \frac{s_2}{d}\right)$ [9.7].

Formula (9-4) is applicable only when fluid flow is perpendicular to the axis of the banks of tubes (angle of attack ψ equals 90°). When the angle of attack $\psi < 90°$ (Fig. 9-10) the change in heat transfer can be accounted for by introducing the correction factor $\varepsilon_\psi = \bar{\alpha}_\psi/\bar{\alpha}_{\psi=90°}$ in formula (9-4), which is the ratio of the heat-transfer coefficient at the actual angle of attack ψ to the heat-transfer coefficient at an angle of attack $\psi = 90°$.

The values of $\varepsilon_\psi = f(\psi)$ can be taken from the graph shown in Figure 9-10.

At values of ψ near to 0° heat transfer is calculated by the formula derived for banks of tubes in parallel flow.

It will be recalled that the cited data pertain to the case $Pr \geqslant 1$. The rate of heat transfer at $Pr \ll 1$ is considered in Chapter 11.

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CHAPTER

10

HEAT TRANSFER BY FREE CONVECTION

10-1. General

Free convection originates due to the non-uniform distribution of mass (volume) forces in the liquid being considered. Forces of this kind include the force of gravity, centrifugal force and the force appearing when a high-intensity electromagnetic field is induced in the liquid. The best understood is free convection caused by gravity.

Gravitational forces are taken into account in the flow equation (4-18) by the term $\rho\vec{g}$, whose unit is force reduced to unit volume. Fluid temperature varies during heat transfer, which leads to the appearance of differences in density and, consequently, of differences in gravitational forces representing a buoyant (descending) force.

In engineering problems the acceleration of gravity does not normally change from one point to another of the space considered. But the mass forces caused by centrifugal effect or an electromagnetic field may vary in a fluid, owing to variation of the vector $\vec{F}$ which represents the ratio of the force acting on a given element of fluid to its mass. $\vec{F} = \vec{g}$, if gravity only is taken into account.

This chapter is devoted solely to consideration of heat transfer by free convection due to gravity. The results obtained are sometimes used to estimate the natural convection caused by other mass forces, the acceleration of gravity being then replaced by its sum with the acceleration g due to the additional mass force (centrifugal acceleration $\frac{w^2}{r}$, for instance). The result so obtained must be considered approximate, since the field of acceleration due to the other forces may differ from that of gravitational acceleration.

10-2. Heat Transfer by Free Convection in a Bulk Volume

In this section we shall deal with free gravitational flow for surfaces of bodies of simplest geometry (vertical plate, horizontal cylinder). The volume of fluid is assumed to be so large that free-convection flow originating near other bodies located in the same volume does not affect the natural convection being considered. Free-convection flow may be both laminar and turbulent, as with forced convection.

(*a*) *Heat Transfer in Free Laminar Flow at a Vertical Plate*

Consider a vertical plate with a constant surface temperature t_w exposed to a liquid or gas. The fluid is stationary at a distance from the plate (there being no forced flow), and its temperature t_0 at a distance from the plate is invariable. To simplify calculations we assume $t_w > t_0$ (the results obtained, however, will hold for the reversed relationship). A layer of ascending heated fluid appears at the plate. At a distance from the plate its velocity becomes zero as before.

Let us place the origin of coordinates at the lower edge of the plate, and the Oy-axis in a direction perpendicular to its surface, as shown in Fig. 10-1. We assume that the plate is of infinite length in the direction of the Oz-axis. The process is a steady-state one.

To simplify solution of the problem posed, we make the following assumptions:

(1) the forces of inertia are negligible compared with those of gravity and viscosity;

(2) transport of heat by convection and conduction along the moving layer of fluid can be ignored;

(3) the pressure gradient is zero;

(4) the physical parameters of the liquid (except density) are constant, and density is a linear function of temperature.

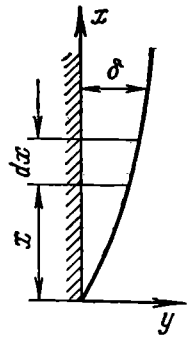

Fig. 10-1. Deriving the calculation formula for heat transfer by the free convection

We further assume that in the moving layer temperature varies according to the following equation:

$$\vartheta = \vartheta_w \left(1 - \frac{y}{\delta}\right)^2 \qquad (10\text{-}1)$$

where $\vartheta = t - t_0$, and $\vartheta_w = t_w - t_0$; ϑ_w is assumed constant.

Equation (10-1) satisfies the boundary conditions that

$$\vartheta = \vartheta_w \text{ at } y = 0, \text{ and } \vartheta = 0 \text{ at } y = \delta \qquad (a)$$

The heat-transfer coefficient is determined by equation

$$\alpha = -\frac{\lambda}{\vartheta_w}\left(\frac{d\vartheta}{dy}\right)_{y=0} \qquad (10\text{-}2)$$

From Eq. (10-1) it follows that

$$\frac{d\vartheta}{dy} = -\frac{2\vartheta_w}{\delta} + \frac{2\vartheta_w}{\delta^2}\, y = -\frac{2\vartheta_w}{\delta}\left(1 - \frac{y}{\delta}\right)$$

and

$$\left(\frac{d\vartheta}{dy}\right)_{y=0} = -\frac{2\vartheta_w}{\delta}$$

Substituting $(d\vartheta/dy)_{y=0}$ in Eq. (10-2), we obtain:

$$\alpha = \frac{2\lambda}{\delta} \qquad (10\text{-}3)$$

The thickness of the moving layer of fluid is variable with height and is associated with the velocity of flow in the layer. The velocity field is described by the equation of flow. With the conditions assumed free fluid flow mainly occurs in the direction of the Ox-axis. Therefore, we use the equation of flow only in the projections upon the Ox-axis. The equation becomes simplified for steady-state flow, taking into consideration the previously made assumptions. As a result Eq. (4-18) is replaced by

$$\mu \frac{d^2 w_x}{dy^2} = -g(\rho_0 - \rho) \qquad (10\text{-}4)$$

With density being in linear relation to temperature

$$\rho = \rho_0 (1 - \beta\vartheta)$$

where $\beta = \text{const}$. Whence

$$\rho_0 - \rho = \rho_0 \beta \vartheta$$

Substituting ϑ according to Eq. (10-1) in (10-4), and taking the last relationship for density into consideration, the equation of flow can be presented as follows:

$$\frac{d^2 w_x}{dy^2} = -\frac{\rho_0 g \beta \vartheta_w}{\mu}\left(1 - \frac{y}{\sigma}\right)^2$$

or

$$\frac{d^2 w_x}{dy^2} = -A\left(1 - 2\frac{y}{\sigma} + \frac{y^2}{\sigma^2}\right)$$

Here

$$A = \frac{\rho_0 g \beta \vartheta_w}{\mu} \neq f(y)$$

Integration gives us:

$$\frac{dw_x}{dy} = -A\left(y - \frac{1}{\delta}y^2 + \frac{1}{3\delta^2}y^3\right) + c_1$$

and

$$w_x = -A\left(\frac{y^2}{2} - \frac{1}{3\delta}y^3 + \frac{1}{12\delta^2}y^4\right) + c_1 y + c_2 \qquad \text{(b)}$$

We assume the following boundary conditions for velocity: $w_x = 0$ at both $y = 0$ and $y = \delta$. It should be noted that, strictly speaking, velocity cannot equal zero at $y = \delta$ ($\vartheta = 0$) because of the effect of viscosity. The moving particles may entrain layers of liquid which are under isothermal conditions.

With the boundary conditions stipulated it follows from equation (b) that

$$c_1 = \frac{A}{4}\delta \quad \text{and} \quad c_2 = 0$$

Substituting these values of c_1 and c_2 in equation (b), and making certain transformations, we obtain the following equation describing distribution of velocity in the moving layer of fluid:

$$w_x = A\left(\frac{\delta}{4}y - \frac{1}{2}y^2 + \frac{1}{3\delta}y^3 - \frac{1}{12\delta^2}y^4\right) \qquad (10\text{-}5)$$

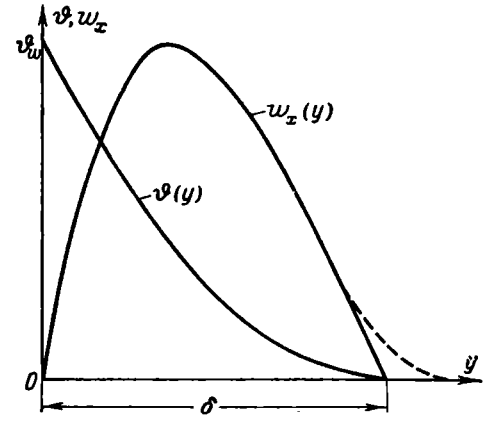

Fig. 10-2. Temperature and velocity distribution following equations (10-1) and (10-5)

The distribution of velocity in accordance with Eq. (10-5) is shown in Fig. 10-2. The same figure presents the temperature curve plotted in accordance with Eq. (10-1). The maximum velocity corresponds to

$$y = 0.38\delta \approx \frac{\delta}{3} \qquad \text{(c)}$$

It will be noted that at $y = \delta$ velocity distribution does not satisfy the condition that $\left(\frac{dw_x}{dy}\right)_{y=\delta} = 0$, the derivative being finite at $y = \delta$ since the solution is only approximate. The actual variation of velocity on the outer boundary of the moving layer of fluid is shown by the dotted line.

In accordance with Eq. (10-5), the integral mean velocity is

$$\bar{w}_x = \frac{1}{\delta}\int_0^{\delta} w_x\,dy = \frac{\rho_0 g\beta\vartheta_w\delta^2}{40\mu} \tag{10-6}$$

To simplify the solution, let us approximately determine the mean temperature of fluid in the layer as the integral mean over the cross-section of the layer*:

$$\bar{\vartheta} = \frac{1}{\delta}\int_0^{\delta}\vartheta\,dy = \frac{1}{\delta}\int_0^{\delta}\vartheta_w\left(1-\frac{y}{\delta}\right)^2 dy = \frac{\vartheta_w}{3} \tag{10-7}$$

With the assumed conditions the mean temperature of the layer is thus independent of the x-coordinate. The rate of fluid flow through its cross-section layer $\delta\cdot 1$ is:

$$G = \rho_0\bar{w}_x\delta\cdot 1 \tag{10-8}$$

and

$$dG = d\,(\rho_0\bar{w}_x\delta) \tag{d}$$

The rate of fluid flow is determined by the density ρ_0 assuming here that fluid of a density ρ_0 entrained in the moving layer, acquires a mean velocity $\bar{w}_x$.

Substituting the mean velocity $\bar{w}_x$ from Eq. (10-6) in formula (d), we obtain:

$$dG = d\left(\frac{\rho_0^2 g\beta\vartheta_w\delta^3}{40\mu}\right) = \frac{3\rho_0^2 g\beta\vartheta_w}{40\mu}\delta^2 d\delta \tag{e}$$

The fluid is set in motion at an initial temperature t_0, and is heated in the moving layer to various temperatures ranging from t_0 to t_w. It may be assumed that the fluid heats on the average to a temperature $\bar{\vartheta}$. The quantity of heat for heating the fluid is:

$$dQ = c_p\bar{\vartheta}\,dG = \alpha\vartheta_w\,dx\cdot 1 = \frac{2\lambda}{\delta}\vartheta_w\,dx\cdot 1 \tag{f}$$

from which it follows that

$$dG = \frac{2\lambda}{\delta c_p}\frac{\vartheta_w}{\bar{\vartheta}}\,dx = \frac{6\lambda}{\delta c_p}\,dx \tag{g}$$

Equating the right sides of equations (e) and (g), we obtain the following differential equation describing the variation in δ with the height of the wall:

$$\frac{3\rho_0^2 g\beta\vartheta_w}{40\mu}\delta^3\,d\delta = \frac{6\lambda}{c_p}\,dx \tag{h}$$

Integrating this equation, we obtain:

$$\frac{3\rho_0^2 g\beta\vartheta_w}{160\mu}\delta^4 = \frac{6\lambda}{c_p}x + c \tag{i}$$

The integration constant c is found from the condition that $\delta = 0$ at $x = 0$. Hence $c = 0$.

---

* Integration in accordance with formula (6-1) shows that

$$\bar{\vartheta} = \frac{\vartheta_w}{3}\frac{135-70\beta\vartheta_w}{126-45\beta\vartheta_w} = \frac{\vartheta_w}{3}\frac{5}{9}\frac{2-\beta\vartheta_w}{1-0.36\beta\vartheta_w}$$

and this differs only slightly from $\vartheta_w/3$ at small values of ϑ_w.

From equation (i) it follows that

$$\delta = 4.23 \sqrt[4]{\frac{\mu\lambda x}{c_p\beta\rho_0^2 g\vartheta_w}} \tag{10-9}$$

According to equation (10-3), $\alpha = \frac{2\lambda}{\delta}$. Substituting the value of δ in Eq. (10-9), we obtain:

$$\alpha = \frac{2\lambda}{\delta} = 0.473 \sqrt[4]{\frac{c_p\beta\rho_0^2 g\vartheta_w\lambda^3}{\mu x}} \tag{10-10}$$

Let us now write Eq. (10-10) in dimensionless form by multiplying its both sides by x and dividing by λ.

With certain transformations we obtain:

$$\mathrm{Nu}_x \equiv \frac{\alpha x}{\lambda} = 0.473 \sqrt[4]{\frac{g\beta\vartheta_w x^3}{\mu^2/\rho_0^2}\frac{\mu c_p}{\lambda}} = 0.473\,(\mathrm{Gr}_x\mathrm{Pr})^{0.25} \tag{10-11}$$

where

$$\mathrm{Gr}_x \equiv \frac{g\beta\vartheta_w x^3}{\mu^2/\rho_0^2} = \frac{g\beta\vartheta_w x^3}{\nu_0^2}, \quad \text{and} \quad \mathrm{Pr} \equiv \frac{\mu c_p}{\lambda}$$

From Eq. (10-11) it follows that $\mathrm{Nu}_x = f(\mathrm{Gr}_x \cdot \mathrm{Pr})$. The theory of similarity yields the same result. The product of the Gr and Pr numbers is often called the Rayleigh number and denoted by Ra.

The temperatures t_w and t_0 are constant in the case considered, hence the temperature difference or potential $\vartheta_w = t_w - t_0$ is also constant, and the heat-transfer coefficients can be averaged by equations (6-21) and (6-22) with identical results.

From equation (10-11) it follows that $\alpha = cx^{-0.25}$, where $c \neq f(x)$. Then

$$\bar{\alpha} = \frac{1}{l}\int_0^l \alpha\,dx = \frac{1}{l}\int_0^l cx^{-0.25}\,dx = \frac{4}{3}cl^{-0.25} = \frac{4}{3}\alpha_{x=l}$$

where $\alpha_{x=l}$ is the local heat-transfer coefficient at the point determined by the coordinate $x = l$.

Mean heat transfer for a vertical plate in laminar flow with a constant wall temperature t_w will, then, be

$$\overline{\mathrm{Nu}}_l = 0.63\,(\mathrm{Gr}_l\mathrm{Pr})^{0.25} \tag{10-12}$$

The proportionality factors in Eqs. (10-11) and (10-12) require some refining. Formulae (10-11) and (10-12) were derived with a number of simplifying assumptions; in particular, the forces of inertia were not taken into consideration. Calculations accounting for the forces of inertia indicate that the proportionality factor of those formulae depends on the Prandtl number. The results of precise solutions of various authors, Poulhousen, Shoe, O. Saunders, Gregg and Sparrow, are illustrated in Fig. 10-3 taken from [10.1]. Here the proportionality factor $c = \mathrm{Nu}_x(\mathrm{Gr}_x\mathrm{Pr})^{-0.25}$. The effect of inertia is most pronounced at low Prandtl numbers. From Fig. 10-3 it follows, in addition, that the intensity of heat transfer at a constant wall temperature is about 7 per cent less than with a constant rate of heat flow at the wall.

Experimental studies show that with Prandtl numbers exceeding 0.7 test results can be described by a formula like (10-11) or (10-12) with constant proportionality factors; but the factors differ somewhat from those in the derived formulae. Other reasons apart, these factors depend on the choice of reference temperature.

The local heat-transfer coefficients for free laminar flow along vertical walls can be calculated by the following formula [10.2]:

$$\mathrm{Nu}_{fx} = 0.60\,(\mathrm{Gr}_{fx}\mathrm{Pr}_f)^{0.25}\,(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25} \qquad (10\text{-}13)$$

Here the reference temperature is that of the liquid outside the moving layer (Pr_w is selected for the local temperature of the wall). The reference dimension is the coordinate along the flow, measured from the point where heat transfer originates. In Fig. 10-4, formula (10-13) is correlated with the experimental data.

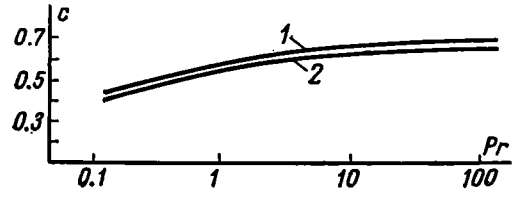

Fig. 10-3. Dependence of heat transfer with free convection on Prandtl number: 1—with q_w = const; 2—with t_w = const

Formula (10-13) was derived on the assumption that q_w = const. Averaging the heat-transfer coefficients by formula (6-1), we find that, with q_w = const, $\bar{\alpha}_l = \frac{5}{4}\,\alpha_{x=l}$. Then the mean heat-transfer coefficients can be calculated by the formula

$$\overline{\mathrm{Nu}}_{fl} = 0.75\,(\mathrm{Gr}_{fl}\mathrm{Pr}_f)^{0.25}\,(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25} \qquad (10\text{-}14)$$

Here, as before, the reference temperature is that of the liquid outside the moving layer, and the reference dimension is the length of the plate measured from the point where heat transfer originates.

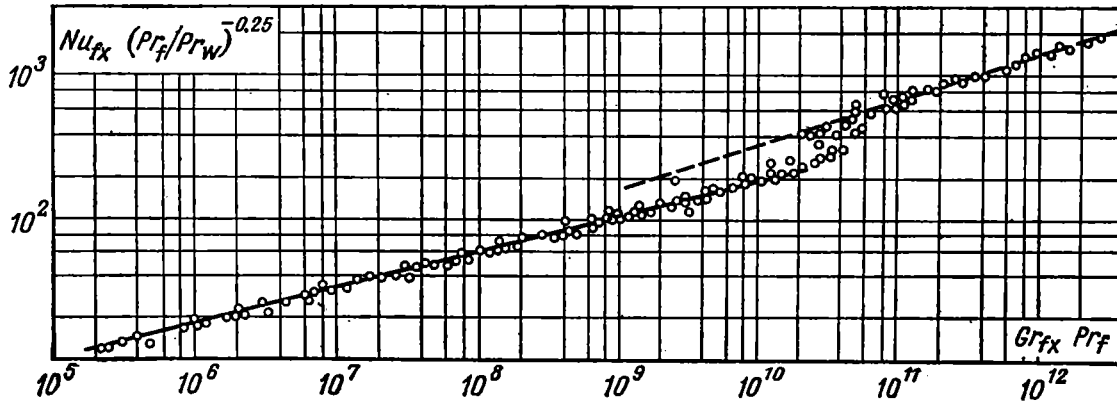

Fig. 10-4. Heat transfer with free convection at vertical surface in a bulk volume of liquid

Formula (10-13) is derived for heat-carrying agents with Prandtl numbers between 0.7 and 3×10^3. The formula is to be used at $10^3 < \mathrm{Gr}_{fx}\cdot\mathrm{Pr}_f < 10^9$.

Eq. (10-13) was derived for q_w = const. From the graph in Fig. 10-3 it follows that for the case t_w = const the proportionality factor in formula (10-13) may be assumed to be about 0.55 as a first approximation. Then $\bar{\alpha}_l = \frac{4}{3}\,\alpha_{x=l}$.

(b) Heat Transfer in Free Turbulent Flow at a Vertical Plate

According to [10.2], turbulent flow becomes fully developed at $\mathrm{Cr}_{fx}\mathrm{Pr}_f \gtrsim 6 \times 10^{10}$ (Fig. 10-4).

The following formula is suggested [10.2] for local heat-transfer coefficients with developed turbulence flow:

$$\mathrm{Nu}_{fx} = 0.15\,(\mathrm{Gr}_{fx}\mathrm{Pr}_f)^{1/3}\,(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25} \qquad (10\text{-}15)$$

The reference temperature and linear dimension are chosen here as in formula (10-13).

The linear dimension is present in both the Nusselt and the Grashof numbers:

$$\mathrm{Nu}_{fx}=\frac{\alpha x}{\lambda_f}, \quad \text{and} \quad \mathrm{Gr}_{fx}^{1/3}=\left(\frac{g\beta\vartheta_w x^3}{\nu_f^2}\right)^{1/3}$$

From this it follows that with developed turbulent flow the heat-transfer coefficient is independent of the linear dimension, and the local heat-transfer coefficient is consequently equal to the mean coefficient of heat transfer.

(c) *Heat-transfer in Free Transition Flow at a Vertical Plate*

The experimental results obtained by various investigators indicate that a transition pattern of flow occurs around $10^9 < \mathrm{Gr}_{fx}\mathrm{Pr}_f < 6 \times 10^{10}$. The transition pattern of flow is marked by instability of both flow and heat transfer, and consequently by considerable scattering of experimental points.

Sometimes, as in the test [10.2] (Fig. 10-4), flow remains laminar even at $\mathrm{Gr}_{fx}\mathrm{Pr}_f$ numbers exceeding 10^9; but developed turbulent flow may also set in at $\mathrm{Gr}_{fx}\mathrm{Pr}_f$ numbers below 6×10^{10}.

The transition region is located along a definite stretch of the wall, where heat transfer increases on average from the value corresponding to and characteristic of laminar flow to that corresponding to turbulent flow.

The largest and smallest heat-transfer coefficients of the transition region can be determined respectively by Eqs. (10-15) and (10-13).

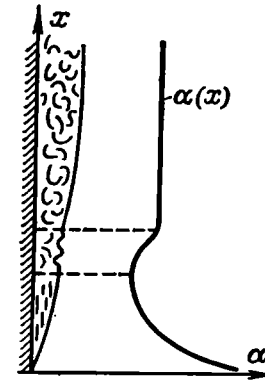

Fig. 10-5. Variation of the heat-transfer coefficient with free flow along a vertical wall

The variation of the heat-transfer coefficient up the height of a vertical wall in free ascending flow, and dependence of this variation on the nature of the flow, are shown in Fig. 10-5.

With laminar flow the heat-transfer coefficient diminishes with height in proportion to $x^{-0.25}$. In the transition region the heat-transfer coefficient is unstable in time and increases on average to values characteristic of turbulent flow. With turbulent flow the heat-transfer coefficient is independent of x.

Figure 10-5 shows only the dependence of α on x. Variations of the physical parameters and of the temperature difference Δt with height may also lead to changes in the heat-transfer coefficients.

(d) *Heat Transfer in Free Flow Around a Horizontal Tube*

The pattern of free flow along a vertical wall already described is also typical of free flow on inclined walls, spheres and round and oval horizontal tubes. Heat transfer near horizontal round tubes is of great practical importance.

The pattern of free-convection flow near hot horizontal tubes is shown graphically in Fig. 10-6. Other things being equal, the larger the diameter of the tubes involved, the more probable is deterioration of laminar flow. With tubes of small diameter this deterioration may occur at a distance from the tube. Mean

coefficients of heat transfer for horizontal tubes in free-convection laminar flow may be calculated from the formula suggested by I.M. Mikheeva [10-3]:

$$\overline{\mathrm{Nu}}_{fd} = 0.50\,(\mathrm{Gr}_{fd}\mathrm{Pr}_f)^{0.25}\,(\mathrm{Pr}_f/\mathrm{Pr}_w)^{0.25} \qquad (10\text{-}16)$$

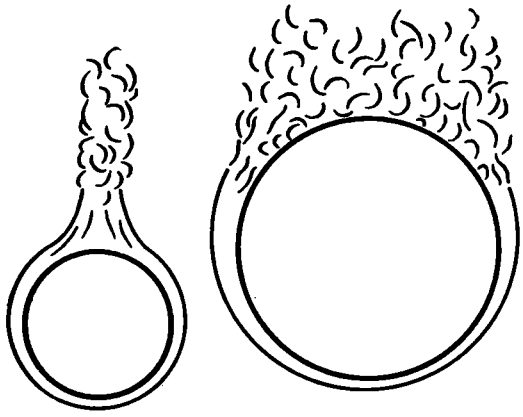

Fig. 10-6. Free flow across horizontal tubes

In this formula the reference temperature is that of the liquid or gas at a distance from the tube, and the reference linear dimension is the tube diameter.

In Fig. 10-7, formula (10-16) is correlated with experimental data.

(e) *Heat Transfer at Very Small Values of the GrPr Term*

The conditions of heat transfer for thin wires ($d = 0.2$-2.0 mm) have been found by Eigenson to be of a special character. The quantity of heat transferred is insignificant, because the surface of the wire is small. At small temperature potentials a stationary film of heated air forms around the wire. This type of heat transfer and free flow is referred to as *film* convection.

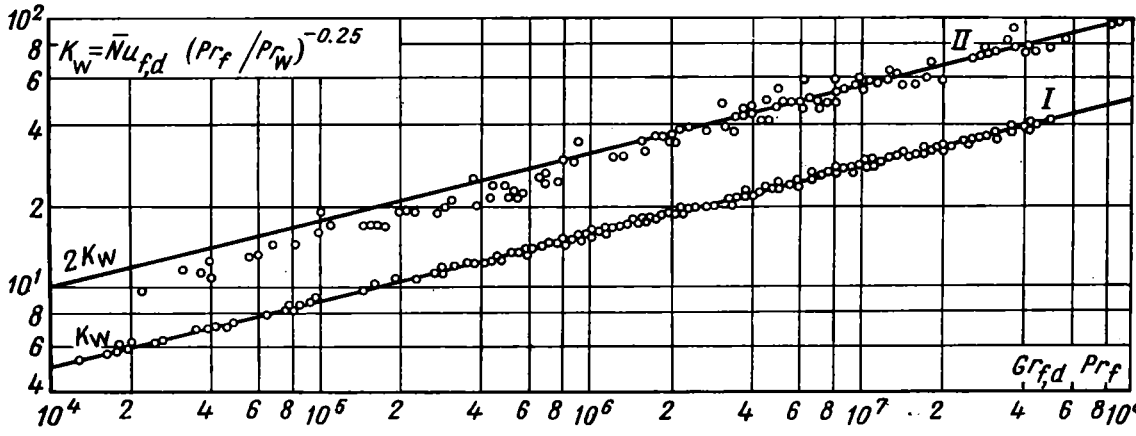

Fig. 10-7. Heat transfer with free flow around horizontal tubes

Film convection was found to occur at magnitudes of the dimensionless term $(\mathrm{Gr}_{wgd}\mathrm{Pr}_{wg}) < 1$, where the subscripts wg and d indicate that the reference parameters are the temperature $t_{wg} = 0.5\,(t_w + t_0)$ and diameter d. For film convection $\mathrm{Nu}_{wgd} = 0.5$, whence $\alpha = 0.5\,\lambda/d$. Heat is transferred by conduction. This type of convection is quite unstable.

Convective streams sometimes appear even at $(\mathrm{Gr}_{wgd}\mathrm{Pr}_{wg}) > 10^{-3}$, and the Nusselt number increases with increase of the dimensionless term GrPr. This pattern of flow is transitional from the film type to laminar, and occurs at values of $(\mathrm{Gr}_{wgd}\mathrm{Pr}_{wg})$ below 5×10^2.

The maximum value of the heat-transfer coefficient for the transition pattern of free-convection flow is described by the Mikheev equation

$$\mathrm{Nu}_{wgd} = 1.18\,(\mathrm{Gr}_{wgd}\mathrm{Pr}_{wg})^{1/8}$$

The minimum value applies to film convection.

10-3. Heat Transfer by Free Convection in a Limited Volume

If the volume of fluid is small, free convection appearing near other bodies or sections of the given body located in the same volume may affect the flow considered. It is very difficult and sometimes impossible to separate these flows and to consider them individually.

Flow and heat transfer depend both on the kind of fluid, its temperature, and the temperature difference, and on the shape and size of the space.

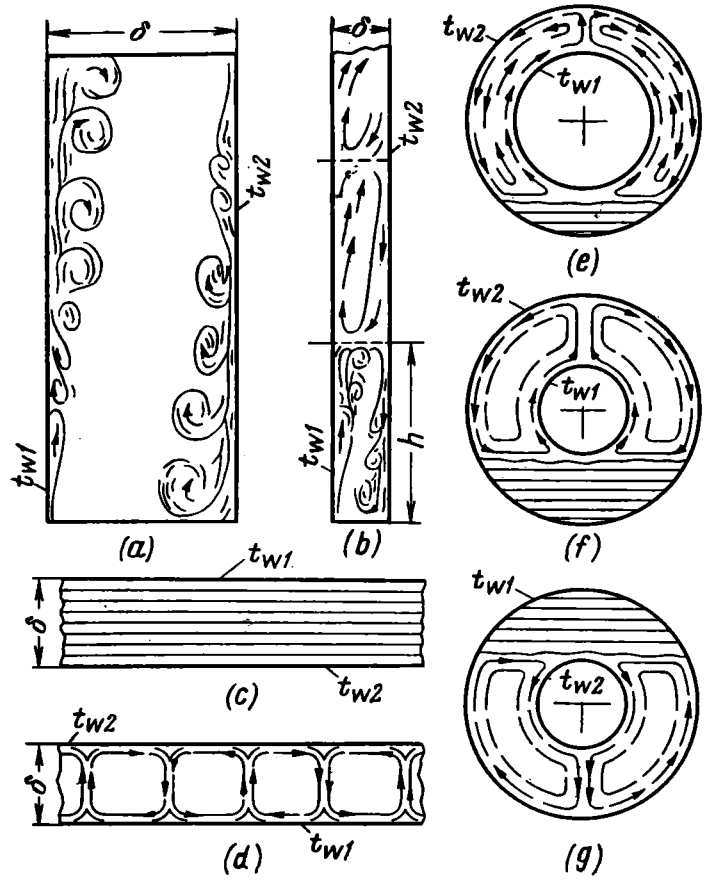

Fig. 10-8. Free convection in an enclosed space

In *horizontal slots* formed by two flat walls the process is determined by the arrangement of the heated and cooled surfaces, and the distance between them and temperature distribution of the wall. There is no free flow of fluid, if the temperature of the upper wall is constant and exceeds that of the lower (Fig. 10-8*c*). This is true for fluids whose density diminishes with increasing temperature. Non-uniform wall temperature contributes to the appearance of convection.

If the lower wall is at temperature higher than that of the upper wall, convective currents arise under certain conditions in the enclosed space. Particles of the hot fluid of smaller density tend to move upward. Alternate rising and descending currents appear in the enclosed space (Fig. 10-8*d*). Regarded from above the field of flow is cellular in structure, with more or less regular hexahedral cells. Fluid moves upward inside these cells, and descends along their periphery. Such flow occurs in the neighbourhood of $GrPr = 10^3$ to 45×10^3. At

$GrPr < 45 \times 10^3$ developed turbulent flow sets in, the cellular structure may remain only near the lower wall.

Circulation may develop differently in *vertical slots*, depending on the distance δ between the walls forming the passage. The rising and descending flows do not interfere with each other if δ is large (Fig. 10-8*a*), and the movement of the fluid is identical here with free-convection flow in a bulk volume. But if δ is small, interference between the two currents gives rise to inner circuits, as illustrated in Fig. 10-8*b*. The height h of these inner circuits is determined by the width of the enclosed space or passage, the type of fluid involved, and the intensity of the process.

The transport of heat between the walls can be calculated by the equations for heat conduction, if $Gr_\delta < 124Pr^{-0.2}\,(0.955 + Pr)\,(l/\delta)$, where l is the height of the layer. The conditions of heat transfer differ only at the ends of the enclosed space, at a height approximately equal to δ.

In spherical and horizontal cylindrical enclosed spaces fluid circulates as shown in Fig. 10-8*e* to *g* depending on diameter ratio. Flow develops only in the region above the lower edge of the heated surface, and below this edge the fluid remains stagnant. But if the outer cylindrical surface is heated, fluid circulation extends to the space beneath the upper edge of the cold surface. With intensive heat transfer circulation of the entire fluid may take place.

In practical calculations it is usually necessary to determine the rate of heat flow through the layers of fluid, and it is common practice to replace the complex process of heat transfer through enclosed spaces by an equivalent process of heat conduction.

The mean rate of heat flow q is conditionally calculated by the formulae of conduction (Chapter 2). For a flat layer

$$q = \frac{\lambda_{eq}}{\delta}(t_{w1} - t_{w2}) \tag{10-17}$$

where λ_{eq} is so-called equivalent thermal conductivity, which accounts for the transport of heat through the passage by both conduction and convection.

The relation $\varepsilon_c = \frac{\lambda_e}{\lambda}$, where λ is the thermal conductivity of the fluid, characterizes the effect of convection. The quantity ε_c is a function of the GrPr term.

The dependence $\varepsilon_c = f\,(GrPr)$ is shown graphically in Fig. 10-9 [10.4], plotted from experimental data obtained by D.L. Boyarintsev, Mull-Reyer, Davis, Beckmann, Kraussold for enclosed vertical and horizontal spaces, annular passages, and closed spherical spaces filled with gas or dropping liquid.

In calculating the dimensionless terms of similarity, the thickness of the enclosed space δ is taken as the reference dimension, and the mean fluid temperature $t_{wg} = 0.5\,(t_{w1} + t_{w2})$ as the reference temperature, irrespective of the shape of the space.

At small values of the argument $[(GrPr)_{wg\delta} < 10^3]$ as follows from the graph in Fig. 10-9 $\varepsilon_c = 1$ and $\lambda_{eq} = \lambda$. This means that heat is transferred here from the hot wall to the cold by conduction. At $10^3 < (GrPr)_{wg\delta} < 10^6$ (curve *1*)

$$\varepsilon_c = 0.105\,(GrPr)^{0.3}_{wg\delta} \tag{10-18}$$

and at $10^6 < (GrPr)_{wg\delta} < 10^{10}$ (curve *2*)

$$\varepsilon_c = 0.40\,(GrPr)^{0.2}_{wg\delta} \tag{10-19}$$

For approximate calculation, according to Mikheev, these two formulae can be replaced by the following relation in the entire range of the argument

$(GrPr)_{wg\delta} > 10^3$:

$$\varepsilon_c = 0.18\,(GrPr)^{0.25}_{wg\delta} \qquad (10\text{-}20)$$

approximating with a certain error the experimental data presented in Fig. 10-9 (curve *3*).

A number of calculation formulae has been suggested for individual problems in free convection in an enclosed space [10.5, 10.6, 10.7, etc.]. In the works enumerated free convection was investigated near a vertical row of horizontal

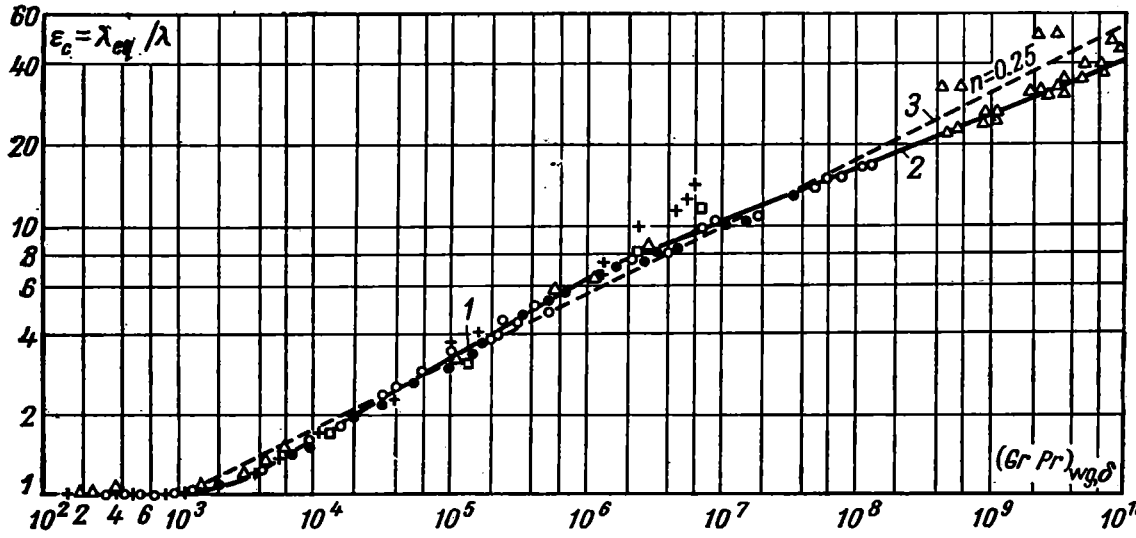

Fig. 10-9. Equivalent thermal conductivity with free convection in an enclosed space: +—flat horizontal gas layer; □—flat vertical gas layer; ○—cylindrical gas layer; ●—cylindrical liquid layer; Δ—spherical gas layer

tubes, near a vertical wall with vertical fins, in a horizontal slot, in a horizontal cylindrical layer with the tubular inner heat source arranged in different positions, and other problems of practical importance, which are beyond the scope of this book.

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CHAPTER

11

SPECIAL PROBLEMS OF CONVECTIVE HEAT TRANSFER IN A SINGLE-PHASE MEDIUM

11-1. Heat Transfer in Liquid Metals

Liquid metals are used where intensive removal of heat must be ensured, or there is a need of a working liquid with a high temperature but low pressure.

With water, which possesses a high heat-transfer coefficient as a heat carrier, it becomes necessary to increase pressure in order to have the working liquid at a high temperature.

Gases can be heated to high temperatures without increase in pressure, but the rate of heat transfer from a wall to gas is very small, and this causes an increase in temperature of the heating surface. As a heat carrier, gas has an essential shortcoming. Since its heat capacity is small, it follows from the equation of heat balance that the rate of gas flow must be increased considerably to remove a given quantity of heat. Consequently, the drop in pressure must also be greater.

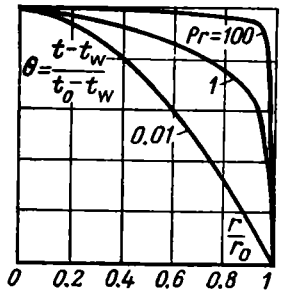

Fig. 11-1. Temperature variation with turbulent fluid flow in a circular tube of radius r for different Prandtl numbers; thermally developed flow:
t—fluid temperature; t_w—wall temperature; t_0—fluid temperature at tube axis; r_0—inside radius of tube

Cooling by means of liquid metals combines the advantages of both gas and water cooling. Liquid metals possess a high boiling point, which permits their temperature to be raised at moderate pressure; they have, in addition, high heat-transfer coefficients.

The most suitable heat carriers of this kind are alkali and heavy metals and their alloys. The physical properties of liquid metals differ essentially from those of the conventional heat carriers—water, oil, etc.; their specific gravity and thermal conductivity are larger, while their heat capacity is smaller, and their Prandtl number is particularly low (Pr $\approx$ 0.005-0.05). The low Prandtl numbers of liquid metals are explained by their higher thermal conductivity; at temperatures ranging from 100 °C to 700 °C, for example, the thermal conductivity of sodium is approximately equal to $\lambda \approx 86$ to 59 W/m·K; of potassium—$\lambda \approx 46$ to 28 W/m·K.

With laminar flow heat is transferred across the flow by conduction, and with turbulent flow by both conduction and convection. Since the thermal conductivity of non-metallic agents is comparatively small, heat is transferred in their turbulent core mainly by convection. The principal thermal resistance to heat transfer across the turbulent flow is their laminar sublayer, and because of this the basic variation of temperature in the cross-section of the fluid flow is concentrated at the wall. In the turbulent core temperature variation is slight (Fig. 11-1). The conductivity of liquid metals is large and it may compete with the process of turbulent transport of heat. Temperature distribution then depends

essentially on conduction. From Fig. 11-1 it follows that liquids with small Prandtl numbers transport heat uniformly over the entire cross-section of the conduit.

Consider turbulent flow in a straight circular tube. With hydrodynamically and thermally stabilized flow and q_w = const, heat transfer can be calculated by Lyon's equation (8-3). Lyon [11.1] obtained a numerical solution of his equation at $\mathrm{Pr}_t = \frac{\varepsilon_s}{\varepsilon_q} = 1$, and has approximated the data obtained for the range of Prandtl numbers characteristic of liquid metals, by means of the formula

$$\mathrm{Nu}_d = 7 + 0.025\mathrm{Pe}_d^{0.8} \tag{11-1}$$

where $\mathrm{Pe}_d = \mathrm{Re}_d\mathrm{Pr} = \frac{\overline{w}d}{a}$ is the Peclet number.

The calculations of R. Seban and T. Shimazaki [11.2] for t_w = const and $\mathrm{Pr}_t = 1$ yield

$$\mathrm{Nu}_d = 5 + 0.025\mathrm{Pe}_d^{0.8} \tag{11-2}$$

Comparing formulae (11-1) and (11-2), we see that the condition t_w = const results in a rather smaller constant term than does at q_w = const.

The presence of two terms on the right side of Eqs. (11-1) and (11-2) is explained by the fact that radial conduction in the flow of liquid metals is taken into account.

Formulae (11-1) and (11-2) were obtained for steady-state turbulent flow in straight circular tubes, without accounting for heat conduction in the direction of flow.

The importance of the fraction of heat transferred in axial direction, i.e. in the direction of flow, increases with diminishing Peclet number.

Labuntsov [11.3] has analysed heat transfer in stabilized laminar flow, taking axial heat transfer into consideration. According to his results [11.3] the dimensionless term $\mathrm{Nu}_d = \frac{\alpha d}{\lambda}$ is a function of Pe_d at t_w = const. This dependence is set out in Table 11-1. The Nusselt number Nu_d does not depend here on the

TABLE 11-1

$\mathrm{Nu}_d = f(\mathrm{Pe}_d)$ in Stabilized Laminar Flow

Pe_d^2	1	10	10^2	10^3	10^4
Nu_d	4.04	3.86	3.74	3.68	3.66

Peclet number, Pe_d at q_w = const and remains constant at:

$$\mathrm{Nu}_d = 4.36 \tag{11-3}$$

The theoretical investigations [11.1-11.3] are based on the assumption that the physical parameters are constant. This condition is comparatively true for liquid metals. The physical parameters of heavy metals and alkali metals depend relatively little on temperature. The temperature difference is usually very small, owing to the high intensity of heat transfer. This explains why no term to account for the variation in physical parameters over the cross-section of the conduits is introduced in the formulae. Other investigators ignore so-called contact thermal resistance.

The effect of contact resistance shows itself in a drop of the heat-transfer coefficient, compared with the value calculated theoretically, and also in an observed instability of heat transfer with time.

Experiments with various liquid metals indicate that thermal contact resistance is the result of a complex process due to a combination of physico-chemical, hydrodynamic, and thermal phenomena occurring at the heat-transfer surface. The most likely reason for the deterioration of heat transfer is the formation of an additional phase (admixtures, oxides) at the interface of the "liquid metal and the wall".

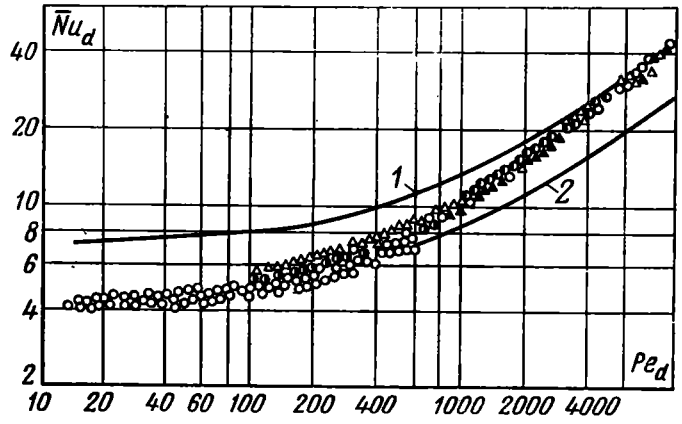

Fig. 11-2. Heat transfer with liquid metals flowing in tubes:
1—by formula $Nu = 7 + 0.025Pe^{0.8}$; *2*—$Nu = 4.8 + 0.014Pe^{0.8}$, experimental points agree with equation (11-4)

The theoretical formulae are applicable to comparatively pure substances with a minimum of admixtures (including oxygen). The magnitude of contact resistance has been shown experimentally to depend also on the choice of material for the wall.

Experimental studies of heat transfer in liquid metals have been conducted by many Soviet and foreign investigators.

Mikheev et al. [11.4] obtained the following calculation formula for mean heat-transfer coefficients of heavy and alkali metals in forced turbulent flow and also their alloys in oxidized steel tubes, without neutral gas protection:

$$\overline{Nu}_{fd} = [3.3 + 0.014Pe_{fd}^{0.8}]\,\bar{\varepsilon}_l \qquad (11\text{-}4)$$

In this formula the mean temperature of the liquid metal and tube diameter* are taken as reference values. The correction $\bar{\varepsilon}_l = 1.72\,(d/l)^{0.16}$ at $\frac{l}{d} < 30$; $\bar{\varepsilon}_l = 1$, if $\frac{l}{d} > 30$.

The intensity of heat transfer has been shown experimentally to vary from the largest value in accordance with formula (11-1) to the smallest by formula (11-4), depending on the cleaning of the circulation conduits.

Certain experimental data on heat transfer in liquid metals flowing in long tubes are presented in Fig. 11-2 [11.5]. Experimental results show that

* Formula (11-4) is recommended [11.4] for heat transfer calculations, because of certain requirements for the cleanliness of the circulation circuit. When used for practical calculations, the constant term 3.3 is replaced in the formula by 4.8.

$Nu_{fd} = 4.36$ at $q_w = \text{const}$ for liquid metals in laminar flow, and this is in full agreement with theory.

The results obtained in investigating heat transfer at $20 < Pe_{fd} < 10^4$ are described according to [11.5] by the interpolated relationship

$$\overline{Nu}_{fd} = 4.36 + 0.025 Pe_{fd}^{0.8} \tag{11-5}$$

The experimental data for developed turbulent flow [11.5] agree with Lyon's equation (11-1).

From Fig. 11-2 it follows that no sharp change is observed in the nature of the dependence of the Nusselt number on the Peclet number at a critical Reynolds number $Re \approx 2000$. After Re reaches critical magnitude the intensity of heat transfer increases monotonously with rising Pe. This nature of the dependence is explained by the fact that molecular diffusion of heat (conduction) is of great importance with liquid metals in turbulent flow, especially at low Peclet numbers. The appearance of turbulent diffusion at $Re > Re_{cr}$ therefore causes a comparatively small increase in the intensity of heat transfer.

Liquid metals have high thermal conductivity, hence, apart from radial heat transfer, conduction in the axial direction must often be taken into account when calculating the cross-sectional mean temperature.

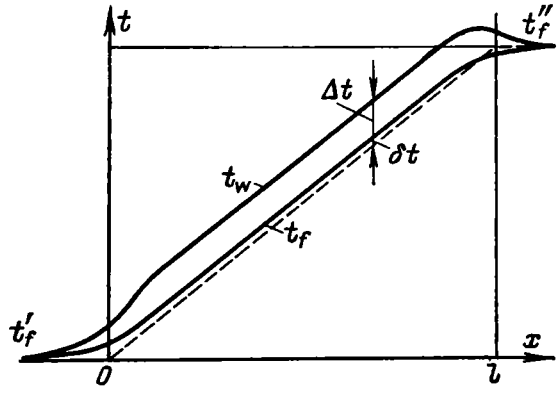

Fig. 11-3. Mean temperature distribution of fluid along a flow, $q_w = \text{const}$

With uniform heat flow from the wall ($q_w = \text{const}$) the temperature distribution is shown in Fig. 11-3. The fluid is heated along the section O-l; the heat-transfer coefficient α is taken as constant. The dotted curve corresponds to the cross-sectional mean temperature of the fluid calculated by formula

$$t_f = t_f' + \frac{q_w F}{G c_p} \tag{11-6}$$

At a sufficient distance from the ends of the section where heat transfer occurs the variation of fluid and wall temperature is linear, and here

$$\Delta t = \frac{q_w}{\alpha} = \text{const}$$

Conduction in the axial direction, however, causes an increase δt in both liquid and wall temperature. As follows from L.G. Genin's work

$$\delta t = \frac{q_w [1 + (\lambda_w f_w / \lambda_f f_f)] \, dP}{Pe_d^2 \lambda_f f_f} \tag{11-7}$$

where P = the circumference of the tube;
d = the inside diameter of the tube;
$\lambda_w, \lambda_f, f_w, f_f$ = thermal conductivities and cross-sectional areas of the tube wall and liquid metal respectively.

We have considered heat transfer in the flow of liquid metals in tubes. Of practical interest is also the process in systems of other geometrical forms, across banks of tubes, in particular.

Heat transfer in banks of in-line and staggered tubes placed in a cross flow of liquid metal (mercury, sodium) has been studied by Borishansky, Andreevsky, Subbotin, Minashin and others [11.6; 11.7]. The experiments have shown the heat-transfer coefficient for deeply arranged rows of tubes to be described by the formula

$$\overline{\mathrm{Nu}}_{fd} = \mathrm{Pe}_{fd}^{0.5} \tag{11-8}$$

Here the reference dimension is the outer tube diameter; the velocity is calculated in the narrowest passage of the bank. The physical parameters are taken according to the temperature of liquid metal.

The heat-transfer coefficients have been determined by experiment as $\alpha = \overline{q_w}/\overline{\Delta t}$. The banks of tubes investigated were comparatively crowded, the relative pitches s_1/d and s_2/d were varied from about 1 to 1.5. The tests did not show the dependence of heat transfer on the type of bank and of its relative pitches. They indicated that the mean rate of heat transfer in the first row was about 20 per cent smaller than the intensity of heat transfer in the deeper rows. Formula (11-8) is recommended for pure liquid metals.

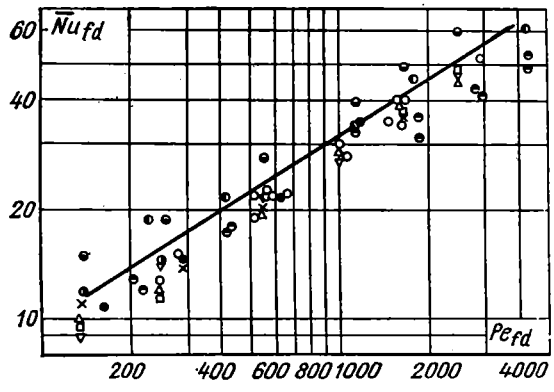

Fig. 11-4. Heat transfer in banks of tubes in a normal flow of liquid metals (staggered and in-line banks of tubes with relative pitches between 1.2 and 1.5; tubes made of nickel and steel 1X18H9T)

Formula (11-8) is graphically correlated with test data in Fig. 11-4 [11.7].

Heat transfer by natural convection in liquid metals is also of practical interest.

In Chapter 10 it was stated that the effect of inertia is marked at small Prandtl numbers. The proportionality factor c in the formulae of the $\mathrm{Nu} = c(\mathrm{GrPr})^m$ kind then proves to be a function of the Prandtl number: at $c =$ const the exponent of Pr is variable. This is also true for free convection in liquid metals.

Research in the problem of natural convection in liquid heavy and alkali metals and alloys in an infinite space has been made at the Krzhizhanovsky Power Institute in Moscow [11.8]. Their work gave the formula

$$\overline{\mathrm{Nu}}_{wg} = c\mathrm{Gr}_{wg}^{n}\mathrm{Pr}_{wg}^{m} \tag{11.9}$$

which describes the mean rate of heat transfer.

Here

$$m = 0.3 + (0.02/\mathrm{Pr}_{wg}^{1/3}) \tag{11-10}$$

The magnitudes of constants c and n depend on the Grashof number. At $\mathrm{Gr}_{wg} = 10^2$ to 10^9, $c = 0.52$ and $n = 0.25$ (laminar flow); at $\mathrm{Gr}_{wg} > 10^9$, $c = 0.105$ and $n = 1/3$ (turbulent flow). As reference is taken so-called mean temperature of the boundary layer $t_{wg} = 0.5(t_f + t_w)$, where t_f is the temperature of liquid at a distance from the wall. The height of vertical plates, and the outside diameter of horizontal tubes were taken as reference dimensions.

The hydraulic resistance to flow (pressure drop) offered by liquid metal is determined by the same formulae as for other liquids.

11-2. Heat Transfer in a Substance in the Supercritical State

High pressures and temperatures are being used more and more in power engineering and other fields. The pressure of the heat carrier used in some equipment exceeds the thermodynamic critical pressure p_{cr}. The temperature of the fluid may then be either higher or lower than the thermodynamic critical temperature t_{cr}.

In the supercritical region a peculiar and substantial change in the physical properties of the fluid occurs, with comparatively small variations in tempera-

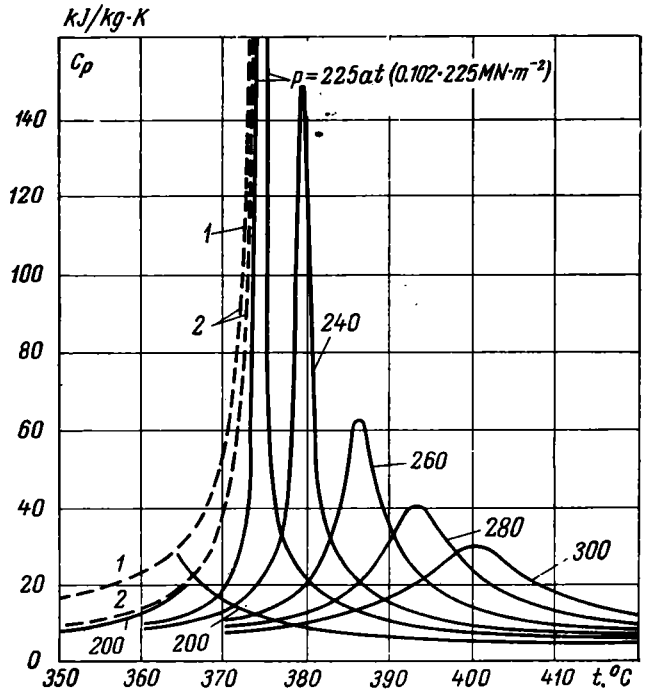

Fig. 11-5. Specific heat of water and steam in the near-critical region

ture and pressure. The change in heat capacity c_p is of particular importance; it may increase by a large factor and approach a maximum (Fig. 11-5) . The temperature t_m corresponding to a maximum heat capacity at p = const is called the *pseudocritical* temperature. Density, coefficients of viscosity and thermal conductivity also change materially in this region. The Prandtl number $\mathrm{Pr} = \frac{\mu c_p}{\lambda}$ changes considerably with it.

From thermodynamics it is known that the transformation from the liquid to the gaseous phase is continuous in the supercritical region. The change in the properties of the fluid is not sharp or abrupt, as can be observed from co-existence of the two phases.

Heat transfer in the supercritical state, therefore, is primarily considered as heat transfer in a single-phase medium, but marked by strongly pronounced variability of the physical properties of the fluid. But with vanishing temperature differences, when the variability of physical properties does not manifest itself practically, the heat-transfer coefficients can be calculated by ordinary formulas, presented above. With increasing temperature difference the discre-

pancy between test data and the data obtained with the aid of these formulas rises and may become intolerable.

In some cases an abruptly reduced rate of heat transfer (the so-called regime of "deteriorated heat transfer") is observed at individual sections of the tube. Then, wall temperature rises considerably which may result in its failure. Deteriorated heat transfer was observed both in horizontal and vertically-arranged tubes at Reynolds numbers equal to or exceeding 10^6. In some of the tests higher values of heat-transfer coefficients were registered, and these patterns of flow may be accompanied by serious velocity fluctuations and noise.

A great number of experimental and theoretical (analytical) studies were conducted with the view of obtaining calculation relationships permitting to determine heat transfer under different conditions of flow. It was shown, in particular, that in the near critical region of state turbulent flow and the associated heat transfer may depend materially on the Grashof number, i.e. on thermal free convection, due to an essential change in density in the region of state of substances being considered.

Notwithstanding the presence of a great number of experimental data serving as a basis for a great number of hypothesis suggested to explain these data, and also the availability of a number of calculation relationships at the present time no sufficiently generalized formulae are available which would permit reliable calculation of heat transfer for all cases. In practice, heat-transfer coefficients can be determined with the aid of experimental data (formulae) complying to the greatest extent with the operating conditions of an industrial outfit.

It will be noted that a number of published works contain data on heat transfer for substance in the supercritical state [11.9-11.14].

11-3. Heat Transfer in Gases at High Velocities

The process of heat transfer in gases at high velocities has been shown by theory and experiment to possess a number of peculiarities which are not to be ignored to avoid considerable errors.

The hydrodynamic and heat-transfer processes are closely interrelated at high velocities. Thermodynamic factors also acquire great importance, since the fluid flow is marked by mutual transformation of its internal and kinetic energy and by expansion of the gas.

For a stream of gas the equation of the first law of thermodynamics can be written as follows:

$$\left(i_1+\frac{w_1^2}{2}\right)-\left(i_2+\frac{w_2^2}{2}\right)=\frac{Q}{G} \qquad (11\text{-}11)$$

where i is the specific enthalpy, J/kg; w is the gas velocity, m/s; Q is the heat flux through the space limited by sections *1* and *2* in watts; and G is the rate of gas flow, kg/s. The subscripts 1 and 2 refer respectively to the initial and final section of flow.

Equation (11-11) is written for one kilogram of gas and the parameters of flow are assumed to be constant over the cross-section of a stream of gas.

With a gas in adiabatic flow ($Q = 0$) its kinetic energy $w^2/2$ may increase only if enthalpy diminishes, and, vice versa, an increase in enthalpy entails a decrease in kinetic energy and, consequently, in velocity. A change in kinetic energy leads in the final analysis to a change in gas temperature.

The enthalpy i at full adiabatic deceleration is called the *enthalpy of adiabatic deceleration* and is equal to:

$$i_0 = i + \frac{w^2}{2} \tag{11-12}$$

The temperature T_{st} of the gas at full adiabatic deceleration is called the stagnation temperature. Substituting into Eq. (11-12) the value of enthalpy for thermodynamically ideal gases, $i = c_p T$, and dividing its right- and left-hand sides by c_p, we obtain:

$$T_{st} = T + \frac{w^2}{2c_p} \tag{11-13}$$

As is known from thermodynamics the ratio of kinetic energy to its enthalpy is:

$$\frac{w^2/2}{c_p T} = \frac{k-1}{2}\frac{w^2}{a^2} = \frac{k-1}{2}\mathrm{M}^2 \tag{11-14}$$

where $k = \frac{c_p}{c_v}$ is the ratio of the specific heat at constant pressure to that at constant volume.

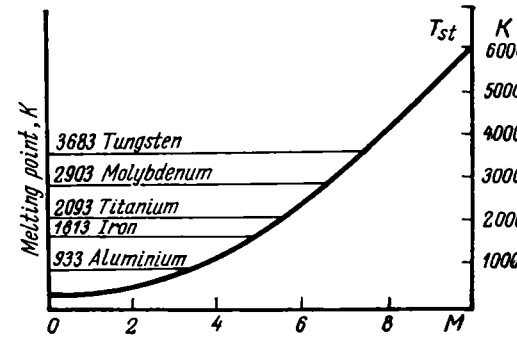

Fig. 11-6. Dependence of stagnation temperature on Mach number M ($k = 1.4$)

The quantity $\frac{w}{a}$ is the ratio of flow velocity to sonic velocity at the same point. This ratio is denoted by M and is called the Mach number. From equation (11-14) it follows that the Mach number characterizes the ratio of the kinetic energy of flow to its enthalpy. In general it may vary from zero to infinity. Flow is said to be *subsonic* if $\mathrm{M} < 1$, *sonic* at $\mathrm{M} = 1$, and *supersonic* at $\mathrm{M} > 1$.

Taking Eq. (11-14) into account, Eq. (11-13) can be rearranged as follows:

$$\frac{T_{st}}{T} = 1 + \frac{w^2}{2c_p T} = 1 + \frac{k-1}{2}\mathrm{M}^2 \tag{11-15}$$

from which it follows, for example, that the stagnation temperature of flow exceeds the thermodynamic temperature by about 1 per cent at a Mach number $\mathrm{M} = 0.25$ and $k = 1.4$ (air).

With a thermodynamic temperature $T = 288$ K sonic velocity in air is about 340 m/s; a velocity around 85 m/s corresponds then to a Mach number $\mathrm{M} = 0.25$. It is usually assumed that at $\mathrm{M} < 0.25$ $T_{st} = T$.

Figure 11-6 illustrates the dependence of stagnation temperature on the velocity of air flow. At a Mach number $\mathrm{M} = 1$, $T_{st} = 1.2T$, at $\mathrm{M} = 5$, $T_{st} = 6T$. Fig. 11-6 also gives the melting points of some metals.

The physical properties of a gas change essentially at very high temperatures. Dissociation of molecules takes place, i.e. distintegration of molecules into two or more particles (atoms, radicals or molecules). In addition, ionization takes place at still higher temperatures, the process consisting in electrons being detached from atoms and molecules, causing the formation of positive ions and free electrons. Effects of this kind manifest themselves at temperatures exceeding thousands of degrees (Fig. 15-1, Sec. 15-1).

The action of a gas at high temperature may bring about destruction of the uncooled surfaces of the body—through melting, evaporation (sublimation). Particles of the failed surface may then react chemically with one another and with the gas of the external flow. The processes of evaporation (or sublimation),

melting, burning, and destruction of surface layer of a body are accompanied by carry-over of the substance by the gas flowing at a high velocity and high temperature and is referred to as *ablation.*

This chapter deals with processes of heat transfer occurring at relatively low velocities (at $M < 5$) and temperatures, in the absence of dissociation and ionization of the gas and ablation. Flow at $1 < M < 5$ will be referred to as supersonic, and at $M > 5$ as *hypersonic.*

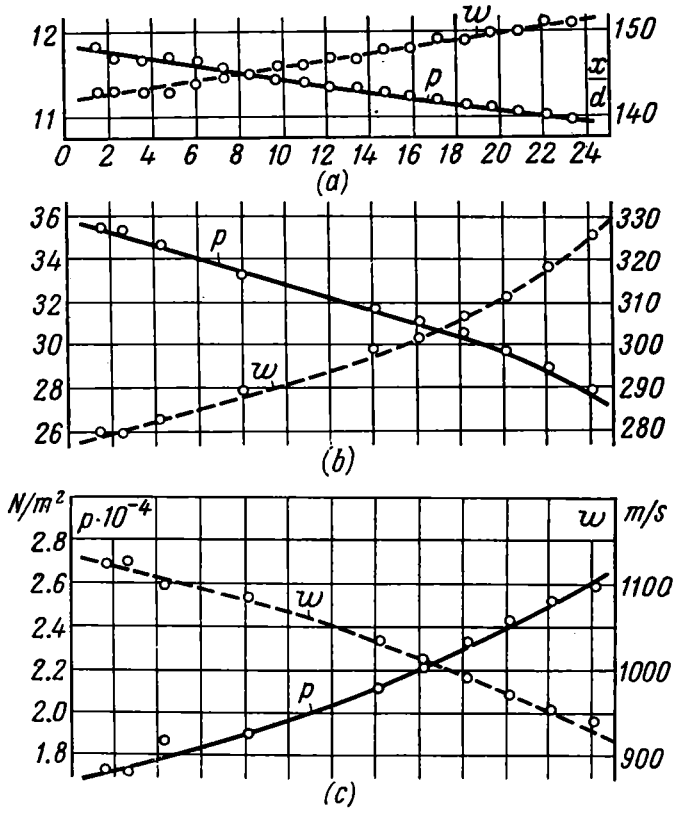

Fig. 11-7. Variation in velocity and pressure of the air along a tube:
a—$Re_{din} = 1.82 \times 10^5$, $M_{in} = 0.42$, $q_w = 0$; *b*—$Re_{din} = 6.51 \times 10^5$, $M_{in} = 0.75$, $q_w = 1.85 \times 10^5$ W/m²; *c*—$Re_{din} = 1.4 \times 10^5$, $M_{in} = 2.4$, $q_w = 1.85 \times 10^5$ W/m²

The kinetic energy of a gas in adiabatic flow can increase only if there is a drop in enthalpy. An increase in velocity of a gas in adiabatic flow is, therefore, linked with a drop in gas temperature. But the density of the gas diminishes with increasing velocity, because the pressure drops faster than temperature. Thus an increase in flow velocity brings about expansion of the gas and a further increase of velocity.

The curves in Fig. 11-7 represent the variation in pressure and velocity of a gas in adiabatic and non-adiabatic flow along a tube at subsonic and supersonic velocities. At small values of the Mach number the flow of compressible gas differs little in practice from the flow of incompressible fluid—the velocity of gas scarcely changes along the conduit, and the decrease of pressure is linear.

In subsonic flow the larger the Mach number, the greater the rise in pressure of a gas along a tube, and the larger the pressure drop.

If, in addition to the work of friction, heat is imparted or removed through the walls of the conduit, then, in the first case, there will be a correspondingly

greater drop in pressure along the stream than with adiabatic flow and in the second case the drop is lesser. Passing along the tube, the supersonic flow decelerates, i.e. its velocity drops along the flow, and pressure rises.

With a gas flowing at high velocities the previously derived equation of energy (4-10) happens to be unsuitable. Account should be taken of the mutual transformations of internal and kinetic energies and also of the compressibility of the gas.

The amount of heat dQ added to a certain unit mass of flow is the sum of heat dQ_1, transferred from the outside by conduction, and frictional heat dQ_{fr}. According to the first law of thermodynamics the heat dQ is expended to change the internal energy du and perform work of expansion (deformation):

$$p\,dv = pd\,(1/\rho) = -\frac{p}{\rho^2}\,d\rho$$

$$dQ_1 + dQ_{fr} = du - \frac{p}{\rho^2}\,d\rho$$

The change in internal energy can be determined as follows:

$$du = di - d\frac{p}{\rho} = di - \frac{1}{\rho}\,dp + \frac{p}{\rho^2}\,d\rho$$

According to the last two equations

$$dQ_1 = di - \frac{1}{\rho}\,dp - dQ_{fr}$$

In deriving the equation of energy (4-10), we believed in principle that $dQ_1 = di$. If, in addition to enthalpy, the terms dp/ρ and dQ_{fr} introduced into the equation of energy, we obtain (at $q_v = 0$):

$$\rho\frac{di}{d\tau} - \frac{dp}{d\tau} = \lambda\nabla^2 t + \mu\Phi \qquad (11\text{-}16)$$

where $di/d\tau$ and $dp/d\tau$ are the total (substantial) derivatives of enthalpy and pressure with respect to time. The term $\rho dQ_{fr}/d\tau$ is denoted by $\mu\Phi$*.

Thus, two additional terms $dp/d\tau$ and $\mu\Phi$ appeared in the equation of energy. They account for the work of expansion and the dissipation of mechanical energy. The dissipative function $\mu\Phi$ was introduced by Rayleigh.

The equation of flow also changes, taking the following appearance for a compressible fluid:

$$\rho\frac{d\vec{w}}{d\tau} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{w} + \frac{1}{3}\mu\ \text{grad div}\,\vec{w} \qquad (11\text{-}17)$$

The equation of continuity

$$\frac{\partial\rho}{\partial\tau} + \text{div}\,(\rho\vec{w}) = 0$$

does not change, apparently.

---

* For fluids obeying Newton's law of viscous friction

$$\Phi = 2\left[\left(\frac{\partial w_x}{\partial x}\right)^2 + \left(\frac{\partial w_y}{\partial y}\right)^2 + \left(\frac{\partial w_z}{\partial z}\right)^2\right] +$$

$$+ \left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right)^2 + \left(\frac{\partial w_y}{\partial z} + \frac{\partial w_z}{\partial y}\right)^2 + \left(\frac{\partial w_z}{\partial x} + \frac{\partial w_x}{\partial z}\right)^2 - \frac{2}{3}(\text{div}\,\vec{w})^2$$

It shall be noted that the evolution of heat is proportional to *squared* derivatives.

It will be noted that in writing Eqs. (11-16) and (11-17) it was assumed for the sake of simplicity that λ and μ are constant.

Equations (11-16), (11-17) and (4-19) are often used together with the equation of state $p = \rho RT$ for the mathematical statement of a problem. If the boundary conditions are assumed to be identical with those used in Sec. 5-2, we can obtain the following equations of similarity:

$$\Theta = f_1(X, Y, \mathrm{Re}, \mathrm{Pr}, \mathrm{M}, k)$$
$$W_x = f_2(X, Y, \mathrm{Re}, \mathrm{Pr}, \mathrm{M}, k) \tag{11-18}$$

etc. If account is taken of the dependence of physical parameters (properties) on temperature, the temperature factor θ_w will also appear in the list of the reference criteria of similarity.

Inasmuch as the gas is fully stagnated directly at the wall, an impression may be produced that in the absence of heat transfer through the wall its surface temperature must be equal to stagnation temperature, which, however, is true only for private cases.

Under real conditions the conversion of mechanical into thermal energy is accompanied by heat and work transfer between adjacent layers of gas. Such transfer of heat and work shall also take place when a solid is heat-insulated and there is no heat transfer between the solid and gas. As a result the temperature of gas particles which are in direct contact with the surface of the heat-insulated wall will exceed the temperature of the gas at a distance from the wall, however, not equal in the general case to the stagnation temperature. The heat-insulated solid will be at the same temperature (a jump in temperature, just as a jump in velocity, may take place at the "solid-gas" interface only in a heavily rarefied gas). This temperature is referred to as the adiabatic, proper or equilibrium temperature.

Thus, by *adiabatic* is meant the temperature which would be shown by a heat-insulated thermometer kept stationary in a high-velocity flow of fluid. The thermometer would indicate the thermodynamic temperature of a non-stagnant gas only if it moved with the gas.

The adiabatic temperature of the wall is determined by the following equation:

$$T_{aw} = T + r\frac{w^2}{2c_p} = T\left(1 + r\frac{k-1}{2}\mathrm{M}^2\right) \tag{11-19}$$

where r is the *recovery factor*. From Eq. (11-19) it follows that

$$r = \frac{T_{aw} - T}{w^2/2c_p} = \frac{T_{aw} - T}{T_{st} - T} \tag{11-20}$$

Equation (11-20) defines the recovery factor r which can be both larger and smaller than unity. If the intensity or rate of heat transfer due to dissipation of mechanical energy exceeds the rate of removal of heat by convection and conduction, then $r > 1$; when $r < 1$, the removal of heat predominates; when $r = 1$, the processes of heat liberation and removal are balanced.

The recovery factor generally depends on the same factors as the temperature field (11-18), and the number of effecting factors diminishes in particular cases.

For a flat plate in longitudinal flow with a laminar boundary layer the recovery factor is

$$r = \sqrt{\mathrm{Pr}} \tag{11-21}$$

For a turbulent boundary layer

$$r = \sqrt[3]{\mathrm{Pr}} \tag{11-22}$$

Formulas (11-21) and (11-22) agree well with experimental data.

The local recovery factor for air in both subsonic and supersonic turbulent flow in a tube can be determined, according to [11.15], by the formula

$$r = \sqrt[3]{\mathrm{Pr}} - \Delta r_x \qquad (11\text{-}23)$$

where $\Delta r_x = 7.16 \times 10^5 \mathrm{Re}_x^{0.4} f(x/d)$ is a correction determined experimentally. At $x/d = 0$-15 (initial section) $f(x/d) = 1$; at $x/d = 15$-27

$$f(x/d) = 1 + 0.0413\left(\frac{x}{d} - 15\right)$$

In calculating Re_x, the mean velocity in the given cross-section is taken, and the reference dimension is the coordinate x, measured from the beginning of the tube. Taken as reference temperature is the thermodynamic mean temperature in the given section. The Prandtl numbers in formulae (11-21)-(11-23) must be chosen for a specially selected reference temperature $T_{\mathrm{ref}} = T + 0.5(T_w - T) + 0.22(T_{aw} - T)$. For many gases the Prandtl number is almost constant in a wide temperature interval, then the selection of the reference temperature is immaterial. The recovery factor for air flow can be taken as 0.92.

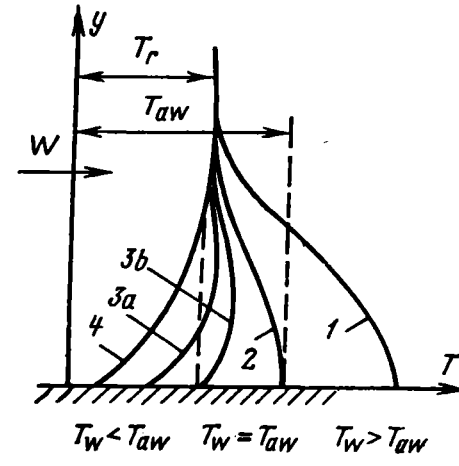

Fig. 11-8. Temperature variation in the boundary layer of a high-velocity gas in a certain section x at different conditions

With liberation of heat, the temperature distribution in the boundary layer changes (Fig. 11-8).

The temperature field in the gas can be considered as the sum of two fields one of which is traced to the evolution of frictional heat and the other is due to the transfer of heat through the wall. These processes, of course, are interrelated, and only in limiting cases $q_w = 0$ and $M = 0$ they manifest themselves "pure".

Let a plate be exposed to the longitudinal flow of a gas at high velocity, the temperature of which is T_g at a distance from the body.

For the case $q_w = -\lambda(\partial t/\partial y)_{y=0} = 0$ (heat-insulated wall surface) $T_w = T_{aw}$ (curve *2*, Fig. 11-8). Heat transfer takes place only inside the gas, since at $y > 0$ the temperature gradient is not equal to zero. The maximum temperature of the gas in the boundary layer is T_{aw} (at $y = 0$). An increase in gas temperature over its temperature T_g is caused by the evolution of heat in the boundary layer, and but for the evolution of this heat, gas temperature would be equal to T_g over the entire thickness of the layer ($q = 0$).

Curves *1*, *3* and *4* (Fig. 11-8) refer to cases when, in addition to heat transfer in the gas, there is heat flow through the wall ($q_w \neq 0$).

Curves *3* correspond to a noticeable evolution of frictional heat in the boundary layer, however, the wall is intensively cooled from the inside, resulting in that $T_w < T_{aw}$ and $q_w < 0$ (i.e. heat flows from the gas into the wall). But at a distance from the wall, beyond the maximum on the temperature curve, frictional heat propagates also into the gas ($q > 0$). A case is feasible when $T_w = T_g$ (curve *3b*), but even then $q_w < 0$ since the temperature in the boundary layer is higher than wall temperature.

Curve *4* corresponds to the case when the evolution of frictional heat is small compared with the intensity of its rejection to the wall, and it practically fails to effect the distribution of gas temperature in the boundary layer. The heat flux is directed toward the wall ($q_w < 0$). The distribution of temperature does not differ from that considered above for a flow of an incompressible gas.

The temperature distribution presented by curve *1* may materialize, if heat is added to the wall from the inside. It is clear that then $T_w > T_{aw}$ and $q_w > 0$.

Thus, in the event of a gas flowing at a high velocity the heat flow q_w will always be directed from the wall into the gas, if $T_w > T_{aw}$, and, on the contrary, from the gas to the wall, if $T_w < T_{aw}$.

With the wall temperature given, the Mach number, at which $q_w = 0$, can be determined from equation (11-19), assuming $T_{aw} = T_w$.

Application of the Newton equation $q_w = \alpha (t_w - t_g)$ to high-velocity flow can lead to inaccurate results. For a heat-insulated body in flow, when $q_w = 0$, for instance, it follows from this formula that $q_w \neq 0$, since $T_g \neq T_w = T_{aw}$. At the same time, when $T_g = T_w$, it follows from the formula that $q_w = 0$, although then $q_w \neq 0$ (curve *3b*). Account must be taken of the fact that with a fluid in high-velocity flow the temperature in the boundary layer rises at the expense of frictional heat. For this purpose T_g is substituted in Newton's equation by the adiabatic wall temperature T_{aw}. Then,

$$q_w = \alpha\ T_{aw} - T_w)$$

or

$$q_w = \alpha \left(T_g + r \frac{w^2}{2c_p} - T_w\right) \tag{11-24}$$

With low velocities, when $r \frac{w^2}{2c_p}$ is much smaller than T_g, this formula turns into the Newton equation since $r \frac{w^2}{2c_p}$ can then be ignored.

Theoretical and experimental studies show that the dimensionless equations for incompressible fluid given earlier, can be used in determinations of α by formula (11-24), to calculate the heat-transfer coefficients for subsonic gas flow. With higher velocities the effect of k and M must be accounted for in the dimensionless equations. The parameters of high-velocity flow change essentially both along the flow and over the cross-section of the conduit. In view of this the knowledge of the local heat-transfer coefficients is of interest.

It has been found by A.S. Sukomel, V.M. Mukhin and V.I. Velichko that the local heat-transfer coefficients for a cooled gas in turbulent flow at supersonic velocity and large temperature differences along a straight circular tube can be determined from the following equation:

$$\mathrm{Nu}_{gd} = 0.022\ \mathrm{Re}_{gd}^{0.8}\ \mathrm{Pr}_g^{0.43}\ (T_g/T_{st})^{0.42}\ \varepsilon_l \tag{11-25}$$

In this formula the physical parameters are reduced to the mean thermodynamic gas temperature T_g in the considered section. The tube inner diameter is taken as the reference dimension. The gas velocity in the Reynolds number is the mean velocity for the cross-section.

The correction ε_l accounts for the variation in the rate of heat transfer along the tube. With a turbulent boundary layer setting in at the inlet of the tube, and at $\frac{x}{d} < 15$, the correction $\varepsilon_l = 1.38\ (x/d)^{0.12}$, where x is the longitudinal coordinate measured from the tube inlet. At $\frac{x}{d} \geqslant 15$, $\varepsilon_l = 1$.

In the experiments [11.15] the air inlet temperature was varied from 200 °C to 800 °C corresponding to the variation in the temperature ratio $\theta_w \equiv T_w/T_{aw}$ from 0.3 to 1. Mach numbers varied from 0.1 to 4 at the tube inlet, and the Reynolds number reached a magnitude of 7×10^5.

The factor

$$(T_g/T_{st})^{0.42} = \left(1 + \frac{k-1}{2} M^2\right)^{-0.42}$$

accounts for the effect of factors characteristic of high-velocity gas flow. The heat-transfer coefficient is determined by formula (11-24).

The experimental dependence of heat transfer on T_g/T_{st} is illustrated in Fig. 11-9. At $M < 1$ $(T_g/T_{st})^{0.42}$ differs only slightly from unity.

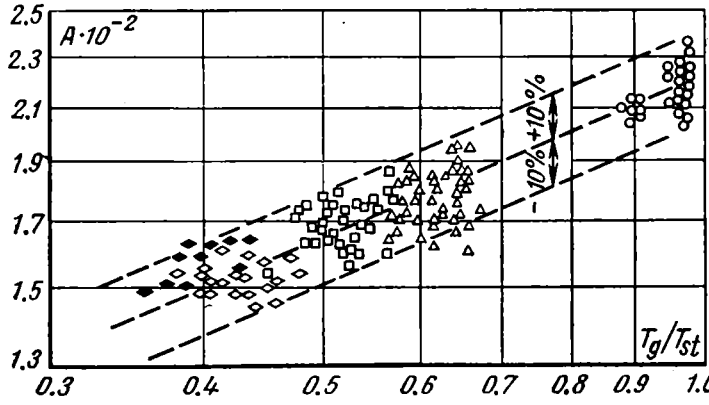

Fig. 11-9. Heat transfer with a compressible gas in turbulent flow in straight circular tubes

According to published work [11.15] when the longitudinal coordinate x is used as the reference dimension the magnitude of the correction changes, and is then equal to $(T_g/T_{st})^{0.38}$ instead of $(T_g/T_{st})^{0.42}$. The calculation formula proposed for the heat-transfer coefficient is similar in other respects to that recommended for a plate in incompressible flow with a turbulent boundary layer (7-39). The development of heat transfer at the tube inlet with a high-velocity gas flow is thus also similar to the development of heat transfer with a plate in flow

The local heat-transfer coefficients for a plate in longitudinal flow with a turbulent boundary layer can here be calculated with the aid of formula

$$\mathrm{Nu}_{gx} = 0.0296\, \mathrm{Re}_{gx}^{0.8}\, \mathrm{Pr}_g^{0.43} (T_g/T_{st})^{0.38} \qquad (11\text{-}26)$$

With a laminar boundary layer the rate of heat transfer is considerably inferior to that for a turbulent boundary layer. Thermal protection of a solid surface exposed to high-velocity and high-temperature gas flow can be ensured by maintaining a laminar pattern of flow in the boundary layer.

The viscosity of a heated gas increases, resulting in that in some apparatus the turbulent boundary layer may become laminar.

11-4. Heat Transfer in Rarefied Gases

In considering the process of heat transfer by convection, we started from the assumption that the gas can be considered as a continuum, i.e. that its discrete structure can be ignored. But with small absolute pressures (or small size of

the bodies involved in the process) the phenomenon of heat transport can only be explained provided that the molecular structure of the gas is taken into consideration. Treatment of the gas as a continuum is then unsuitable. The boundary conditions change as well. The gas directly adjoining the surface of the body exposed to flow does not have the velocity and temperature of the latter, i.e. "slipping" of the gas and a jump in temperature occur at the interface.

The gas, of course, can be represented as a host of molecules in rectilinear motion in all directions and continuously colliding with one another. Each collision results in a change in the velocity and direction of molecular motion. As pressure falls, collisions become less and less frequent.

The mean molecular path $\bar{l}$ between two consecutive collisions is known as the mean free path. From the kinetic theory of gases it follows that

$$\bar{l} = \frac{\bar{w}}{\varkappa} = \frac{1}{n\sigma\sqrt{2}} \tag{11-27}$$

where $\bar{w}$ is the mean velocity of gas molecules, m/s; $\varkappa$ is the number of collisions per second; n is the number of molecules per unit volume; σ is effective molecular cross-section:

The effective cross-section is determined by the distance of approach between molecules at the moment of collision, i.e. by the distance at which molecular interaction brings about a just perceptible change in direction.

If we envisage the colliding molecules as rigid balls, the effective cross-section σ will be πd^2, where $d = r_1 + r_2$, and r_1, r_2 are the radii of molecules.

The mean free path for ordinary molecular gases under standard conditions (pressure about 0.1 MN/m^2 and a temperature of 273 K) is about 10^{-7} m. As follows from Eq. (11-27), $\bar{l}$ is inversely proportional to the number of molecules per unit volume, and, consequently, inversely proportional to the pressure. The main free path becomes very large at low pressures. Thus, for example, at a pressure $p = 0.133$ N/m^2 the mean free path is about 7 cm, and at $p = 0.133 \cdot 10^{-5}$ N/m^2 about 7 km.

Thus the lower the pressure of the gas, the larger the mean free path $\bar{l}$ and the smaller the probability of collision. At low pressures molecules may cover the distance from wall to wall of the conduit without colliding.

According to Eq. (11-27), the mean free path $\bar{l}$ is inversely proportional to the effective molecular cross-section σ. The value of σ depends on molecular velocity, because fast molecules undergo a smaller deflection from their path than slow ones with one and the same force of interaction. Therefore, the larger their velocity, the smaller the force required to bring about molecular deflection, the smaller the distance between colliding molecules and, consequently, the smaller the calculated cross-section. Since the kinetic energy of molecules is a measure of gas temperature, the value of σ diminishes with increasing T, and the mean free path increases.

Thus the mean free path of molecules increases with diminishing pressure and rising temperature of the gas.

Molecules may also collide with the surface of the body. It is suggested that an impinging molecule may be either mirror-reflected from the surface or remain on the surface for a certain time, and then pass back into the volume of the gas. The time during which it rests on the surface depends on the location and nature of the latter, and on nature of the molecule itself, the surface temperature, the kinetic energy of the molecule, and other factors. This phenomenon is known as adsorption. The re-emission of adsorbed molecules occurs in such a

way that they scatter symmetrically in all directions in a hemisphere, i.e. *diffuse* emission occurs*.

The nature of the interaction between the molecules of a gas and the wall is of great importance for estimation of heat transfer.

Heat transfer is due to an exchange of energy between the molecules and the surface of the body. When a molecule strikes the surface of a solid it may impart a certain amount of energy to the latter. Complete transfer of heat is believed to occur when the period of retention of the molecules on the surface is much larger than the period of oscillation of the molecules of the body. If the delay or contact time is short, heat transfer is incomplete. The degree of "completeness" is characterized by an *accommodation coefficient* which is determined by the following expression:

$$\gamma = \frac{E_{im} - E_0}{E_{im} - E_w} \tag{11-28}$$

Here E_{im} is the mean energy of a molecule striking the surface of the body; E_0 is the mean energy of the same molecule after contact with the body ("emission" from the body); E_w is the mean energy of a molecule corresponding to the wall temperature.

The accommodation coefficient is thus the ratio of the actual exchange of energy to the maximum possible.

With instantaneous specular reflection of a molecule its energy changes, i.e. $E_{im} = E_0$ and $\gamma = 0$. With complete exchange of energy $E_0 = E_w$ and $\gamma = 1$. In intermediate cases E_0 varies between E_{im} and E_w and the accommodation coefficient lies between 0 and 1.

The accommodation coefficient can usually be determined for different kinds of molecular energy (translational motion, rotational or angular, and oscillation). Since a great many collisions are required to excite an oscillatory degree of freedom, the accommodation coefficient of vibrational or oscillation energy is usually taken as zero.

The concept of the accommodation coefficient is also used to characterize the energy exchange in a molecular *aggregate*. By the terms E_{im}, E_0 and E_w in Eq. (11-28) are then meant the respective energies of the molecular aggregate, and not those of a single molecule.

Since gas temperature is considered to be proportional to the mean kinetic energy of the molecular aggregate**, the expression for the accommodation coefficient (11-28) is often replaced by

$$\gamma = \frac{T_{im} - T_0}{T_{im} - T_w} \tag{11-29}$$

The change from (11-28) to (11-29) is sound for a monatomic ideal gas; for others it has to be considered an approximation.

From definition it follows that the accommodation coefficient depends on the surface temperature of the solid and the difference between the temperatures of the gas and surface or, more precisely, on the difference between the energy of the incident molecules and the energy they would have possessed at the

* With diffuse emission molecular density is maximum in the direction normal to the body. The molecular density in any direction of the hemisphere is equal to $I_d \cos \varphi$, where I_d is the molecular flux density in the normal direction, and φ is the angle measured from the normal to the body.

** The temperature T_p, measured in units of power, e.g. joules, is connected with the temperature T_{deg}, measured in K, by the relationship $T_p = kT_{deg}$. Here k is the Boltzmann constant numerically equal to 1.38×10^{-23} J/K.

surface temperature. The accommodation coefficient depends on the nature of the gas and the surface on which accommodation takes place. The accommodation coefficient has been shown to depend on the velocity of gas flow in relation to the wall, on the presence and amount of adsorbed gas on this wall (which depends in many respects on the prehistory of the process). The probability of adsorption and the moment when it sets in depend on the number of molecules falling on the surface, since the state of the adsorbed layers depends on the latter. This makes it very difficult to determine accommodation coefficients theoretically, so that γ is currently determined experimentally.

The values obtained range between 0.02 and 1. Present-day studies usually lead to results according to which the accommodation coefficient is close to unity.

The collision of molecules with the surface of the solid also leads to exchange of momentum between the molecules and wall, an exchange characterized by the *momentum exchange factor f*. By definition

$$f = \frac{mw_{xim} - mw_{x0}}{mw_{xim}} \tag{11-30}$$

Here w_{xim} is the component of the velocity of the incident molecule directed along the wall; w_{x0} is the same component but referred to the reflected molecule (the wall is assumed to be stationary); m is the molecular mass. In practice f is determined for a large number of molecules considered simultaneously.

With specular reflection $w_{xim} = w_{x0}$ and $f = 0$. With diffuse reflection $w_{x0} = 0$ on average, and, consequently, $f = 1$ on average. With a mixture of the two forms of reflection the momentum exchange factor has a value between 0 and 1.

Like the accommodation coefficient γ, the momentum exchange factor f depends on a large number of factors, reflecting the actual conditions of interaction between gas molecules and the wall. In particular the factor f depends on the nature of the gas and of the wall, and the velocity of the gas flow taken as a whole.

The discrete structure of the gas and the associated nature of the interaction between molecules and the wall bring about, as already mentioned, a jump in temperature and a jump in velocity (slip) at the surface.

Slip flow along walls is due to the greater length of the free path compared with the reference dimensions of the solid. The molecules of a rarefied gas, unlike of a dense gas, may fail to collide with others, or the number of collisions may be minimal. Consequently, their tangential velocities are not on average equal to zero, as they approach the wall. But once reflected from the wall, they may scatter at random and their tangential velocity on average will equal zero. The mean tangential velocity of all gas molecules at the wall (both incident and reflected) is therefore, not equal to zero, and apparent slip flow is observed. The mean free path of gas molecules at normal pressure is small and no slip is observed in practice.

The jump in temperature at the surface of the body exposed to the flow of a rarefied gas can be explained by analogy. The molecules of a very rarefied gas approaching the wall may on average be at a temperature differing considerably from that of the surface, owing to the absence of the levelling action of mutual collisions. The molecules emitted by the wall, however, may on average be at a temperature close (or equal) to its temperature. This results in the mean temperature of the gas in the layer adjoining the wall being different from the wall temperature.

A number of formulae have been suggested to determine velocity slip and temperature jump. All are based on certain assumptions and, consequently, cannot be considered as mathematically strict. They may be written as follows:

$$w_j = c_1 \bar{l} \left(\frac{\partial w_x}{\partial y} \right)_{y=0} \qquad (11\text{-}31)$$

and

$$\Delta T_j = c_2 \bar{l} \left(\frac{\partial t}{\partial y} \right)_{y=0} \qquad (11\text{-}32)$$

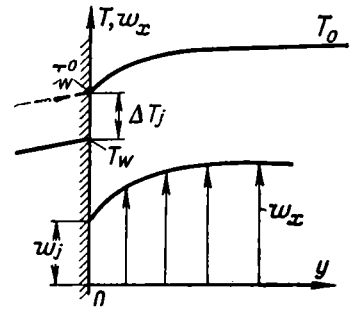

Fig. 11-10. Velocity and temperature jumps at the surface of a body

Here w_j and ΔT_j are respectively the velocity slip and temperature jump on the wall (Fig. 11-10); velocity and temperature are assumed to vary only in the direction of the Oy-axis [11.16]. The coefficients c_1 and c_2 reflect the specific conditions of the interaction between gas and wall. According to the elementary conclusions of kinetics

$$c_1 = \frac{2-f}{f}$$

where f is the momentum exchange factor.

The product $c_1\bar{l}$ is called the *slip factor*; its unit is the unit of length, as follows from equation (11-31).

A number of expressions have been proposed for c_2, in particular

$$c_2 = \frac{2-\gamma}{\gamma} \frac{2k}{k+1} \frac{1}{\mathrm{Pr}}$$

where γ is the accommodation coefficient, and $k = c_p/c_v$ is the ratio of specific heats at constant pressure and volume.

It is very difficult theoretically to determine the exact values of c_1 and c_2, since they depend on f and γ, and these, in turn, depend on many factors. They are therefore determined experimentally.

With $\bar{l} \to 0$ it follows from Eqs. (11-31) and (11-32) that

$$w_j \to 0 \quad \text{and} \quad \Delta T_j \to 0$$

Since $\bar{l}$ may be very small with dense gases, but still greater than zero, the boundary conditions $w_j = 0$ and $\Delta T_j = 0$, previously assumed for a continuum, apparently hold, though only approximately, but with an accuracy sufficient for practical purposes.

The aerodynamic degree of gas rarefaction, considered as the degree of deviation from the state of continuum, is determined, as described above, by the Knudsen number $\mathrm{Kn} \equiv \bar{l}/l_0$, where $\bar{l}$ is the mean free molecular path; and l_0 the reference dimension.

If $\mathrm{Kn} < 0.001$, the gas can be considered a continuous medium (continuum), and the calculation formulae previously derived are applicable.

If $\mathrm{Kn} > 10$, the gas is considered a *free molecular flux*. Flow and heat transfer are then calculated by means of equations from the kinetic theory of gases.

With $\mathrm{Kn} < 0.001$ molecular collisions prevail over collisions with the wall. With free-molecule flow, on the contrary, collisions with the wall predominate.

A rarefied gas can neither be considered an absolutely continuous medium nor a free-molecule flux at Knudsen numbers between 10^{-3} and 10. Two patterns are distinguished for this range of Knudsen numbers: *slip flow* ($10^{-3} < \mathrm{Kn} < 1$) and *transition flow* ($1 < \mathrm{Kn} < 10$).

The change from continuous flow to free-molecule flow is gradual. The boundary Knudsen numbers cited are, therefore, conditional. It is believed, for instance, that free-molecule flow must set in with a convex body at Knudsen numbers smaller than with a similar but concave body.

Free-molecule flow. Heat transfer in a free-molecule flow can be calculated from the kinetic theory of gases.

The data quoted in this section are based on the assumption that incident molecules of the gas do not collide with molecules reflected from the surface. It is therefore assumed that a macroscopic velocity of gas flow is superimposed on a Maxwell velocity distribution of the thermal motion of the gas molecules. The energy of the incident molecules is determined by taking into account both the macroscopic velocity and the velocity of molecular thermal motion. The energy imparted to the wall is determined by the accommodation coefficient (see Eq. 11-28).

The effect of the velocities is accounted for by the parameter $s = w/\bar{w}_m$, where w is the velocity of gas flow; and $\bar{w}_m$ is the maximum probable molecular velocity. The magnitude of $\bar{w}_m$ can be determined as follows:

$$\bar{w}_m = \sqrt{2RT}$$

Then, taking into consideration the fact that sonic velocity $a = \sqrt{kRT}$, we may write:

$$s = \frac{w}{\bar{w}_m} = \frac{w}{\sqrt{2RT}} = \frac{w}{a}\sqrt{\frac{k}{2}} = \mathrm{M}\sqrt{\frac{k}{2}}$$

where $k = c_p/c_v$ is the ratio of specific heats at constant pressure and volume; and M is the Mach number.

From the kinetic theory of gases we obtain

$$\mathrm{St} \equiv \frac{\mathrm{Nu}}{\mathrm{Re\,Pr}} \equiv \frac{\alpha}{\rho w c_p} = \frac{k+1}{k}\gamma F_1(s)$$

The function

$$F_1(s) = \mathrm{St}\,\frac{k\gamma^{-1}}{k+1}$$

is shown graphically in Fig. 11-11.

The curves plotted in the graph are based on theoretical calculations and pertain to cases in which the front parts of various bodies (plates, cylinders, spheres) are exposed to flow, and for the case of a plate in parallel flow.

The rate of heat flow for a rarefied gas is calculated by the generalized formula (11-24)

$$q = \alpha\,(T_{aw} - T_w)$$

For low-velocity flow this formula becomes the ordinary Newton's formula.

According to the kinetic theory of gases, the recovery factor r can be calculated by the equation

$$r = \frac{k}{k+1}F_2(s)$$

The values of the function

$$F_2(s) = \frac{k+1}{k} r$$

are given in the chart in Fig. 11-12, from which it follows that $F_2(s)$ becomes practically constant and equal to 2, as soon as s equals 5. At lower relative velocities s the value of r depends materially on the shape of the body exposed to flow.

Slip flow. The differential equations describing the process of non-isothermal slip flow of a gas with a jump in temperature differ from those previously derived for convection by heat exchange. Problems of slip flow, however, are often

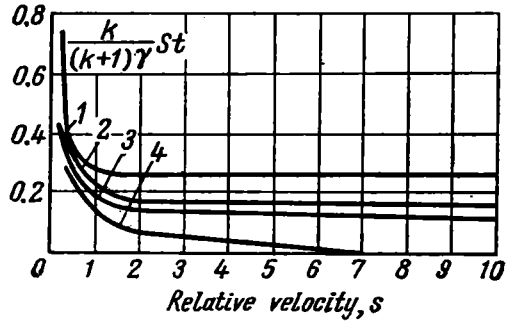

Fig. 11-11. Dependence of $\frac{k}{(k+1)\gamma}$ St on relative velocity s:
1—flat plate normal to flow; 2—cylinder in cross flow (tube;) 3—sphere; 4—flat plate in parallel flow

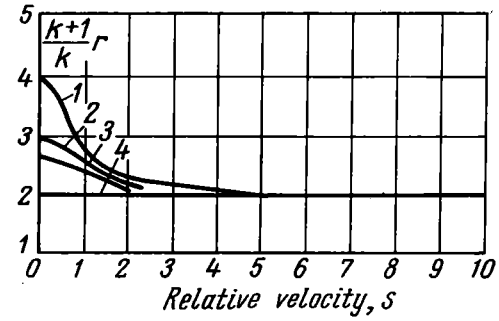

Fig. 11-12. Dependence of $\frac{k+1}{k} r$ on s (notations are the same as those in Fig. 11-11)

currently solved by means of the system of differential equations derived in Chapter 4, temperature jump and velocity slip being taken into consideration. In general, this approach is not a strict one, but the solutions obtained agree fairly well with experimental results.

The heat-transfer coefficient for slip flow can be determined, as a first approximation, by introducing a correction in the value of α obtained for a continuum with the same Reynolds number.

With a non-rarefied gas the rate of heat flow at the wall is

$$q_w = \lambda \left(\frac{\partial t}{\partial y}\right)_{y=0} = \alpha^0 (t_0 - t_w^0) \qquad (11\text{-}33)$$

where the zero superscript denotes the state of continuity (Fig. 11-10).

For a rarefied gas the rate of heat flow is

$$q_w = \lambda \left(\frac{\partial t}{\partial y}\right)_{y=0} = \alpha (t_0 - t_w) \qquad (11\text{-}34)$$

The term $(\partial t/\partial y)_{y=0}$ in this formula is the temperature gradient in the layer of gas directly adjoining the surface of the solid.

Comparison of Eqs. (11-33) and (11-34) shows the temperature gradients of rarefied and non-rarefied flow to be equal at the wall (λ = idem), provided that the rates of heat flow are equal. In the subsequent argument it will be assumed that both rates of heat flow q_w are equal.

From Eqs. (11-33) and (11-34) it follows that

$$\alpha^0 (t_0 - t_w^0) = \alpha (t_0 - t_w)$$

and, since $\Delta T_j = t_w^0 - t_w$, then

$$\frac{\alpha^0}{\alpha} = \frac{t_0 - t_w}{t_0 - t_w^0} \frac{(t_0 - t_w^0) + \Delta T_j}{t_0 - t_w^0} = 1 + \frac{\Delta T_j}{t_0 - t_w^0}$$

Since $\Delta T_j / (t_0 - t_w^0) > 0$, then $\alpha^0 > \alpha$.

Substituting the value of ΔT_j from Eq. (11-32) in this last equation, and assuming that

$$\left(\frac{\partial t}{\partial y}\right)_{y=0} = \frac{\alpha^0}{\lambda}(t_0 - t_c^0)$$

we obtain:

$$\frac{\alpha^0}{\alpha} = 1 + \frac{c_2 \bar{l}}{\vartheta} \alpha^0$$

After simple rearrangements we can obtain:

$$\alpha = \frac{1}{1/\alpha^0 + \delta_s/\lambda} = \frac{1}{R^0 + R_s} \tag{11-35}$$

where $\delta_s = c_2 \bar{l}$, m; $R^0 = 1/\alpha^0$; and $R_s = \delta_s/\lambda$.

The heat-transfer coefficient of a rarefied gas can thus be represented as a value depending on two thermal resistances: the thermal resistance in a dense gas R^0 and the fictitious thermal resistance R_s due to the temperature jump. It is assumed here that the presence of the fictitious thermal resistance R_s does not affect R^0.

Equation (11-35) can be reduced to dimensionless form:

$$\frac{\alpha l_0}{\lambda} = \frac{\alpha^0 l_0/\lambda}{1 + \frac{\alpha^0 l_0}{\lambda} \frac{\delta_s}{l_0}} \quad \text{or} \quad \mathrm{Nu} = \frac{\mathrm{Nu}^0}{1 + \mathrm{Nu}^0 c_2 \mathrm{Kn}} \tag{11-36}$$

The last equation is used to treat the results of experiments on heat transfer in rarefied gases. The magnitude of c_2 is given a more precise value, according to the experimental results.

Many experimental data on heat transfer through various bodies exposed to a slip flow of rarefied gases are available, but detailed consideration of this material is outside the scope of our study.

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3 HEAT TRANSFER WITH PHASE CHANGES AND CHEMICAL TRANSFORMATIONS

CHAPTER

12

HEAT TRANSFER IN CONDENSING PURE VAPOUR

12-1. Fundamentals

Condensation is the process of transition from a vapour (gas) to the liquid or solid state (a change of phase of the first kind).

Condensation is often encountered in practice. In condensers of steam turbines steam condenses on cooled tubes; condensation of steam is utilized in desalination plants, and in many heat exchangers.

Condensation is accompanied by the liberation of heat, owing to the change in phase, and the process is therefore intimately associated with heat transfer.

Condensation is only possible when a gas (vapour) is in a subcritical state; it may be caused by cooling of the gas, or by compressing it to a temperature and pressure at which the condensate phase is thermodynamically more stable than the gaseous. If the temperature and pressure of the compressed gas exceed those corresponding to the triple point of a given substance, a liquid condensed phase forms; but if the pressure and temperature of the compressed gas are lower than the triple point, it assumes the solid state.

Condensation can take place both within the volume of the vapour and on a cooled heat-transfer surface. In the first case the condensed phase may form spontaneously at a considerable subcooling of the vapour in respect to the saturation temperatures and on the cold liquid or solid particles introduced into the vapour.

The processes most commonly encountered in power engineering and in many other branches of engineering and industry are those of vapour condensing to the liquid state on cooled heat-transfer surfaces. This chapter will be devoted mainly to consideration of this problem, assuming that the vapour condensing contains no other vapours or gases, i.e. that it is pure.

Saturated or superheated vapour condenses on a solid heat-transfer surface if the temperature of the latter is lower than the saturation temperature at the pressure given.

The thickness of the forming film of condensate may greatly exceed the distance of effective action of intermolecular forces. The surface of the body may often be covered with individual bubbles of condensate.

The first form of condensation, during which the liquid phase forms on the heat-transfer surface in a continuous stable film is referred to as *film* condensation, and the second kind characterized by the formation of drops as *dropwise*

condensation. Film condensation occurs when the condensate wets the heat-transfer surface. If the condensate fails to wet the surface, condensation is said to be dropwise.

Wettability is usually characterized by the contact angle θ, formed by the surface of the liquid and surface of the solid, which are contiguous to a certain third medium, vapour in the case being considered (Fig. 12-1).

Assuming that only capillary forces are acting, the state of equilibrium is determined by the following equation*

$$\sigma_{wg} = \sigma_{wf} + \sigma_{fg} \cos \theta \qquad (12\text{-}1)$$

If $\sigma_{wg} > \sigma_{wf}$, i.e. if the surface tension arising between the gas and wall exceeds that appearing between the wall and the liquid, the contact angle

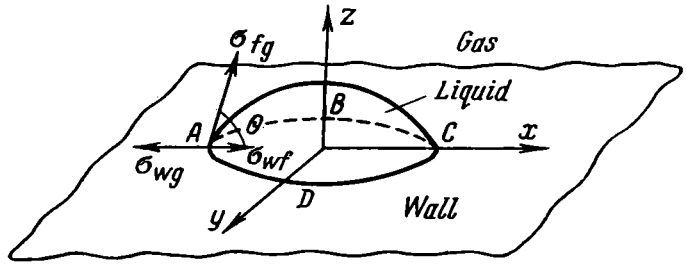

Fig. 12-1. Determining a contact angle

$\theta < 90°$; the liquid is then said to wet the solid surface. With $\sigma_{wg} < \sigma_{wf}$ the contact angle $\theta > 90°$, the liquid does not wet the wall. Absolute wettability and non-wettability are characterized respectively by the contact angles $\theta = 0$ and $\theta = 180°$.

Intermediate cases of partial wetting ($\theta < 90°$) or partial non-wetting ($\theta > 90°$) are most common.

Observations have shown that, with condensers in steady-state operation, the condensate as a rule wets the heat-transfer surface, and film condensation takes place. Dropwise condensation is observed when a heat exchanger is put into service, with its walls covered by various contaminations, including oil films, e.g. with mercury vapour condensation and in some other cases.

Dropwise condensation can be provoked artificially by means of special substances, *lyophobic agents* (*hydrophobic agents* with steam condensation). These substances are applied to the heat-transfer surface or introduced into the vapour.

The intensity of heat transfer with dropwise condensation of steam is many times larger than with film condensation. This is explained by the fact that the film of condensate offers a large thermal resistance to transfer of the heat liberated due to the change of phase, from the liquid surface to the wall. This thermal resistance is considerably less with dropwise condensation, because of the break in the film.

In the general case, in addition to the thermal resistance offered by the condensate, an additional resistance can be singled out.

The condensing surface will be assumed in the further arguing to be flat (or nearly flat), and the thickness of the condensate film much greater than the range of intermolecular action (Fig. 12-2).

* In the general case the contact angle is determined not only by the capillary, but also by the other forces applied to the drop, for instance, by the force of gravity, dynamic head of the moving gas, etc.

Thermal resistance to heat transfer from the vapour to the wall may be represented as the sum of two items:

$$R \equiv \frac{t_v - t_w}{q} \equiv \frac{1}{\alpha} = R_c + R_\varphi \qquad (12\text{-}2)$$

where t_v and t_w are respectively the temperature of vapour and wall; q is the rate of heat flow; α is the coefficient of heat transfer from vapour to wall.

The first term (R_c) of Eq. (12-2) is the *thermal resistance offered by the film of condensate*. The second term (R_φ), which we shall call *thermal resistance to phase transition* (interfacial thermal resistance) is not a thermal one, as usually conceived. The appearance of this resistance is due to the temperature jump on the vapour-liquid interface determined by the resultant velocity of vapour inflow to the condensing surface.

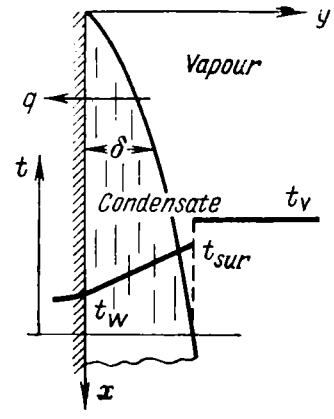

Fig. 12-2. Temperature profile with film condensation of pure vapour

Let us consider the physical nature of the temperature jump. The apparent condensation results from the processes involving condensation of vapour molecules which strike the surface of the liquid and are captured by it (condensing) and evaporation of molecules, separating from the same surface at the same time. Apparent condensation takes place when the number of entrapped molecules exceeds the number of evaporated molecules. Not all the molecules that reach the surface are captured by it. A part of them may reflect from the surface and return into the vapour. The energy of the reflected molecules may generally be less than that of the molecules striking the surface (Sec. 11-5).

As a result two uneven layers of molecules which are at different temperatures (energies) move in the surface layer of vapour of a thickness equal to about the mean free path. Such a layer is referred to as the *Knudsen* layer. The temperature of this layer of vapour differs on the average from the surface temperature of the liquid. The theory of continuous media regards this temperature drop in so thin a layer as a jump. Outside the limits of this layer temperature evens out due to molecular collision.

The more molecules striking the liquid are reflected, failing to condense, the greater the temperature jump. This is accounted for by the *condensation coefficient*, which is the ratio of the number of captured molecules to the total number striking the surface of condensate. In the general case the condensation coefficient may vary from zero to unity.

The flow of vapour passing through the Knudsen layer to the surface of the liquid will be $\rho_v \bar{c}_{yv}$, from the liquid into the vapour—$\rho_{sur} \bar{c}_{ysur}$, where $\bar{c}_y$ is the component of mean molecular velocity normal to the surface of the liquid. The resultant flow, reduced to unit area, is $j = (\rho_v \bar{c}_{yv} - \rho_{sur} \bar{c}_{ysur})$, kg/m^2·s.

It follows from the kinetic theory of gases that $\bar{c}_y = \sqrt{R_v T/2\pi}$. Multiplying the resulting flux j by the condensation coefficient k and substituting $\rho_v \bar{c}_{yv}$

and $\rho_{sur}\bar{c}_{ysur}$, taken by the parameters of vapour and surface of liquid, we obtain Hertz-Knudsen's equation:

$$j = \frac{k}{\sqrt{2\pi R_v}}\left(\frac{p_v}{\sqrt{T_v}} - \frac{p_{sur}}{\sqrt{T_{sur}}}\right) \tag{12-3}$$

where T_v and T_{sur} = respectively vapour and condensate surface temperatures;
p_v, p_{sur} = saturated vapour pressure at temperatures T_v and T_{sur} respectively;
R_v = vapour gas constant;
k = condensation coefficient;
$\rho_v = p_v/R_vT_v$;
$\rho_{sur} = p_{sur}/R_vT_{sur}$.

Formula (12-3) was derived for a comparatively simple molecular model. Further studies have shown that in this formula the condensation coefficient must be replaced with the function $k/(1 - 0.4k)$ [12.1].

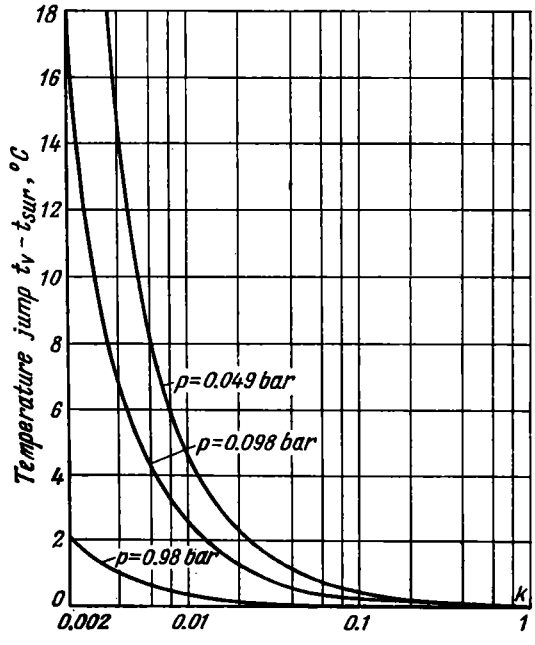

Fig. 12-3. Effect of condensation coefficient and vapour pressure on temperature jump $t - t_{sur}$

The interfacial thermal resistance is determined as

$$R_\varphi = \frac{T_v - T_{sur}}{q} \tag{12-4}$$

Assuming that the heat q transferred through the interface is only the heat due to the change in phase, we can write $q = rj$, where j is determined from Eq. (12-3), and r is the heat of phase transition.

Fig. 12-3 shows the magnitude of the temperature jump $t_v - t_{sur}$ depending on the pressure of the condensing water vapour (steam) and condensation coefficient k at a rate of heat flow $q \approx$ $\approx$ 29 000 W/m² [12.2]. It follows from the curves that at a low condensation coefficient a considerable temperature jump may take place, especially at low pressures. The thermal resistance R_φ then can be compared with the thermal resistance of the condensate film R_c and may even exceed the latter considerably. The temperature jump also increases with a rising rate of heat flow q.

It follows from a number of experimental studies that with pure water vapour, condensing at a pressure $p > 10^4$, Pa, it may be assumed with sufficient approximation that there is no temperature jump at the interface and, that as follows from Eq. (12-2),

$$R \equiv \frac{t_v - t_w}{q} \equiv \frac{1}{\alpha} \approx R_c \tag{12-5}$$

Owing to the insufficiency of data on the condensation coefficient, this last relation is often used to calculate the condensation of other vapours as well.

The thermal resistance of the condensate film R_c depends on the pattern of flow. Across a film of a condensate in laminar flow heat is transferred by conduction, and in turbulent flow by convection as well. The change from laminar flow of the film to turbulent is determined by the Reynolds number of the film:*

$$\mathrm{Re} = \frac{\bar{w}\delta}{\nu}$$

where $\bar{w}$ = mean velocity of film flow in the cross-section;
δ = thickness of the film in the same section;
ν = kinematic viscosity coefficient of the condensate.

The Reynolds number calculated in this way is a local value.

Experimental data obtained by various researchers indicate that the critical Reynolds number may change from around 60 to 500. The following value is considered the most probable for a practically stagnant vapour condensing on a vertical wall:

$$\mathrm{Re}_{cr} = 400$$

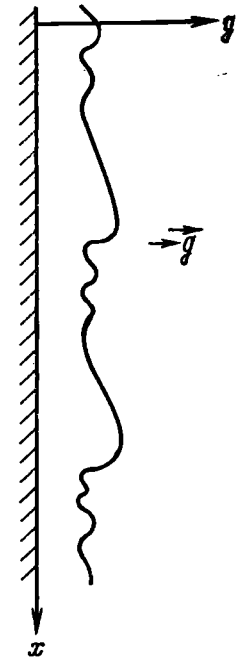

Fig. 12-4. Condensate film in wave flow

Laminar flow of the liqiud film may be accompanied by wave motion (Fig. 12-4). The particles of liquid at the surface of the film may be shifted by the action of accidental perturbations, which cause deformation of the surface and deviation from the state of equilibrium. Forces then appear tending to restore equilibrium. When films flow down, of great importance is the force due to surface tension. Under the action of the restoring forces, the fluid particles tend to return to equilibrium; as, however, they pass through the position of equilibrium by inertia, they are again subjected to the action of the restoring forces, and so on. But this flow is superimposed by the forces of gravity [12.3]. This leads to capillary waves appearing on the surface of the film subjected to accidental disturbance. Waves appearing uniformly at different places owing to accidental disturbance give the process a complex three-dimensional pattern, as they superimpose. The film in laminar flow is unstable in respect to the disturbances of sufficient wavelength ($\gg \delta$). With small Reynolds numbers the disturbances appearing in the layer are carried away with the flow. But if the Reynolds number of the film exceeds some limiting Re_w, steady-state wave flow sets in.

* In the Reynolds number encountered in the literature for reference dimension is sometimes taken the equivalent diameter $d_{eq} = 4\delta$. The Reynolds number of the film is then equal to $\frac{\bar{w}\delta}{\nu}$.

According to [12.4] for a film with periodic rippling and flowing down a vertical surface under gravity

$$\mathrm{Re}_w = 0.56 \left(\frac{\sigma}{\rho_l g^{1/3} \nu^{4/3}} \right)^{3/11} \tag{12-6}$$

From this it follows that the Reynolds number of rippling for water, for example, is $\mathrm{Re}_w \approx 5$ at a temperature $t = 15\,^\circ\mathrm{C}$. This value Re_w agrees well with the experimental results [12.5, 12.6].

The rate of condensate flow is closely linked with heat flow. Condensing dry saturated steam gives up the heat liberated during phase transition r, measured in J/kg. In addition, inasmuch as wall surface temperature is lower than that of the condensate film which is in contact with the vapour phase, a fraction of heat of the condensate is also transferred to the wall. The condensate is subcooled to an average temperature between that of the film (on the side of the vapour) and of the wall.

In many important practical applications the heat of subcooling is negligible compared with that liberated in phase transition.

The heat flux (or the rate of heat flow) then does not change over the thickness of the condensate film. Ignoring the heat of subcooling of the condensate, we can write:

$$Q = rG \tag{12-7}$$

where Q is the heat flow, W; and G is the quantity of condensate formed per unit time, or mass rate of flow, kg/s.

The rate of condensation in any cross-section of a moving film is $G = \rho_l \bar{w} f = \rho_l \bar{w} \delta l_z$, where l_z is the dimension of the wall perpendicular to the plane illustrated in Fig. 12-2. From this it follows that $\bar{w}\delta = G/\rho_l l_z$. Along the section between the points $x = 0$ and x plotted upstream of the considered cross-section, condensate is formed at a rate of G, kg/s and the heat transferred per unit time is $Q = \bar{q}F = \overline{\alpha \Delta t} F \approx rG$. Hence

$$\bar{w}\delta = \frac{G}{\rho_l l_z} = \frac{\bar{q}x}{r\rho_l}$$

and

$$\mathrm{Re} = \frac{\bar{w}\delta}{\nu_l} = \frac{G}{\rho_l \nu_l l_z} = \frac{\bar{q}x}{r\rho_l \nu_l} = \frac{\overline{\alpha \Delta t} x}{\mu_l r} \tag{12-8}$$

Under these conditions the Reynolds number is thus not only the usual hydrodynamic criterion, but also determines the intensity of heat transfer.

Since $Q = \bar{q}F = \bar{q}x l_z = rG = r\rho_l \bar{w}\delta l_z$, then, considering that

$$\bar{q} = \frac{1}{x} \int_0^x q\, dx, \quad \text{and} \quad \bar{w} = \frac{1}{\delta} \int_0^\delta w_x\, dy$$

we can write:

$$\frac{Q}{l_z} = \int_0^x q\, dx, \quad \text{and} \quad \frac{rG}{l_z} = r\rho_l \int_0^\delta w_x\, dy$$

(r and ρ_l being assumed to be constant).

Equating the right sides of the last two equations, and differentiating both sides of the equation obtained for x, we obtain the following equation for the

local rate of heat flow:

$$q = r\rho_l \frac{d}{dx} \int_0^{\delta} w_x \, dy \qquad (12\text{-}7')$$

Equation (12-7') enables us to calculate the density of heat flow, if the velocity distribution in the film is known.

A distinction will be made between the condensation of moving and stagnant vapour. With longitudinal moving of vapour, the forces of friction appearing at the phase interface may either decelerate or accelerate the film of condensate, depending on the respective directions of flow of vapour and condensate.

A condensing vapour cannot be absolutely stagnant, if the density of the liquid phase ρ_l differs from that of the vapour phase ρ_v. At states differing considerably from the critical, $\rho_l \gg \rho_v$. The volumes of the vapour condensing at the wall are at once replenished by new portions of vapour coming from the main mass. Strictly speaking, therefore, the vapour is in constant motion.

When vapour condenses (or liquid evaporates) the vapour is though drawn (or injected) through the interface. If drawn off, the following amount of momentum is transported per unit area in unit time

$$s_j = j_{ysur} w_0 = (\rho_v w_y)_{sur} \, w_0 = 2c_j \frac{\rho_v w_0^2}{2}$$

where $j_{y\,sur} = (\rho_v w_y)_{sur}$ = density of the cross flow of substance at the interface, kg/m^2·s;

w_0 = longitudinal velocity of vapour along the boundary beyond the boundary vapour layer, measured from the velocity of the interface;

$c_j \equiv w_{ysur}/w_0$ = discharge coefficient.

The hydrodynamic drag or water resistance determined by the above formula is sometimes called the *sink resistance.*

Allowing also for the friction due to molecular viscosity, we can write:

$$s_{sur} = s_f + s_j = c_f \frac{\rho_v w_0^2}{2} + 2c_j \frac{\rho_v w_0^2}{2} = c_{fj} \frac{\rho_v w_0^2}{2}$$

where $c_{fj} = c_f + 2c_j$ corresponds to the friction factor accounting both for molecular and convective transport of momentum.

When a flat surface is exposed to an infinite flow and vapour is drawn off uniformly through this surface the inequality $s_j \gg s_f$ is valid provided [12.7]

$$c_j \sqrt{\frac{w_0 x}{\nu}} > 2 \quad \text{or} \quad \mathrm{Re}_x > \frac{4w_0^2}{w_{ysur}^2}$$

If steam film flows along a condensate film at a low velocity the quantity s_{sur} is also small, i.e. the vapour does not exert any appreciable dynamic impact on the condensate film and may be considered stationary, and the term "stationary vapour" will be used below in this sense.

In considering practical problems in heat transfer the condensing vapour will usually be considered as dry and saturated. The condensation of wet and superheated vapour will be treated apart.

12-2. Heat Transfer in Stagnant Vapour Undergoing Film Condensation

(a) *Vertical Wall*

Laminar film flow. Dry saturated vapour condenses on a vertical wall which is at a temperature t_w. The film of condensate is in laminar flow. Let us consider a steady-state problem, assuming the wall to be infinite in the direction of the Oz-axis (Fig. 12-2).

In considering this problem the following assumptions are made:

(a) the forces of inertia appearing in the condensate film are negligible compared with viscosity and gravity;

(b) the heat transferred by convection in the film, and by conduction along it are disregarded, only conduction across the film is taken into account;

(c) there is no friction at the liquid-vapour interface;

(d) the outer surface of the film is at a constant temperature equal to the saturation temperature t_s;

(e) the physical parameters of the condensate are independent of temperature;

(f) surface tension in the free surface of the film does not affect the nature of its flow;

(g) vapour density is small in comparison with that of the condensate.

These assumptions permit an essential simplification of the mathematical description. The left side of the energy equation (4-10) is equal to zero, because the process is steady-state and the transport of heat by convection is not taken into account. The derivatives $\partial^2 t/\partial x^2$ and $\partial^2 t/\partial z^2$ are equal to zero, since heat transfer by conduction along the condensate film is negligibly small and the wall is infinite in the direction of the Oz-axis.

The projections of the flow equations (4-12) and (4-13) on the Oy- and Oz-axes are ignored, as the film moves in the direction of the Ox-axis. The projection of the left side of the energy equation (4-11) on the Ox-axis is equal to zero, since the problem considered is steady-state, and inertia is assumed to be negligible. Since film thickness δ is small $\partial^2 w_x/\partial y^2 \gg \partial^2 w_x/\partial x^2$ and the derivative of $\partial^2 w_x/\partial x^2$ can be ignored (Sec. 4-4). The derivative $\partial^2 w_z/\partial z^2$ also equals zero since the wall is infinite in the direction of the Oz-axis. The derivative $\partial p/\partial x = 0$, because velocity variation along the Ox-axis is determined by change in the hydrostatic pressure of the vapour. Since vapour density is small, compared with the density of the condensate, and the walls are not high the pressure gradient can be ignored.*

As a result we have the following system of equations:

$$\frac{d^2t}{dy^2} = 0$$

and

$$\mu_l \frac{d^2 w_x}{dy^2} = -\rho_l g$$

with boundary conditions:

$$\text{at} \quad y = 0, \quad t = t_w \quad \text{and} \quad w_x = 0$$

$$\text{at} \quad y = \delta, \quad t = t_s \quad \text{and} \quad \frac{\partial w_x}{\partial y} = 0$$

* If vapour density is large, $\rho_l g$ is replaced in the flow equation by $g(\rho_l - \rho_v)$, where the subscripts l and v refer respectively to liquid and vapour. Free convection in the film can usually be ignored.

From the assumption that friction on the phase interface is negligible it follows that $\partial w_x/\partial y = 0$; from $s = \mu (\partial w_x/\partial y)$ it follows that at $dw_x/dy = 0$, $y = \delta$, since $s_{y=\delta} = 0$ and $\mu \neq 0$.

Integration of the energy equation in accordance with the boundary conditions assumed gives (see Sec. 2-1):

$$\frac{dt}{dy} = \frac{t_s - t_w}{\delta}$$

The coefficient of heat transfer

$$\alpha \equiv \frac{q_w}{t_s - t_w} = \frac{\lambda_l \dfrac{dt}{dy}}{t_s - t_w} = \frac{\lambda_l (t_s - t_w)}{\delta (t_s - t_w)} = \frac{\lambda_l}{\delta}$$

Thus, if heat is transferred across the film only by conduction, then

$$\alpha = \frac{\lambda_l}{\delta} \tag{12-9}$$

i.e. at a definite point on the heat-transfer surface the heat-transfer coefficient is directly proportional to thermal conductivity and inversely proportional to the thickness of the film at that point.

The thickness of the film will depend on the quantity of vapour condensed or on the rate of condensate flow in the considered cross-section. The quantity of condensate flowing per unit time through a cross-section of the film whose thickness is unity is expressed by the following formula:

$$G = \rho_l \bar{w} \delta \cdot 1$$

The quantity of liquid flowing through a cross-section at a distance dx downstream of the considered cross-section exceeds G by

$$dG = d(\rho_l \bar{w} \delta) \tag{a}$$

This increment in flow is due to condensation. Assuming that the heat imparted to the wall is that of phase transition, i.e. ignoring the subcooling of the condensate, we have

$$dG = \frac{q_w}{r} dx \cdot 1$$

where q_w/r is the quantity of condensate formed on a unit area of the heat-transfer surface per unit time.

On the other hand,

$$q_w = \lambda_l \frac{dt}{dy} = \frac{\lambda_l}{\delta} (t_s - t_w)$$

and

$$dG = \frac{\lambda_l}{\delta} \frac{1}{r} (t_s - t_w) \tag{b}$$

Equating the right sides of (a) and (b), we obtain:

$$d(\rho_l \bar{w} \delta) = \frac{\lambda_l}{\delta} \frac{1}{r} (t_s - t_w)\, dx \tag{c}$$

In order to integrate Eq. (c), we must first determine the mean velocity $\bar{w}$.

In the equation of flow $d^2 w_x/dy^2 = -\rho_l g/\mu_l$ the dimensionless term $\rho_l g/\mu_l$ is constant in accordance with the conditions assumed.

Integration then yields:

$$\frac{dw_x}{dy} = -\frac{\rho_l g}{\mu_l} y + c_1, \quad \text{and}$$

$$w_x = -\frac{\rho_l g}{2\mu_l} y_0 + c_1 y + c_2$$

Using the boundary conditions, we obtain:

$$c_1 = \frac{\rho_l g}{\mu_l} \delta, \quad \text{and} \quad c_2 = 0$$

The velocity distribution will then be expressed by the formula:

$$w_x = \frac{\rho_l g}{\mu_l} \delta y - \frac{\rho_l g}{2\mu_l} y^2 \tag{12-10}$$

The mean flow velocity $\overline{w}$ at x is:

$$\overline{w} = \frac{1}{\delta} \int_0^\delta w_x \, dy = \frac{1}{\delta} \int_0^\delta \frac{\rho_l g}{\mu_l} \delta y \, dy - \frac{1}{\delta} \int_0^\delta \frac{\rho_l g}{2\mu_l} y^2 dy = \frac{\rho_l g}{3\mu_l} \delta^2$$

Substituting the value of $\overline{w}$ obtained in (c), we get:

$$d\left(\frac{\rho_l^2 g \delta^3}{3\mu_l}\right) = \frac{\lambda_l}{\delta} \frac{1}{r} (t_s - t_w) \, dx$$

or

$$\frac{r\rho_l^2 g}{\lambda_l \mu_l} \delta^3 \, d\delta = (t_s - t_w) \, dx$$

Integrating the last equation, we obtain

$$\frac{r\rho_l^2 g}{4\lambda_l \mu_l} \delta^4 = (t_s - t_w) \, x + c$$

From the condition that $\delta = 0$ at $x = 0$, it follows that $c = 0$. Solving the last equation for δ, and taking the magnitude of the integration constant into account, we get:

$$\delta = \sqrt[4]{\frac{4\lambda_l \mu_l (t_s - t_w) \, x}{r\rho_l^2 g}} \tag{12-11}$$

Substituting the value of δ in the expression $\alpha = \lambda_l / \delta$, we obtain the following equation for the local heat-transfer coefficient:

$$\alpha = \sqrt[4]{\frac{r\rho_l^2 g \lambda_l^3}{4\mu_l (t_s - t_w) \, x}} \tag{12-12}$$

In our problem the temperature difference $\Delta t = t_s - t_w$ does not change along the x-axis. The methods previously used to average the heat-transfer coefficients (6-21) and (6-22) yield similar results here. We find that $\overline{\alpha} = 4/3 \alpha_{x=h}$, and, consequently,

$$\overline{\alpha} = \frac{4}{3} \sqrt[4]{\frac{r\rho_l^2 g \lambda_l^3}{4\mu_l (t_s - t_w) \, h}} = 0.943 \sqrt[4]{\frac{r\rho_l^2 g \lambda_l^3}{\mu_l \Delta t h}} \tag{12-13}$$

Eq. (12-13) was first derived by Nusselt in 1916 [12.8]. Variation of the surface temperature of a vertical wall does not effect the mean rate of heat transfer, if the heat-transfer coefficient is calculated by formula $\overline{\alpha} = \overline{q}/\Delta \overline{t}$ [12.9].

From Eqs. (12-11), (12-12), and (12-13) it follows that $\delta = c' \sqrt[4]{x}$, $\alpha = c'' \sqrt[4]{1/x}$, and $\bar{\alpha} = c''' \sqrt[4]{1/h}$.

This dependence is shown graphically in Fig. 12-5. The heat-transfer coefficients diminish with rising x or h, but the thickness of the film, on the contrary, increases. This increase explains the decrease in the heat-transfer coefficients, since the thermal resistance of the film increases with thickness.

From Eqs. (12-11), (12-12) and (12-13) it follows that α, $\bar{\alpha}$, and δ depend on the temperature difference Δt in the same way as they depend on x and h.

The fact, however, that α diminishes with an increase in Δt does not mean that an increase in Δt is accompanied by reduction in the rate of heat flow q_w. According to Newton's law, $q_w = \alpha \Delta t$, and at the same time $\alpha = c\Delta t^{-1/4}$. Thus

$$q_w = c\Delta t^{3/4}$$

The Nusselt solution must be regarded approximate, owing to the assumptions admitted in stating the problem.

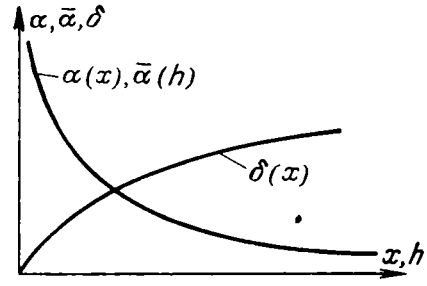

Fig. 12-5. Variation of heat-transfer coefficient and of film thickness for condensate in laminar flow along a vertical wall

The solutions obtained by Kruzhilin and Labuntsov [12.10, 12.11], who took into account inertia and convection heat transport in the film, show that with $K \equiv \frac{r}{c_{pl}\Delta t} > 5$ and $1 < \mathrm{Pr} < 100$ there is sufficient agreement between more accurate solutions and the Nusselt solution. The difference in the calculated heat-transfer coefficients is only several per cent, and may be ignored in practical calculations.

The heat-transfer coefficient increases considerably, compared with α calculated with the aid of Nusselt's equation, only at great temperature difference or in the near-critical region where r/c_{pf} diminishes sharply.

There is also a difference in the values obtained for α with small Prandtl numbers (liquid metals). The rate of heat transfer is then smaller, over a wide range of temperature differences, than the rate of heat flow calculated by the Nusselt formulae (see Sec. 12-6).

The Nusselt solution does not take into account variation in the physical properties of the condensate. According to D.A. Labuntsov [12.12] the dependence of thermal conductivity λ and viscosity μ on temperature is determined by multiplying the right side of Eq. (12-12) or Eq. (12-13) by $\varepsilon_t = [(\lambda_w/\lambda_s)^3 (\mu_s/\mu_w)]^{1/8}$, where the subscripts w and s indicate that the given thermal conductivities must be chosen respectively for the temperature of the wall and saturation temperature. The physical parameters in Eqs. (12-12) and (12-13) must be selected for the saturation temperature.

According to the experimental data made available by N.V. Zozulya et al. [12.13] the correction ε_t accounts sufficiently well for effect of the variation of the physical properties of condensate on heat transfer; numerically, it is close to the correction $(\mathrm{Pr}_s/\mathrm{Pr}_w)^{0.25}$. In deriving Eqs. (12-12) and (12-13) the effect of the wave motion of the film was ignored. P.L. Kapitsa [12.14] assumed the wave motion to be periodic and steady, described for any cross section x by the sinusoidal distribution of film thickness in time. He found that with wave motion the effective layer thickness δ_{ef} which should be introduced into Eq. (12-9) is smaller than the thickness δ calculated from Nusselt's equation. With a small condensate film thickness the thermal resistance of condensate R_c drops at a

rate greater than that at which R_c increases with a greater film thickness, then the mean heat-transfer coefficient increases by 21 per cent compared with α calculated by formula (12-13).

The condensate film is in intricate wave motion depending on the Reynolds number and other factors [12.3]. Labuntsov [12.12] has shown that the correc-

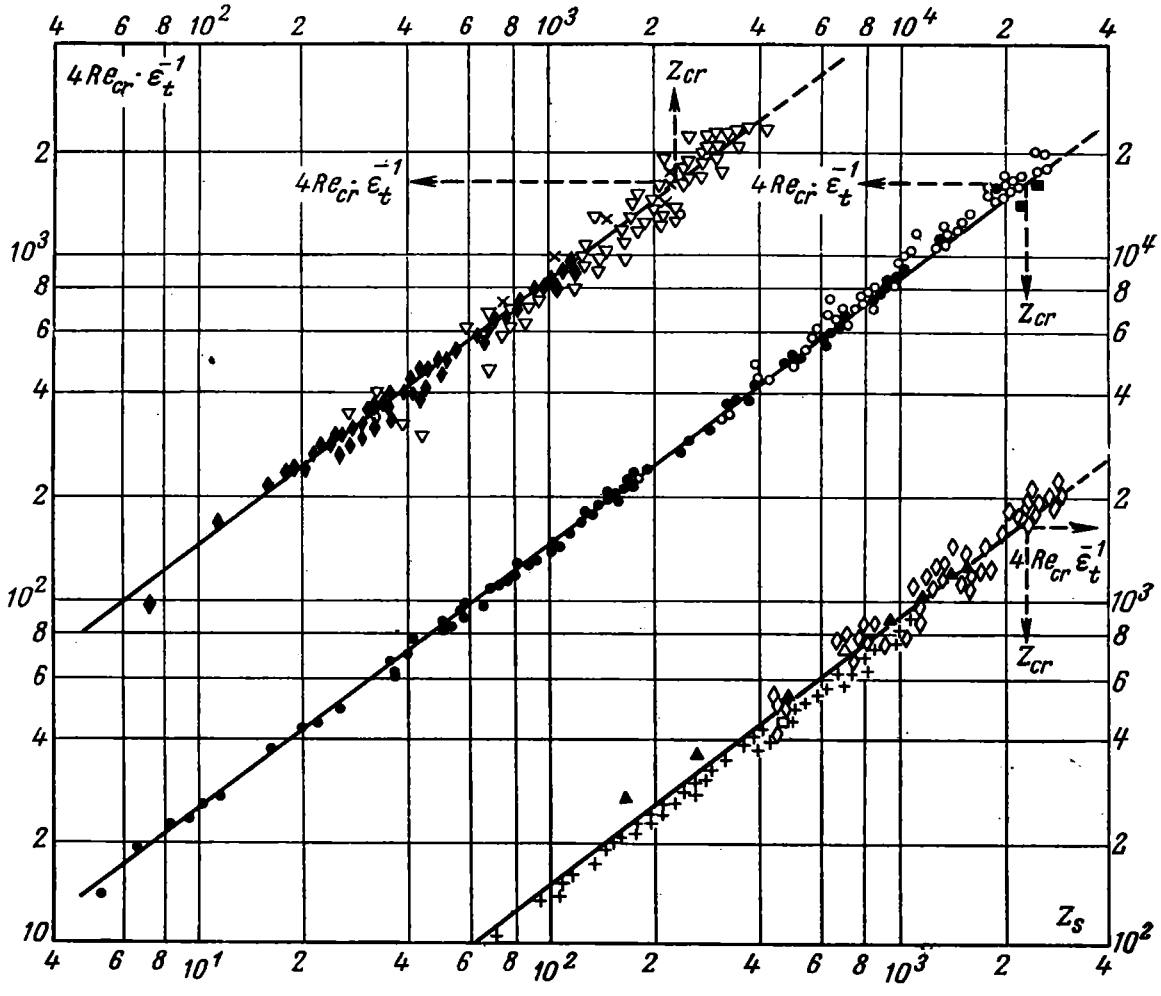

Fig. 12-6. Heat-transfer data for film condensation of stationary vapour on a vertical wall with laminar flow of the film

tion accounting for rippling can then be regarded with sufficient accuracy to be a function of only the Reynolds number:

$$\varepsilon_v = \mathrm{Re}^{0.04}$$

At small values of Re the correction ε_v is close to unity; it increases with increasing Re and reaches a value of 1.27 at Re = 400. The Reynolds number pertains to the lower section of the condensate film in the direction of flow.

For a practically stagnant pure vapour condensing on vertical surfaces the mean heat-transfer coefficients can thus be calculated by the following formula:

$$\bar{\alpha} = \bar{\alpha}_N \varepsilon_t \varepsilon_v \qquad (12\text{-}14)$$

where $\bar{\alpha}_N$ = heat-transfer coefficient calculated by the Nusselt formula (12-13) choosing all physical parameters for the saturation temperature.

The Nusselt equation (12-13), and Eq. (12-14), can be presented in dimensionless form. Having chosen $\mathrm{Re} = \frac{\bar{\alpha}\Delta t h}{r\mu_l}$ for the defined term we obtain:

$$\mathrm{Re} = 0.943 Z^{3/4}$$

where

$$Z = \mathrm{Ga}^{1/3}\frac{\lambda_l \Delta t}{r\mu_l}, \quad \mathrm{Ga} = \frac{gh^3}{\nu_l^2}$$

and the subscript l indicates that the physical parameters in the dimensionless term are those of the condensate.

Reducing Eq. (12-14) to dimensionless form, assuming that $\varepsilon_t^{1.04} \approx \varepsilon_t$ and taking into account that $\varepsilon_v = \mathrm{Re}^{0.04}$, we obtain [12.12]:

$$\mathrm{Re}_s = 0.95 Z_s^{0.78}\varepsilon_t \qquad (12\text{-}15)$$

The subscript s indicates that the physical parameters of the condensate in the dimensionless terms Re and Z are chosen by the saturation temperature. The correction ε_v is implicit in Eq. (12-15). Formula (12-15) is correlated with experimental data in Fig. 12-6.

If the rate of heat flow $\bar{q}_w$ is given, formula (12-14) can be presented in a more convenient form. Assuming $\varepsilon_v = 1.21$, equations $\alpha = \lambda_l/\delta$ and $\bar{w} = g\delta^2/3\nu_l$ yield:

$$\left[\frac{\alpha}{\lambda_l}\left(\frac{\nu_l^2}{g}\right)^{1/3}\right] = 0.85\mathrm{Re}_s^{-1/3}\varepsilon_t \qquad (12\text{-}16)$$

where $\mathrm{Re}_s = \bar{q}_w x/r\mu_l$. Here the mean rate of heat flow $\bar{q}_w$ depends on x.

Turbulent film flow. Flow of the condensate film becomes turbulent at $\mathrm{Re} \geqslant 400$, but in the upper part of the film where $\mathrm{Re} < 400$ it remains laminar. The condensate will be in a combined pattern of flow on the wall.

Ignoring subcooling heat, dissipation of mechanical energy and in the absence of inner heat sources, heat transfer through the film of condensate takes place at a constant rate of flow across the film, i.e.

$$q = q_w = \mathrm{const} \neq f(y) \quad \text{or} \quad \partial q/\partial y = 0 \qquad \text{(a)}$$

where y is the coordinate perpendicular to the wall (see Fig. 12-2). This distinguishes, for instance, the considered case of heat transfer for a homogeneous fluid flowing in tubes previously, where the rate of heat flow q changes from maximum at the wall to zero at the tube axis (with boundary conditions being constant over the cross-section of flow).

The local heat-transfer coefficient for film condensation can be found from (a) and

$$q = (\lambda_l - \rho_l w_{pl} c_q)\frac{dt}{dy} = \lambda_l\left(1 + \frac{\mathrm{Pr}}{\mathrm{Pr}_t}\frac{\varepsilon_s}{\nu_l}\right)\frac{dt}{dy}$$

$$\int_{t_w}^{t_s} dt = t_s - t_w = q\int_0^{\delta}\frac{dy}{\lambda_l\left(1+\frac{\mathrm{Pr}}{\mathrm{Pr}_t}\frac{\varepsilon_s}{\nu_l}\right)}$$

Taking into account that $\alpha = q_w (t_s - t_w)$, we then get

$$\alpha = \frac{1}{\int_0^{\delta} \frac{dy}{\lambda_l \left(1 + \frac{\Pr}{\Pr_t} \frac{\varepsilon}{\nu_l}\right)}} \qquad (12\text{-}17)$$

Thus determination of the heat-transfer coefficient consists in finding the value of the integral found in the denominator of equation (12-17). Labuntsov used the equations of the turbulent diffusivity of momentum ε_s, suggested by Lin and Schlinger. The physical parameters of the condensate were also assumed to be constant, and the integration results were approximated within the range of $1 \leqslant \Pr_l \leqslant 25$ and $1.5 \times 10^3 \leqslant \mathrm{Re} \leqslant 6.9 \times 10^4$ by the following equation:

$$\frac{\alpha}{\lambda_l} \left(\frac{\nu_l^2}{g}\right)^{1/3} = 0.0325 \mathrm{Re}^{0.25} \Pr^{0.5} \qquad (12\text{-}18)$$

where $\frac{\alpha}{\lambda_l} \left(\frac{\nu_l^2}{g}\right)^{1/3} = \frac{\mathrm{Nu}}{\mathrm{Ga}^{1/3}}$, α is the local heat-transfer coefficient.

In calculating mean heat transfer for turbulent flow, it is necessary to take account of the fact that a certain layer of condensate has already formed on the upper part of the wall, within the section of laminar flow.

With constant physical parameters heat transfer in the laminar section is described by Eq. (12-15) with $\varepsilon_t = 1$:

$$\mathrm{Re} = 0.95 Z^{0.78}$$

With a critical Reynolds number $\mathrm{Re}_{cr} = 400$ we have

$$Z_{cr} = \left(\mathrm{Ga}^{1/3} \frac{\lambda_l \Delta t}{r \mu_l}\right)_{cr} = 2300$$

Having written formula (12-7′) in dimensionless form as

$$\frac{\alpha}{\lambda_l} \left(\frac{\nu_l^2}{g}\right)^{1/3} = \frac{d\mathrm{Re}}{dZ}$$

and taking account of Eq. (12-18), we obtain:

$$\frac{d\mathrm{Re}}{dZ} = 0.0325 \mathrm{Re}^{0.25} \Pr^{0.5}$$

Separating the variables Re and Z in the last equation (assuming Pr = const), and integrating from Z_{cr} to Z, and correspondingly from Re_{cr} to Re, we obtain:

$$\frac{4}{3} (\mathrm{Re}^{3/4} - \mathrm{Re}_{cr}^{3/4}) = 0.0325 \Pr^{0.5} (Z - Z_{cr})$$

Solving this for Re, and assuming $\mathrm{Re}_{cr} = 400$, $Z_{cr} = 2300$, we obtain the following equation:

$$\mathrm{Re} = [89 + 0.024 \Pr^{0.5} (Z - Z_{cr})]^{4/3} \qquad (12\text{-}19)$$

Equation (12-19) describes the mean heat transfer for a vertical wall on which there are both laminar and turbulent sections of condensate flow.

To account for their dependence on temperature in the turbulent region of flow, the coefficient 0.0325 found in Eq. (12-18), or the coefficient 0.024 in Eq. (12-19), must be multiplied by the correction $(\Pr_s/\Pr_w)^{0.25}$, where the subscripts s and w indicate that the Prandtl numbers of the condensate are selected for the temperature of the wall and the saturation temperature. All the other physical

parameters in Eqs. (12-18) and (12-19) must be chosen for the saturation temperature.

Taking these remarks into consideration, the calculation formula for the mean heat-transfer coefficient for pure stagnant vapour condensing on a vertical surface in conditions of mixed film flow acquires the following form [12.12]:

$$\mathrm{Re}_s = [89 + 0.024\,(\mathrm{Pr}_s/\mathrm{Pr}_w)^{0.25}\,\mathrm{Pr}_s^{0.5}\,(Z - 2300)]^{4/3} \qquad (12\text{-}20)$$

Where the rate of heat flow $\bar{q}_w$ is given, it is more convenient, taking into consideration that $\mathrm{Re} = \bar{q}_w h/r\mu_l$, to use Eq. (12-20) rearranged as follows:

$$\left[\frac{\bar{\alpha}}{\lambda_l}\left(\frac{\nu_l^2}{g}\right)^{1/3}\right]_s = \frac{\mathrm{Re}_s}{2300 + 41\mathrm{Pr}_s^{-0.5}\,(\mathrm{Re}_s^{3/4} - 89)\,(\mathrm{Pr}_s/\mathrm{Pr}_w)^{0.25}} \qquad (12\text{-}21)$$

In Eq. (12-21) the reference temperature is the saturation temperature t_s (except for Pr_w), and the reference dimension is the height of the wall h. All physical parameters are those of the condensate.

Where the vapour is at a high pressure and its density is commensurable with that of the condensate the term $\rho_l g$ of the energy equation is replaced by $g\,(\rho_l - \rho_v)$.

The replacement results in the Galilei number $\mathrm{Ga} = \frac{gh^3}{\nu_l^2}$ present in the equations for turbulent and laminar film flow being replaced by the Archimedean number $\mathrm{Ar} \equiv \frac{gh^3}{\nu_l^2}\left(1 - \frac{\rho_v}{\rho_l}\right)$.

The term $\frac{\alpha}{\lambda_l}\left(\frac{\nu_l^2}{g}\right)^{1/3}$ can be written down in this case in the following form:

$$\frac{\alpha}{\lambda_l}\left[\frac{\nu_l^2}{g\left(1 - \frac{\rho_v}{\rho_l}\right)}\right]^{1/3}$$

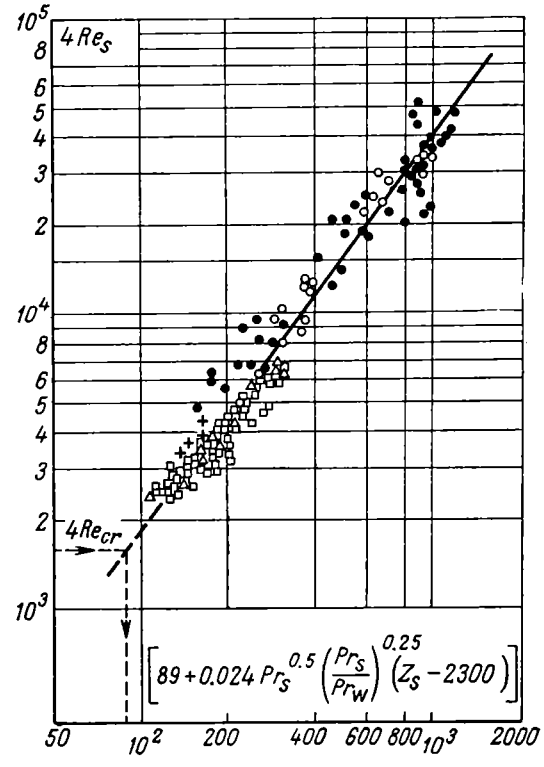

Fig. 12-7. Heat-transfer data for film condensation of stationary vapour on a vertical wall with mixed (laminar and turbulent) flow of the tube:
+—water, $h = 3.66$ m [Meissenburg]; □—water, $h = 6.1$ m [Stroebe]; ○—water, $h = 2.9$ m [Salikov]; Δ—water, $h = 3.66$ m [Hubbard]; ●—diphenyl [Badger]

and the term $Z = \mathrm{Ga}^{1/3}\,(\lambda_l \Delta t/r\mu_l)$ will be replaced by

$$Z = \mathrm{Ar}^{1/3}\,\frac{\lambda_l \Delta t}{r\mu_l} = \left[\frac{gh^3}{\nu_l^2}\left(1 - \frac{\rho_v}{\rho_l}\right)\right]^{1/3}\frac{\lambda_l \Delta t}{r\mu_l} \qquad (12\text{-}22)$$

As before, the subscripts l and v indicate that the value refers to the liquid and vapour phases.

Formula (12-20) is correlated with experimental data in Fig. 12-7.

The chart in Fig. 12-8, based on Eqs. (12-14) and (12-19), can be used to determine the mean heat-transfer coefficient $\bar{\alpha}$ for steam condensing on vertical

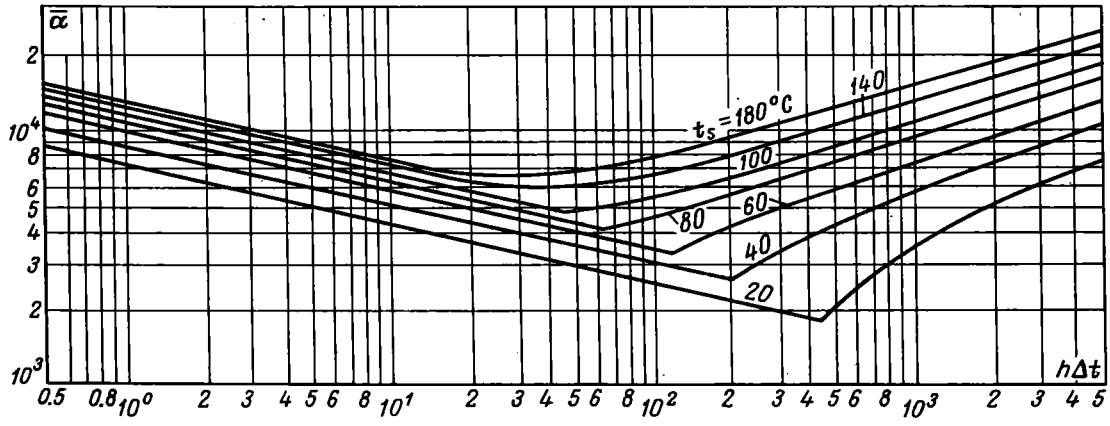

Fig. 12-8. Calculation graph for the mean heat-transfer coefficient of stationary vapour film condensing on vertical surfaces:
$\bar{\alpha}$—kcal/m²·h·K; h—m; Δt—°C; 1 kcal/m²·h·K = 1.16 W/m²·K

surfaces. It makes it possible to find $\bar{\alpha}$, when h, the height of the heat transfer surface, $\Delta t = t_s - t_w$ the temperature difference, and the saturation temperature are known.

(*b*) *Horizontal Tube*

The formulae derived in the preceding section hold for cases where vapour condenses on vertical flat walls. With an inclined wall the gravity acceleration g present in the basic equation must be replaced by a projection of the gravity acceleration vector on the Ox-axis

$$g_x = g \cos \varphi$$

Here φ is the angle formed between the direction of gravity and the Ox-axis which runs in the direction of film flow.

This yields the following formula for inclined walls

$$\alpha_{inc} = \alpha_{ver} \sqrt[4]{\cos \varphi} \tag{12-23}$$

The angle φ is variable for curvilinear surfaces, and for a horizontal cylinder in particular. Allowing for the variable angle, and assuming that $\delta \ll d$, where d is the diameter of a cylinder, Nusselt obtained the following formula for the circumferential mean heat-transfer coefficient of a tube with the condensate film in laminar flow:

$$\bar{\alpha} = 0.728 \sqrt[4]{\frac{\lambda_l^3 \rho_l^2 g r}{\mu_l (t_s - t_w) d}} \tag{12-24}$$

Formula (12-24) differs from (12-13), derived for vertical walls, in that its proportionality factor is equal to 0.728 instead of 0.943; and the reference diameter d replaces h. Equation (12-24) was obtained with the same simplifications assumed when deriving formula (12-13).

As stated above, inertia and convection can be neglected at $K > 5$ and $Pr > 1$. Variation of the physical parameters of the condensate can be accounted for by

the factor ε_t previously introduced; the physical parameters in formula (12-24), which can be found from handbooks, are based on saturation temperature.

The development of rippling requires a certain section of flow several wavelengths long; hence there is no time for rippling to develop on tubes of small diameter. A correction is introduced only when the tube diameter satisfies the following approximate relationship

$$d > 20\,(\delta/\rho_l g)^{0.5} \tag{12-25}$$

Equation (12-24) has been derived, assuming $t_w = \text{const}$. At $q_w = \text{const}$ the proportionality factor equal to 0.728 must be replaced by 0.693 [12.9].

12-3. Heat Transfer Upon Film Condensation of Vapour Flowing Inside Tubes

Inside tubes the volume of condensing vapour is limited by the walls. The tubes may be of considerable length, and a large quantity of vapour can therefore condense in them. Vapour flow becomes oriented, and its velocity can be very high (up to 100 m/s and higher). The friction at the vapour-condensate interface can therefore be considerable.

If the direction of vapour flow coincides with that of the condensate due to gravity, friction leads to acceleration of the condensate flow, thinning of the film and increase of the heat-transfer coefficient. If vapour flows in a direction opposed to that of the condensate, the film may be decelerated, its thickness increased, and the rate of heat transfer reduces. An increase in vapour velocity can bring about entrainment of the film by vapour and its partial separation from the wall, which leads to an increase in heat transfer.

Depending on the magnitude of the forces of gravity and friction, three basic cases can be distinguished:

(a) gravity prevails over the dynamic effect of the vapour, which can for practical purposes be considered stagnant (problems of this kind were considered in the preceding section);

(b) gravity and the dynamic effect of the vapour are commensurable;

(c) the dynamic effect of the vapour prevails over gravity; condensate is entrained by the vapour, and heat transfer is practically independent of the position of the tube.

There is, of course, no drastic difference between the three regimes of vapour flow in tubes.

Vapour velocity does not remain constant because the vapour fraction decreases along the tube in the direction of flow while the condensate fraction increases, and velocity is largest at the inlet. Calculation of the mean inlet velocity of the vapour is quite simple for dry saturated steam which condenses completely in the tube.

As follows from the relationship

$$Q = \bar{q}_w \pi\, dl = rG_v = r\rho_v \bar{w}_{v0} \frac{\pi d^2}{4} \tag{12-26}$$

the heat of condensate subcooling being ignored, the mean inlet vapour velocity $\bar{w}_{v0}$ is:

$$\bar{w}_{v0} = \frac{4G_v}{\rho_v \pi d^2} = \frac{4Q}{r\rho_v \pi d^2} \tag{12-27}$$

where d and l are respectively the inside diameter and length of the tube.

As vapour condenses, a fraction of the tube cross-section becomes filled with condensate, the mean velocities of vapour and liquid not being equal.

It is convenient for calculations to take constant and given magnitudes of velocity. For this we can use the so-called *circulation velocity*, or *rate of circulation*, w_c, determined from the following relationship:

$$w_c = \frac{G_{mix}}{\rho_l f} = \frac{4G_{mix}}{\rho_l \pi d^2} \tag{12-28}$$

where $G_{mix} = G_v + G_l$ is the overall mass rate of vapour and condensate flow.

The velocity w_c is a fictitious quantity, which equals the real velocity of flow only in the cross-section where condensate fills the tube completely, i.e. where $G_{mix} = G_l$ and $G_v = 0$.

Complete or partial condensation of vapour in a tube is possible, depending on the conditions. With complete condensation the vapour outlet velocity is zero and there is no secondary vaporization. If the tube is long enough, and condensation is intensive, the end section of the tube may become completely filled with condensate.

The flow of condensate and vapour can be both laminar and turbulent. Inlet flow of vapour can be turbulent. As vapour condenses, its velocity decreases and turbulent flow can change into laminar. If complete condensation takes place, the axial vapour velocity will be zero at the end of the section. The rate of condensate flow across the tube increases simultaneously, and laminar flow of condensate can change to turbulent flow. Separation of drops from the film surface can also take place under certain conditions.

The combination of these conditions makes the problem of heat transfer in vapour condensing in tubes very complex and complicates a strict and full solution. There are still many unanswered questions in this field; in particular the boundaries of the different patterns are not clear. Many of the original theses used in theoretical solutions lack adequate experimental corroboration. For that reason the concrete data on the process and the design recommendations offered below are rather limited.

Condensate film in laminar flow. Figure 12-9 presents the results of an experimental study of heat transfer upon condensation of dry saturated water vapour

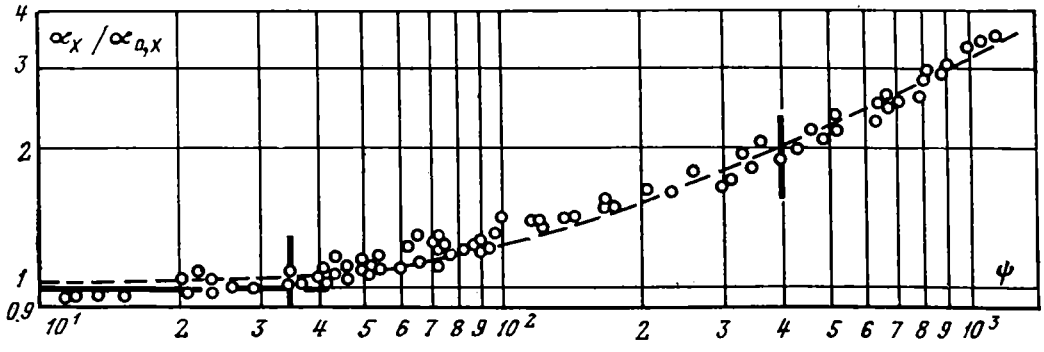

Fig. 12-9. Heat-transfer data for steam condensing in a vertical tube; condensate film in laminar flow

(steam) in a vertical tube [12.15]. The vapour enters the tube at the top. In the graph shown in Fig. 12-9 the relative coefficients of heat transfer α_x/α_{0x} are

plotted as a function of

$$\psi \equiv \frac{\rho_v}{\rho_f}\left(\frac{\nu_v}{\nu_f}\right)^2 \frac{\mathrm{Re}_{vd}^2}{\mathrm{Ga}_{fd}^{2/3}}\,\mathrm{Re}_{fx}^{-0.28}\varepsilon_t$$

where $\mathrm{Re}_{vd} = \overline{w}_v d/\nu_v$;
$\mathrm{Ga}_{fd} = gd^3/\nu_f^2$;
$\mathrm{Re}_{fx} = q_w x/r\mu_f$;
$\overline{w}_v$ = the mean vapour velocity in cross-section x.

The physical parameters of vapour and condensate are denoted respectively by the subscripts v and f and were selected by the saturation temperature.

The heat-transfer coefficients α_x are the average over comparatively short sections of the tube which permits as a first approximation to consider them as local heat-transfer coefficients. The values of α_{0x} were calculated by formulae derived for practically stagnant (stationary) vapour condensing on a vertical wall.

The relative heat-transfer coefficient can be calculated by formula

$$\frac{\alpha_x}{\alpha_{0x}} = \sqrt{0.005\psi + \sqrt{(0.005\psi)^2 + 1}} \qquad (12\text{-}29)$$

The form of Eq. (12-29) was determined theoretically, making some simplifying assumptions, and the constants were checked by the data of tests conducted with water vapour at about atmospheric pressure. The Reynolds number Re_{vd} was changed from 1800 to 17×10^3 with a corresponding change in vapour velocity from 3.6 to 33.5 m/s. The temperature difference varied from 8 to 60 K.

With $\psi \leqslant 35$ the calculation formulae for stationary vapour are suitable.

Condensate in turbulent flow. Heat transfer in a film of condensate in turbulent flow has been studied [12.16-12.18]. Heat transfer was investigated theoretically and experimentally [12.16] in the presence of a predominant effect of vapour friction. Theoretical study was based on the analogy between heat transfer and friction resistance and gave the following semi-empirical formula describing the local heat-transfer coefficients:

$$\mathrm{Nu}_{ds} = c\mathrm{Re}_{cds}^{0.8}\mathrm{Pr}_{ls}^{0.43}\sqrt{1 + x\left(\frac{\rho_l}{\rho_v} - 1\right)_s} \qquad (12\text{-}30)$$

where x is the mass vapour fraction in the considered section.

The mean heat-transfer coefficient for incomplete vapour condensation from a water-vapour mixture is described by formula:

$$\overline{\mathrm{Nu}}_{ds} = c\mathrm{Re}_{cds}^{0.8}\mathrm{Pr}_{ls}^{0.43}\frac{1}{2}\left[\sqrt{1 + x_1(\rho_l/\rho_v - 1)_s} + \sqrt{1 + x_2\left(\frac{\rho_l}{\rho_v} - 1\right)_s}\right] \qquad (12\text{-}31)$$

where x_1 and x_2 are the mass vapour fractions in the inlet and outlet cross-sections of the tube section: $x_1 = (G_v/G_{mix})_{in}$ and $x_2 = (G_v/G_{mix})_{out}$. For steel tubes $c = 0.024$, for copper $c = 0.032$.

In formulae (12-30) and (21-31)

$$\overline{\mathrm{Nu}}_{ds} = \frac{\overline{\alpha} d}{\lambda_l}, \quad \mathrm{Nu} = \frac{\alpha d}{\lambda_l}$$

$$\mathrm{Re}_{cds} = \frac{w_c d}{\nu_l} = \frac{4G_{mix}}{\pi d \mu_l}$$

All physical parameters are taken for the saturation temperature. The subscripts l and v indicate, as before, that the given quantity is a physical parameter respectively of liquid and vapour.

Equations (12-30) and (12-31) are derived for the general case in which $1 \geqslant x_1 > 0$ and $1 > x_2 \geqslant 0$. It is possible to single out particular individual regimes, depending on the magnitude of x_1 and x_2: (a) $x_1 = 1$, $x_2 = 0$ for complete condensation of dry vapour in a tube; (b) $x_1 = 1$, $1 > x_2 > 0$ for incomplete condensation of vapour; (c) $1 > x_1 > 0$, $x_2 = 0$ for complete condensation of vapour from a water-vapour mixture flowing into a tube. At $x_1 = x_2 = 0$ formulae (12-30) and (12-31) acquire the structure of equations used to calculate processes of heating and cooling in single-phase liquids.

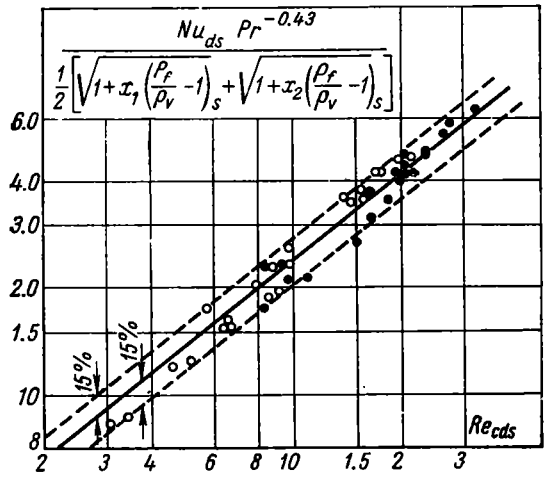

Fig. 12-10. Heat transfer data for steam condensing in a tube with the forming condensate in turbulent flow
●—$d = 16$ mm; ○—$d = 20$ mm

Formula (12-31) is correlated in Fig. 12-10 with test data [12.16] obtained for $Re_{cds} > 5 \times 10^3$, $Pr_{ls} \approx 1$, $1 > x_1 > 0$; $1 > x_2 > 0$; $p_v = 1.22$ to 8.82 MN/m^2.

If $Re_{cds} > 5 \times 10^3$ the incline angle of the tube does not affect the intensity of heat transfer. This is indicative of the predominance of dynamic effect of the vapour over gravity.

With flow at low velocities heat transfer in vapour condensing in a tube is very complicated, because the forces of both gravity and friction have to be taken into account. The problem has been solved approximately [12.19], and the formulae derived are still quite intricate, because it is necessary to take into consideration the fact that condensate flows not only along the tube but also circumferentially. The pattern of vapour and condensate flow may also differ along different sections of the tube.

Another difficulty is that no data are available to determine the borderlines of sections with different patterns of film flow. All these peculiarities of the process greatly complicate the derivation of exact relationships.

12-4. Film Condensation of Vapour Moving Around a Single Horizontal Tube and Across Banks of Tubes

The problem of heat transfer in stagnant vapour condensing on the outside surface of a single horizontal tube was considered in Sec. 12-2. Data concerning heat transfer in condensing moving vapour are of the greatest importance for industrial application. Theoretical and experimental studies have shown that the nature of the heat transfer on a horizontal tube changes with moving vapour. In conducted tests [12.20] saturated steam was in downward flow past a single horizontal tube exposed to it. Some test results are shown in Fig. 12-11, where the term $\frac{\bar{\alpha}}{\alpha_N}$ is plotted as a function of (Re_v, Δt). Here $\bar{\alpha}$ is the experimental heat-transfer coefficient for condensing moving vapour, $\bar{\alpha}_N$ is the heat-transfer coefficient calculated by the Nusselt formula (12-24) for still vapour, $Re_v =$

$= \frac{\bar{w}_v d}{\nu_v}$ where $\bar{w}_v$ is the mean vapour velocity in the contracted section of the passage; and d is the outside diameter of the tube.

As follows from Fig. 12-11, the intensity of heat transfer increases with Re_v.

The tests [12.20] were conducted at $p_v = (0.032 \text{ to } 0.98) \cdot 10^5$ Pa; $\bar{w}_v = 0.26$ to 17.6 m/s; $\Delta t \approx 0.6$ to 12 K; $\mathrm{Re}_v = 46$ to 864, and a mean air content in the vapour between 0.008 and 0.017 per cent by volume.

Generalization of the test data [12-20] made it possible to derive the following formula for the mean heat-transfer coefficient:

$$\frac{\bar{\alpha}}{\bar{\alpha}_N} = B \left(\frac{\rho_v \bar{w}_v^2}{\rho_l dg} \right)^{0.08} \left(\frac{\rho_l g d_r^3}{\lambda_l \nu_l \Delta t} \right)^{-0.125} \qquad (12\text{-}32)$$

The physical parameters of the condensate in the formula were chosen for saturation temperature.

From Eq. (12-32) it follows that the heat-transfer coefficient is not as dependent on the temperature difference with moving vapour as with stagnant vapour; with stagnant vapour $\alpha \sim \Delta t^{-0.25}$, and with moving vapour

$$\alpha \sim \Delta t^{-0.25} \qquad (12\text{-}33)$$

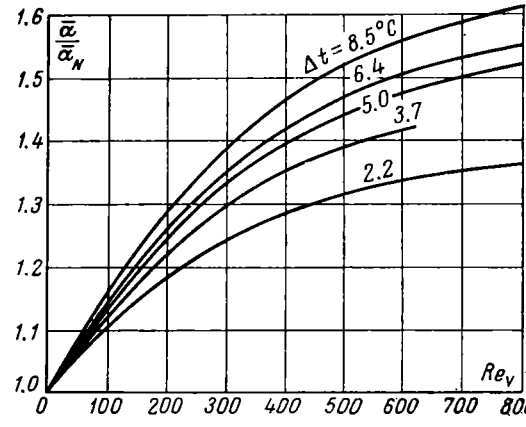

Fig. 12-11. Dependence of the relative heat-transfer coefficient $\bar{\alpha}/\bar{\alpha}_N$ for a single horizontal tube on the Reynolds number of steam with $p = 8.6 \cdot 10^3$Pa

As a rule condensers incorporate a bank of tubes, and not a single one. The tubes are usually staggered or arranged in-line as illustrated in Fig. 9-8.

Condensation on the outside surface of a single horizontal tube differs from that on the outside of similar tubes of a bank. For pure vapour the difference in mode of condensation is traced to two factors: a decrease in vapour velocity owing to partial condensation as it moves across the bank, and an increase in film thickness as the condensate flows successively from one tube to the tube directly below it without splashing.

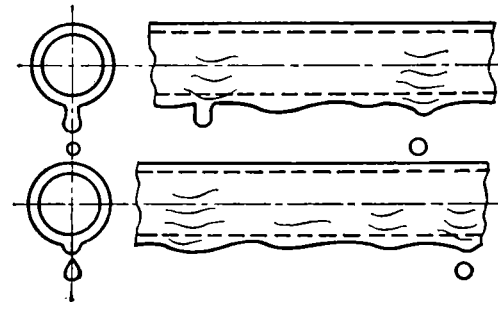

Fig. 12-12. Flow of condensate in a bank of horizontal tubes

The drop in velocity, as the vapour flows across the bank results in a decrease of the rate of heat transfer from row to row of the bank in the direction of flow. The same effect is caused by the flow of condensate from upper tubes onto those below. The way in which condensate flows downward in a horizontal bank differs, however, from condensate flow along a continuous vertical wall.

Condensate does not flow from a tube in a continuous film, but in individual drops or streams (see Fig. 12-12). The drops reaching the top of the tube directly below make the film there temporarily thicker, on the one hand

and, on the other hand, disturb flow of the film as they spread over its surface; this may contribute to the appearance of rippling or even of turbulent flow. The spots where drops separate and fall shift continuously along the tube. Shifting is intensified if the tube is even slightly inclined. An undulatory stream of condensate in this case moves along the bottom section of the tube. Condensate flow of this kind leads to a certain increase in thermal resistance, owing to an increase in film thickness; the increase is compensated for to a great extent by disturbances appearing in the course of flow.

The effect created by the flow onto the top of a tube may be accounted for by the parameter $\sum_{i=1}^{i=n} G_i/G_n$, where $\sum_{i=1}^{i=n} G_i$ is the total quantity of condensate flowing from the tube of the nth row; G_n is the quantity of condensate formed on the tube considered ($i = n$) [12.21]. Here n is the number of rows of tubes in an in-line bank, or one half of the number of rows in a staggered bank.

Experiments have yielded

$$\frac{\bar{\alpha}_n}{\bar{\alpha}_1} = \left(\frac{\sum_{i=1}^{i=n} G_i}{G_n} \right)^{-0.07} \tag{12-34}$$

where $\bar{\alpha}_n$ is the mean heat-transfer coefficient in the nth row, reduced to the vapour velocity and temperature difference in the first row of tubes; and $\bar{\alpha}_1$ is the mean heat-transfer coefficient for the topmost row in the bank [12.22].

In dealing with downward vapour flow across a bank of equal cross-section along its entire height, heat transfer can be calculated using formulae (12-32) and (12-34). The calculation is complicated by the fact that the heat-transfer coefficient for each horizontal row of tubes depends on the local temperature difference across the given row, the vapour velocity and the pressure, and that the laws governing variation in these parameters are unknown. Heat transfer must then be calculated successively for each row, beginning with the first, the inlet one.

The method indicated is very laborious; it can be facilitated considerably by introducing some assumptions that permit an approximate formula to be obtained for the mean heat-transfer coefficient on the vapour side.

Assuming the temperature difference and vapour pressure to be constant along the height of the bank, Berman [12.23] obtained the following approximate calculation formula for the mean heat-transfer coefficient $\bar{\alpha}_v$ of the entire bank:

$$\frac{\bar{\alpha}_v}{\bar{\alpha}_N} = \frac{\bar{\alpha}_1}{\bar{\alpha}_N} = \frac{0.84\varepsilon}{[1-(1-\varepsilon)^{0.84}]\, n^{0.07}} \tag{12-35}$$

In formula (12-35) the term $\bar{\alpha}_1/\bar{\alpha}_N$ is the relative heat-transfer coefficient for the first row, calculated by Eq. (12-32); n is the number of rows in an in-line bank in the direction of flow, or one half of the number of rows in a staggered bank; $\varepsilon = \frac{G_{in}-G_{out}}{G_{in}}$ rate of vapour condensation where G_{in} and G_{out} are respectively the inlet and outlet mass vapour fractions.

12-5. Heat Transfer in Dropwise Condensation of Vapour

Dropwise condensation takes place if the condensate does not wet the surface of the wall, and the latter is then covered with individual bubbles of condensate.

High-speed photography with microscope magnification has shown that small drops grow very rapidly, then the rate of growth becomes insignificant. The

growing in size drops coalesce continuously, freeing a certain part of the wall surface. Due to multiple coalescence and continuous condensation the drops grow to a size at which they detach from the surface and roll down under gravity (or are separated from the surface by the moving vapour if its velocity is high).

It follows from what has been said above that dropwise condensation is strictly speaking a transient process, but it can be considered as steady-state, however, if the time-averaged characteristics of the process remain constant.

Upon the initial contact of vapour with the bare wall an adsorption layer forms, the rapidly proceeding condensation causes the formation of a polymolecular liquid film.

While the film is very thin it is arranged in the power molecular field formed between the two neighbouring phases (solid wall and vapour), resulting in that the film acquires special properties differing from these of a bulk of the same liquid (at a distance from the interface). Such film will be referred to as *thin* and its thickness amounts to fractions of a micron.

In the general case the thickness of the film varies over the surface and it is commensurable with or smaller than the height of the inherent surface irregularities of a solid.

The thin film is under an excess pressure which is referred to as *disjoining pressure* [12.24], which is inversely proportional about to cubed film thickness:

$$\Pi \sim \delta^{-3}$$

In the event of non-wetting the effect of the disjoining pressure manifests itself in that the surfaces of the thin film are attracted to each other, with the degree of attraction being the higher the thinner the film. Such disjoining pressure is assigned the minus sign. The local thinning of the film (for instance, on a surface irregularity, due to local structural defects of the solid, change of lyophobicity, a smaller local rate of condensation, etc.) leads to an increase of the adjoining pressure at the particular spot, compared with the adjacent. As a result the liquid is rapidly displaced to neighbouring sections where the initial drops form of a size exceeding the effective range of intermolecular forces [12.25].

With a positive adjoining pressure (the surfaces of a thin film repulse each other) film condensation sets in (a thick film forms); the surface of the solid is said to be wetted by the liquid involved.

Inasmuch as with intensive dropwise condensation primary drops form continuously on the bared surface of the wall, it may be assumed that on the average a thin and a thick (drops) film exists on the wall. The simultaneous existence of a thin film and visible drops should be visualized only as a continuously changing phenomenon.

Thus, the effects produced by second-order capillarities (thin films) are important for the formation of initial drops. In large drops the effects due to the second-order capillarities may manifest themselves only at the root of a drop (at the triple vapour-liquid-wall boundary).

The equilibrium pressure of saturated vapour is known to be greater on top of a convex vapour-condensate interface than above a flat one. Condensation of vapour on a spherical drop of radius R, suspended in the vapour, is possible only if $R > R_{cr}$, where R_{cr} is the critical (minimum possible) radius of curvature of the vapour-condensate interface. With a thin spherical film the effect of its thickness must also be taken into account.

With account taken of the capillary effects of the first and second order, the pressure drop in a thin spherical layer of condensate and in the vapour can be described by equation

$$\Delta p = p_f - p_v = 2\sigma/R - \Pi \tag{12-36}$$

where σ = surface tension coefficient at the liquid-vapour interface;
R = radius of layer.

Accordingly, in the event of spherical films, the critical subcooling of vapour ΔT_{cr} is described by equation

$$\Delta T_{cr} = \frac{2\sigma T_s}{r\rho_f R_c} - \frac{\Pi T_s}{r\rho_f} \tag{12-37}$$

The first term in the right-hand side of Eq. (12-37) accounts for first-order capillary effects (surface tension), the second—for the adjoining pressure. If the actual subcooling of the vapour ΔT exceeds the calculated ΔT_{cr}, condensation is thermodynamically possible (here $\Delta T = T_s - T_{sur}$, where T_s and T_{sur} respectively are the absolute temperatures of saturated vapour at the given pressure and the temperature of the layer of condensate). If $\Delta T < \Delta T_{cr}$, the layer must evaporate, i.e. from the thermodynamic viewpoint the formation of such a layer is impossible. At $\Delta T = \Delta T_{cr}$ the system is in equilibrium.

For the case $\Pi \to 0$ Eqs. (12-36) and (12-37) reduce into the known Laplace and Thomson's equations:

$$\Delta p = \frac{2\sigma}{R} \tag{12-38}$$

$$\Delta T_{cr} = \frac{2\sigma T_s}{r\rho_f R_c} \tag{12-39}$$

It follows from Eq. (12-37) that the required subcooling of saturated vapour above a thin non-wetting film ($\Pi < 0$) will exceed the subcooling of the vapour above the spherical surface of a large drop. It also follows from Eq. (12-37) that with condensation taking place on the wall between surface irregularities the required subcooling is less than with condensation on the surface irregularities ($R > 0$). As a result, under otherwise equal conditions, the formation of the condensed phase is most likely to take place in the microrecesses of the solid.

In practice the surface of a solid is not absolutely homogeneous, plain (smooth), in particular. Even when machining to a 7 to 11th grade of surface finish the square mean magnitude of surface microirregularities is between 0.05 and 1.5 micrometer, and the local radii of microirregularities may vary in a wide range.

The larger the temperature difference $t_s - t_w$, the more microirregularities able to serve as centres of nucleation (condensation), and the number of initial drops increases.

Critical subcooling ΔT_{cr} also depends on the physical properties of the media involved in heat transfer. In particular, this dependence manifests itself for the adjoining pressure as well.

Upon condensation of the vapour of organic liquids the required amount of subcooling ΔT_{cr} is usually small. For mercury vapour the required subcooling is very high. The condensation of water vapour occupies an intermediate position. An intensive formation of condensate of organic-liquid vapours at high temperature differences may cause filling of the wall surface with the condensate and an increase in the thermal resistance to heat transfer (an effect causing al-

most the same result as thickening of the film at film condensation). When mercury vapour condenses on steel surfaces a comparatively small number of drops form, condensation proceeds at a low rate, and then the heat-transfer coefficient may be smaller than in the event of film condensation of the same vapour [12.26].

It follows from what has been said above that in the case of dropwise condensation the liquid phase is in peculiar motion resulting from the fact that variable forces are applied to it, with the capillary forces being of material importance.

The theory of heat transfer upon dropwise condensation is based on a statistical description of the process.

In the general case at each moment of time there exist on the wall a multitude of drops the radius of which changes from the critical R_c to the separation radius R_0. The range of drop sizes can be described by the differential size distribution function $\varphi(R) \equiv dn/dR$, m^{-3}, where dn is the number of drops with a radius between R and $R + dR$, per unit wall area.

If the process of dropwise condensation is quazi-stationary, then $\varphi(R) = \text{const}$. The continuous increase of the drops in size due to condensation and coalescence is compensated for by the formation of new (primary) and the vanishing of large drops (reaching the separation dimension).

The increase in the volume of a drop in unit time due to condensation is determined by equation

$$\Delta V = F \frac{dR}{d\tau} = Fw(R)$$

where F = surface of a drop turned into vapour;
$w(R) \equiv dR/d\tau$, m/s = function of the rate of drop growth

During the process the heat of phase transition evolves.

Let us consider a drop to be a hemisphere. For drops with radii between R and $R + dR$ the evolving heat of phase transition, reduced to unit wall area and unit time, is determined by equation

$$dq_R = r\rho_f 2\pi R^2 \frac{dR}{d\tau} dn = r\rho_f 2\pi R^2 w(R)\, \varphi(R)\, dR$$

Then, the mean rate of heat flow will be

$$q_m = 2\pi \int_{R_c}^{R_0} r\rho_f R^2 w(R)\, \varphi(R)\, dR + q_{c0} \qquad (12\text{-}40)$$

where the term q_{m0} accounts for the heat evolving during the formation of primary drops, which is usually negligibly small. In dropwise condensation the temperature differences are usually also small which makes it possible to ignore the heat expended in subcooling the drops in respect to saturation temperature t_s.

Thus, to calculate q_m or $\alpha = q_m/\Delta t$, we must know the growth and size distribution functions for the drops, $w(R)$ and $\varphi(R)$ [12.27]. These functions are determined for various conditions both analytically and by experiment. According to experimental data and calculations, for instance, the growth of a hemispherical drop on condition that the main thermal resistance is that offered by the drop to heat transfer by conduction is determined by equation

$$dR/d\tau \approx 2 \frac{\lambda_f (t_s - t_w)}{r\rho_f R}$$

In addition to the approach described, heat-transfer coefficients are determined by direct experimentation.

The mean heat-transfer coefficient for dropwise condensation of stagnant vapour is described by the following equations [12.28]:

$$\text{at } \mathrm{Re}_* = 8 \times 10^{-4} \text{ to } 3.3 \times 10^{-3}$$

$$\overline{\mathrm{Nu}} = 3.2 \times 10^{-4}\, \mathrm{Re}_*^{-0.84} \Pi_c^{1.16} \mathrm{Pr}^{1/3} \tag{12-41}$$

$$\text{at } \mathrm{Re}_* = 3.3 \times 10^{-3} \text{ to } 3.5 \times 10^{-2}$$

$$\overline{\mathrm{Nu}} = 5.0 \times 10^{-6}\, \mathrm{Re}_*^{-1.57} \Pi_c^{1.16} \mathrm{Pr}^{1/3} \tag{12-42}$$

Here

$$\overline{\mathrm{Nu}} = \frac{\bar{\alpha} R_{cr}}{\lambda_l} = \frac{\bar{\alpha} 2\sigma T_s}{\lambda_l r \rho_l (t_s - t_w)}$$

$$\mathrm{Re}_* = \frac{w_* R_{cr}}{\nu_l} = \frac{\lambda_l (t_s - t_w)}{r \mu_l}$$

$$\Pi_c = \frac{\xi \sigma R_{cr} (t_s - t_w)}{\rho_l \nu_l^2} = \frac{2\xi \sigma^2 T_s}{r \rho_l^2 \nu_l^2}; \quad \mathrm{Pr} = \frac{\nu_l}{a_l}$$

The physical parameters of condensate in formulae (12-41) and (12-42) are taken for the saturation temperature t_s.

The scales of the linear dimensions and velocity are the critical radius R_{cr} determined by formula (12-39)*, and the fictitious rate of condensation (with the heat of phase transition being removed by conduction):

$$w_* = \frac{\lambda_l \Delta t}{r \rho_l R_{cr}}, \text{ m/s}$$

The dimensionless number Π_c accounts for the effect of capillary forces. Its component term $\xi (t_s - t_w)$ reflects the dependence of surface tension on temperature. Here

$$\xi = \frac{1}{\sigma} (\partial \sigma / \partial t)$$

is the temperature coefficient of surface tension. As a result the Π_c number can be looked upon as the ratio of thermo-capillary to viscosity forces and it can be presented in the following form:

$$\Pi_c = \xi (t_s - t_w) \frac{\sigma R_{cr}}{\rho_f \nu_f^2} = \xi (t_s - t_w)\, \mathrm{Lp}$$

where $\mathrm{Lp} = \sigma R_{cr} / \rho_f \nu_f^2$ is the Laplace number.

In Fig. 12-13 formulae (12-41) and (12-42) are correlated with experimental data obtained with saturated vapour (steam) condensing on vertical walls up to 0.61 m high, on the walls of vertical tubes and horizontal banks of tubes. In the experimental data used the dimensionless term Π_c ranged from 0.98×10^{-2} to 4.5×10^{-2}, the Prandtl number was between 1.75 and 3.65 and the steam pressure $p_v \approx (0.12 \text{ to } 1)\ 10^5$ Pa.

It follows from formulae (12-41) and (12-42) that to calculate the heat-transfer coefficients it is sufficient to know the saturation temperature and the temperature difference. The chart shown in Fig. 12-14 can be used to determine the

* At the initial moment of film or drop growth the temperature difference between surfaces of liquid and wall is slight. Consequently, T_s is replaced by the wall temperature T_w in formulae (12-41) and (12-42).

heat-transfer coefficient for dropwise condensation of stagnant saturated steam. The chart is based on calculations and graphical presentation of formulae

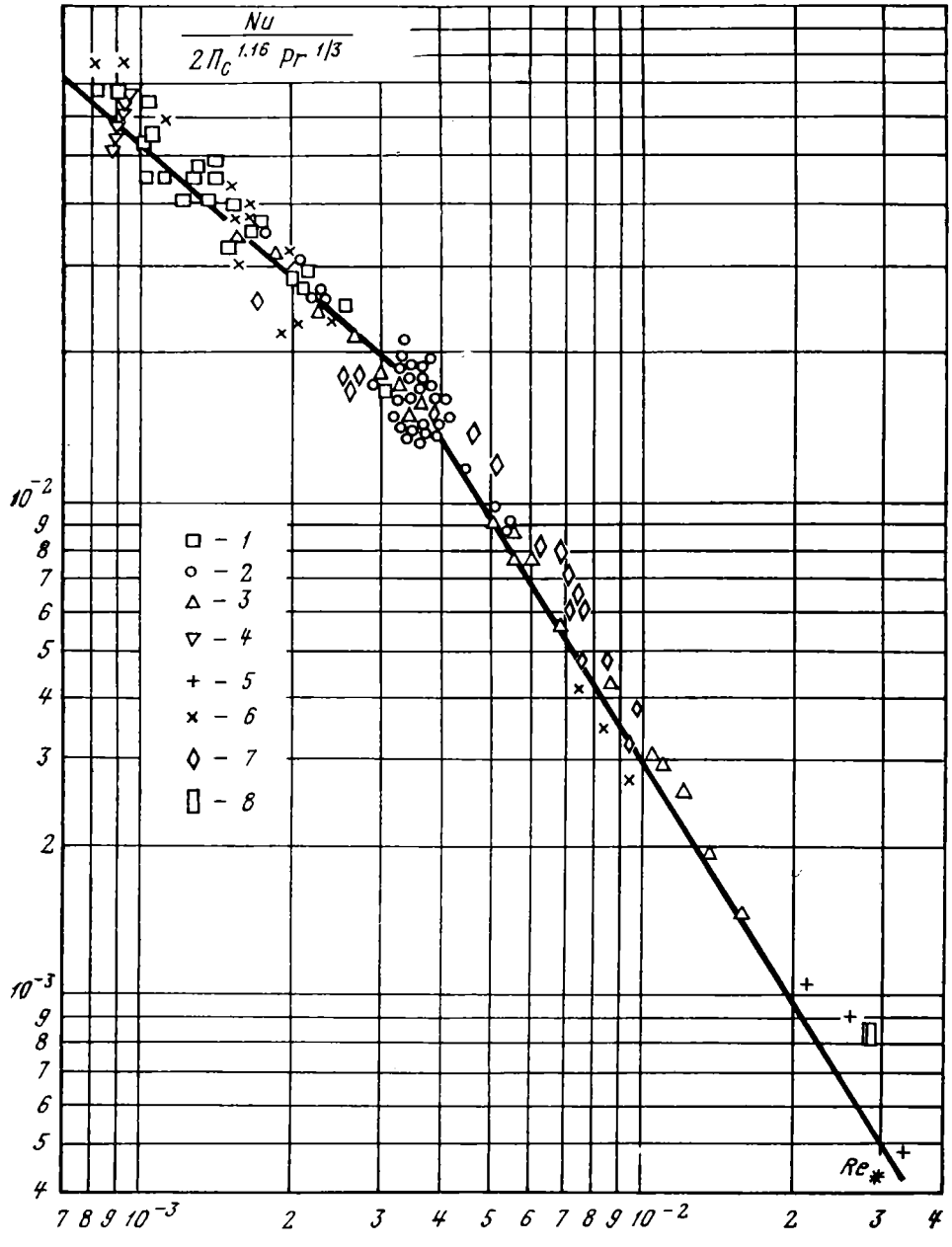

Fig. 12-13. Heat transfer for dropwise condensation of vapour:
□—Gnam; ○—Nagle et al; △—Shea and Krase; ▽—Lapshin and Konfederatov; +—Schmidt et al; ×—Kutateladze; ◇—Isachenko and Solodov; II—Mitsushina et al

(12-41) and (12-42). In accordance with formulae (12-41) and (12-42)

$$\text{at } \mathrm{Re}_* < 3.3 \times 10^{-3},\ \alpha \sim \Delta t^{0.16} \text{ and } q_w \sim \Delta t^{1.16};$$

$$\text{at } \mathrm{Re}_* > 3.3 \times 10^{-3},\ \alpha \sim \Delta t^{-0.57} \text{ and } q_w \sim \Delta t^{0.43}$$

From these relationships and Fig. 12-14 it follows that the local heat-transfer coefficient increases with an increase in Δt at small temperature differences, and, on the contrary, diminishes at large temperature differences Δt. The positive relationship $\alpha(\Delta t)$ becomes negative at Re_* approximately equal to 3.3×10^{-3}.

The change in the nature of the relationship $\alpha(\Delta t)$ may be explained as follows. With $Re_* < 3.3 \times 10^{-3}$ the rate of condensation is restricted above all by the necessity to supersaturate the condensing steam. The higher the tempe-

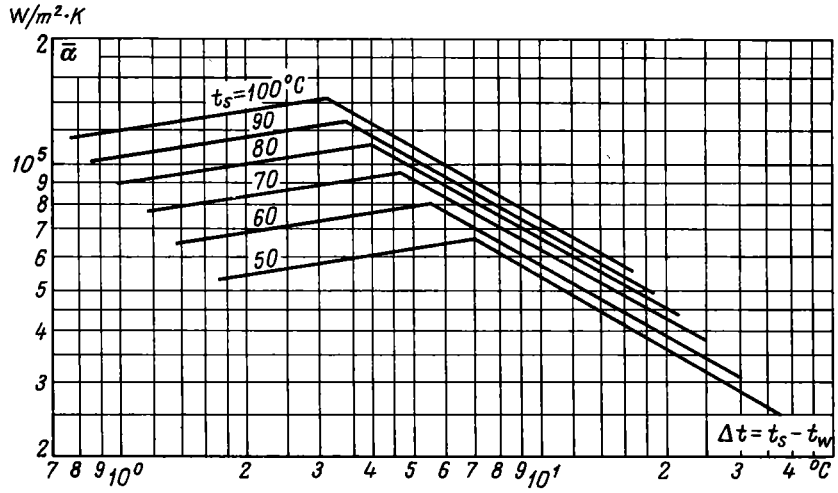

Fig. 12-14. Heat-transfer coefficient for dropwise condensation of steam according to saturation temperature and temperature difference:
with positive dependence of α on Δt: $\alpha = 2.61 \times 10^2 t_s^{4/3} \Delta t^{0.16}$; with negative dependence of α on Δt: $\alpha = 2.79 \times 10^4 t_s^{1/3} \Delta t^{-0.57}$

rature difference Δt, and the greater the supersaturation of the steam, the more intensive is the process of condensation, while the rising thermal resistance of the condensate is still comparatively small.

The increase in rate of condensation with rising Δt, and the increase in quantity of condensate on the heat-transfer surface result in that the effect of the thermal resistance of the condensate acquires greater importance for heat transfer.

At $Re_* > 3.3 \times 10^{-3}$ the rate of condensation depends above all on the thermal resistance offered by the already formed condensate to heat flow and a crisis of dropwise condensation occurs.

With dropwise condensation the rate of heat transfer becomes dependent on vapour velocity at its comparatively low quantity.

At the same time the intensifying effect of vapour velocity, causing the appearance of a large amount of condensate of the wall, may precipitate the crisis of dropwise condensation.

The dependence of the mean heat-transfer coefficients on steam velocity is shown in Fig. 12-15. Atmospheric pressure steam condensed on a vertical plate, flowing downward along it. On the chart $\Pi_w = 2\rho_v \overline{w_v^2}/\rho_f (g\nu_f)^{2/3}$, where $\overline{w_v}$ is the mean velocity of steam over the calculated section and α_0 is the heat-transfer coefficient calculated by formula (12-41).

The test data presented in Fig. 12-15 were obtained with Re_* ranging from 8×10^{-4} to 3.3×10^{-3}, and can be described by equation

$$\frac{\bar{\alpha}}{\alpha_0} = 6.9 \times 10^{-3} Re_*^{-0.7} \Pi_w^{0.25} \qquad (12\text{-}43)$$

where saturation temperature is taken as reference.

It follows from formula (12-43) that $\bar{\alpha} \sim w_v^{0.5}$ and $\bar{\alpha} \sim \Delta t^{-0.54}$. According to the test data presented in Fig. 12-15, $\bar{\alpha}/\alpha_0 > 1$. The maximum increase in the rate of heat transfer amounted to $\bar{\alpha}/\alpha_0 = 3.5$ ($\bar{\alpha} = 425 \times 10^3$ W/m^2·K) and occurred at $\Delta t = 1$ K and $\bar{w}_v = 6.6$ m/s.

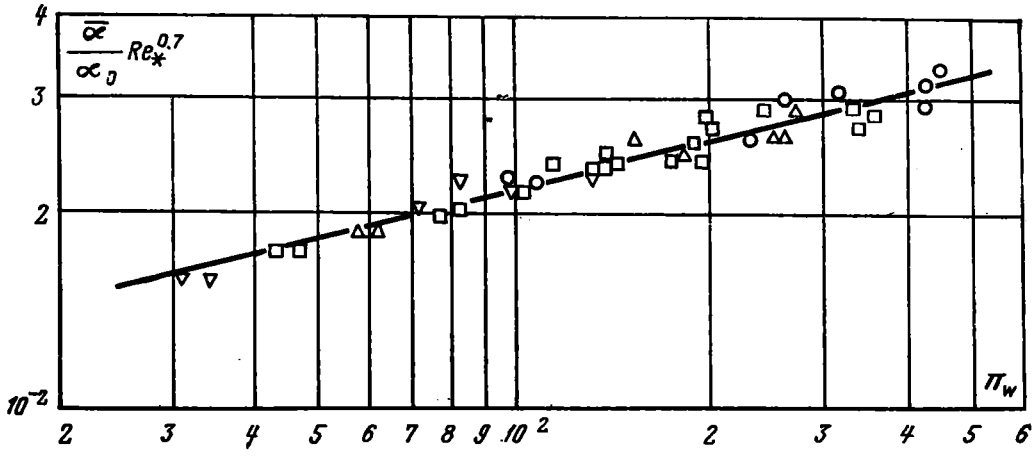

Fig. 12-15. Dependence of heat transfer on the velocity of vapour in dropwise condensation. Treatment of test data by Shea and Krase at $Re_* = 8 \times 10^{-4} - 3.3 \times 10^{-3}$; ○—plate 117 mm long; □—234 mm; △—351 mm; □—468 mm; ▽—585 mm

The intensifying effect of vapour velocity also manifests itself at $Re_* > 3.3 \times 10^{-3}$. With condensation of moving vapour, however, at considerable temperature differences the mean heat-transfer coefficient differs but little from α_0.

It has been shown by experiments that with dropwise condensation the flow of condensate from an upper upon a lower tube leads to a slight decrease in the heat-transfer coefficient. With the value of $\sum_n G_i/G_n$ ranging from 2 to 12 the rate of heat transfer diminished by about 10-15 per cent.

The intensity of heat transfer in dropwise condensation greatly depends on the admixture of noncondensing gases (Sec. 14-6).

12-6. Special Problems of Heat Transfer in Condensing Vapour

Condensation of superheated steam takes place if the wall temperature is lower than the saturation temperature; but if $t_w > t_s$, no condensation occurs, and the developing heat transfer process is similar to that occurring in a single-phase fluid (vapour).

The temperature of condensing superheated steam drops gradually at the wall and, in essence, condensation of saturated steam takes place. The heat of superheat is transferred to the condensate by ordinary convection. Thus, the

condensing superheated steam transfers both the heat of phase transition or latent heat and its superheat to the condensate. In addition, the fraction of steam which fails to condense in the heat exchanger, gives up part of its superheat by means of ordinary convection, with an accompanying drop in steam temperature.

With complete condensation of superheated steam, each kilogram of it gives up the following amount of heat:

$$r_{sup} = r + c_{pv}\delta t_{sup}$$

where $\delta t_{sup} = t_{sup} - t_s$; t_{sup} is the temperature of the superheated steam.

The problem of heat transfer in condensing superheated steam has not yet been fully investigated. Some of the test data available however on film and dropwise condensation of stagnant steam make it possible to consider that heat-transfer coefficients for complete condensation can be calculated by the formulae derived for dry saturated steam, with an accuracy sufficient for practical applications. The enthalpy r is then replaced in the formulae by r_{sup}.

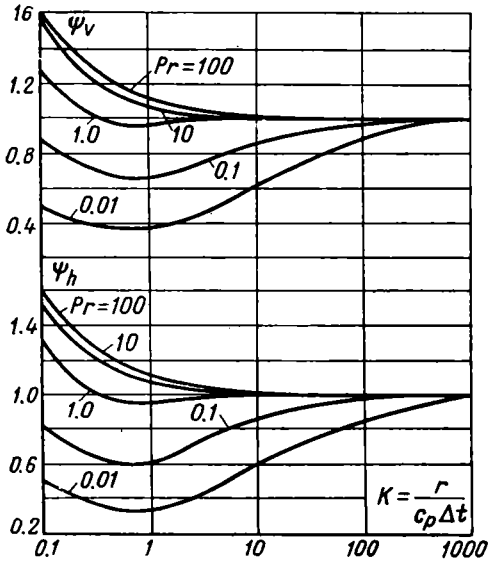

Fig. 12-16. Effect of inertia forces and heat transfer by convection on the transfer of heat in condensate film in laminar flow

Condensation of wet steam. A fraction of the moisture contained in wet steam falls out during condensation together with the condensing steam. The maximum quantity of moisture that can be deposited on the surface of the condensate film is assumed to equal the moisture content of the condensed steam. Approximate estimates show that the effect of moisture can be neglected when dealing with wet steam of a moisture content ranging from 10 to 20 per cent (by mass). The effect of moisture has not yet been sufficiently investigated.

Condensation of liquid-metal vapour. Both film and dropwise condensation of liquid-metal vapour are possible.

The thermal resistance of liquid metal is very small and the process of heat transfer in a condensing vapour may therefore be greatly affected by the thermal resistance to phase transition and the contact thermal resistance resulting from wall contamination. With the type of condensation (film or dropwise) the rate of heat transfer affects to a considerably lesser extent.

Dropwise condensation is observed in the condensation of mercury vapour. The vapours of alkali metals (sodium, potassium), it would seem, mainly condense film-wise.

The Prandtl number of liquid metals $Pr \ll 1$; the inertia and convection not taken into account can therefore lead to considerable errors.

The results obtained from theoretical calculation [12.11] of film condensation with the film in laminar flow are presented in Fig. 12-16. Here $\psi_v = \frac{\alpha_v}{\alpha_{Nv}}$, and $\psi_h = \frac{\alpha_h}{\alpha_{Nh}}$; α_v and α_h are the local heat-transfer coefficients for vapour conden-

sation on a vertical wall and horizontal tube respectively, taking inertia and convection into consideration; α_{Nv} and α_{Nh} are the coefficients calculated by Nusselt's formula for a vertical wall and horizontal tube. From the curves in Fig. 12-16, the intensity of heat transfer at $Pr \approx 0.01$ can drop by about 60 per cent compared with the data obtained by means of the Nusselt formula.

According to available data [12.29], for a number of liquid-metal coolants the condensation coefficient is close to unity.

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CHAPTER

13

HEAT TRANSFER IN BOILING SINGLE-COMPONENT LIQUIDS

13-1. Mechanism of Heat Transfer in Liquids in Nucleate Boiling

(*a*) *Kinds of Boiling*

Boiling is defined as the process of intensive vapourization in the entire volume of a liquid at the saturation temperature or somewhat superheated with respect to the saturation temperature, accompanied by the formation of vapour bubbles. Processes of boiling are of great practical importance in the field of heat and nuclear power generation, chemical engineering, and a number of other fields of modern engineering.

Boiling is possible in the entire temperature interval between the triple point and critical points of a substance. Heat of evaporation is absorbed during the change of phase, i.e. as the liquid turns into vapour. Boiling, therefore, is always linked with the supply of heat to the boiling system.

Two kinds of boiling are distinguished: boiling of liquid on a solid heat-transfer surface to which heat is applied from outside (local boiling), and boiling in the whole volume of a liquid (bulk boiling).

With boiling on a solid surface the formation of vapour bubbles is observed at individual spots on the surface. In bulk boiling the vapour phase appears directly within the liquid in the form of individual vapour bubbles. Bulk boiling is possible only if the liquid phase is superheated in relation to the saturation temperature at the given pressure. Considerable superheating can be ensured, for instance, by causing a rapid reduction of system pressure. Bulk boiling can take place in the presence of inner heat sources in a liquid.

Processes of boiling on solid heating surfaces (tube surfaces, walls of channels, etc.) are commonly encountered in modern power engineering and in other branches of engineering and technology. It is this kind of boiling that will mainly be considered here.

The mechanism of heat transfer in nucleate boiling differs from that of convection in a single-phase liquid in there being molecular transport of mass and heat by the vapour bubbles from the boundary layer inside the boiling liquid.

This leads to a higher intensity of heat transfer in a boiling liquid, compared with the rate of heat transfer by convection in a single-phase fluid.

For boiling to originate, two conditions must be fulfilled: (a) there must be superheating of the liquid with respect to the saturation temperature, and (b) the presence of starting points of vapourization.

Superheating of the liquid is maximum directly at the heated surface of heat transfer. The starting points are located on this surface—individual surface irregularities, air bubbles, specks of dust, etc.—hence the reason that vapour bubbles mainly form directly on the heat-transfer surface. Two kinds of boiling are distinguished: *nucleate* and *film*.

Boiling during which vapour forms in the form of periodically forming and growing bubbles is called nucleate boiling. With an increase in the heat flow to a certain value individual vapour bubbles coalesce, forming a continuous vapour layer at the wall, breaking periodically into the bulk of liquid. The kind of

boiling characterized by the presence of a film of vapour on the surface, separating it from the liquid, is referred to as film boiling [13.1].

The intensity of heat transfer upon film boiling is considerably inferior to that upon nucleate boiling.

The mechanism of heat exchange upon nucleate boiling and film boiling will be considered below.

(*b*) *Minimum Bubble Radius*

The process of vapourization can be divided into individual stages. The initial stage involves the formation of bubbles at some starting points on the heating surface. Then the bubbles grow and separate from these centres. The final stage includes movement of the bubbles in the bulk of superheated liquid. When the moving vapour bubbles reach the surface of the liquid they burst. Then, new bubbles form and the process repeats. The minimum size of a vapour bubble at the moment of formation is called the *critical radius* (R_{cr}), and it corresponds to the size of the irregularities of the heat-transfer surface (centres of vapourization). In the presence of centres of vapourization the steam bubbles form at a slight superheating of the liquid above the saturation temperature. If there are a few centres, the liquid begins to boil at considerable superheat temperatures. The critical radius is determined from the conditions of thermodynamic equilibrium of phases.

For a vapour bubble to form and continue to exist, the vapour pressure inside the bubble should not be less than the sum of all external forces acting upon it. These are mainly two: the pressure exerted by the surrounding liquid p, and the surface tension of the bubble itself. The condition of equilibrium of these two forces for a spherical bubble is described by the Laplace equation

$$\Delta p = p_1 - p = \frac{2\sigma}{R_{cr}} \tag{13-1}$$

At an excess pressure $\Delta p \geqslant 2\sigma/R_{cr}$ the bubble is free to exist and grow; at $\Delta p < 2\sigma/R_{cr}$ it condenses.

The formation of a vapour bubble with a critical radius R_{cr} is possible only if the surrounding liquid is superheated, i.e. if the temperature of the liquid T_l is higher than the saturation temperature T_s (at a liquid pressure p) by a certain value $\Delta t = T_l - T_s$. The temperature of the vapour in a bubble of critical radius, $T_{v\,cr}$, must be equal to the temperature of the surrounding layers of liquid, T_l. Therefore the temperature $T_l = T_{v\,cr}$ can be determined approximately (if we neglect the small correction for the effect of interface curvature on saturation pressure) as simply the saturation temperature, but at the vapour pressure inside the bubble $p_1 = p + \Delta p$. From this it follows that the relation between the pressure drop Δp and the required superheat Δt is determined by formula

$$\Delta p = p_1 - p = p'\Delta t + \frac{1}{2!}p''\Delta t^2 + \ldots \approx p'\Delta t$$

If we also take into account the dependence of pressure on the curvature of the water-vapour interface, then

$$\Delta p = p'\Delta t\,\frac{\rho_l - \rho_v}{\rho_l}$$

The quantity p' is the derivative of pressure with respect to temperature at the saturation line, which is expressed in accordance with the Clapeyron-

Clausius law by the equation

$$p' = \left(\frac{\partial p}{\partial T}\right)_s = \frac{r\rho_l\rho_v}{T_s(\rho_l - \rho_v)}$$

Hence,

$$\Delta p = \frac{r\rho_v \Delta t}{T_v} \tag{13-2}$$

Relationships (13-1) and (13-2) permit the critical radius of a bubble to be expressed by

$$R_c = \frac{2\sigma T_s}{r\rho_v (T_l - T_s)} \tag{13-3}$$

This relationship is similar to that for the critical radius of a spherical drop (12-39).

Relationship (13-3) determines the critical radius of a spherical bubble in the bulk of a superheated liquid at a distance from the heating surface.

The layers of the liquid which are in direct contact with the heating surface are at a temperature T_f, equal to wall temperature T_w. Therefore, if we replace in formula (13-3) T_f by T_w, then

$$R_c = \frac{2\sigma T_s}{r\rho_v (T_w - T_s)} \tag{13-4}$$

When presented in this form the quantity R_c characterizes the curvature of the external surfaces of the bubbles forming on the heat-transfer surface. At the same time the critical radius R_c determines the order of magnitude of the surface irregularities which may serve as vapourization centres or starting points under given conditions (pressure, superheat temperature, etc.).

With rising liquid superheat at a given pressure R_c diminishes. The minimum critical radius R_c also diminishes with increasing pressure at a given superheat temperature, since an increase in pressure results in a greater derivative p' and surface tension decreases.

A higher superheating of the liquid and pressure lead to a smaller R_c, and subsequently, to a greater number of acting centres of vapourization, to intensive mixing of the liquid in the boundary layer and a greater rate of heat transfer.

If the liquid involved wets the wall, it boils practically without any superheating above the saturation temperature. Thus, the factors effecting the formation of a vapour bubble with a radius R_c include: Δt, p and the nature of the physical and chemical interaction between the vapour and liquid and the solid wall.

(c) *Minimum Work Expended on the Formation of a Vapour Bubble of Critical Radius*

An analysis of the work to be expended on the formation of vapour bubbles of critical radius makes it possible to determine the heat-transfer surface irregularities which are most likely to serve as starting points of vapourization.

The work expended on the formation of a vapour bubble of critical radius R_c depends on whether it forms in the bulk of the liquid at a distance from the heating surface or directly on that surface. In addition, the work expended on bubble formation will vary depending on the shape of the starting points, which may be recesses in the rough surface and also different cracks, grooves, etc. The conditions most favourable for bubble formation are these with which a minimum amount of work is expended.

The work expended on the formation of a vapour bubble in the bulk of boiling liquid [13.2] is expressed by the relation

$$L = -\Delta pV + \sigma F$$

where V and F are the volume and surface of bubbles.

The term σF represents the work expended on forming the vapour-liquid interface F.

The smaller the magnitude of work L, the more bulk boiling is probable since the probability is proportional to $\exp(-L/kT_s)$, where $k = 1.38 \times 10^{-23}$, J/K, is Boltzmann's constant.

The volume and surface of a bubble are respectively equal to $\frac{4}{3}\pi R_c^3$ and $4\pi R_c^2$. Taking formulae (13-1), (13-2) and (13-3) into consideration, the expression for L can therefore be presented in the form:

$$L = \frac{4}{3}\pi\sigma R_{cr}^2 = \frac{16\pi\sigma^3}{3\Delta p^2} = \frac{16}{3}\frac{\pi\sigma^3}{(r\rho_v)^2}\left(\frac{T_s}{\Delta t}\right)^2$$

From that it is clear that the greater superheat of the liquid $\Delta t = T_l - T_s$ (and the larger Δp or the smaller R_c), the smaller the work L. Thus the probability of bulk boiling of liquid in volume increases with increase of superheat.

The physical conditions in which bubbles form on the heating surface are in many ways similar to those considered above for bulk boiling. The main difference consists in that, in addition to the interface, the vapour bubble will have a solid-liquid interface on which molecular adhesion is more or less weakened. Therefore the term in the expression for L representing the work of formation of new liquid-vapour interfaces is quantitatively smaller. This results in a lower minimum of work expended on bubble formation; generation of a new bubble is accordingly more probable.

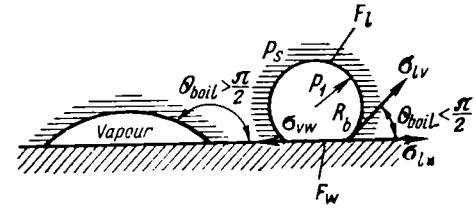

Fig. 13-1. Determination of minimum radius R_{cr} of a vapour bubble and of the wetting contact angle θ_{boil}:

p_s—saturation pressure; p_1—pressure inside the bubble

The phenomenon of molecular adhesion at the solid surface is closely connected with the phenomenon of wetting treated in Sec. 12-1.

Fig. 13-1 schematically illustrates a vapour bubble of critical size, formed on a flat heating surface under the condition that the liquid wets the surface and that the contact angle $\theta_{boil} < \pi/2$. The bubble shown next to it is formed on an unwetted surface and at a contact angle of $\theta_{boil} > \pi/2$. Let us consider the case with a contact angle $\theta_{boil} < \pi/2$. The surface F of the bubble consists of a section F_l along which the vapour is in contact with the liquid, and of a base F_w. Let us determine the work involved in the formation of new interfaces due to the appearance of such a vapour bubble on the heating surface. Before the bubble appears surface F_l did not exist and the section F_w was exposed to the liquid. The surface energy on this section was equal to $F_w\sigma_{lv}$. After the vapour bubble appeared surface F_l formed and an amount of work $F_l\sigma_l$ was expended to create it. In addition, liquid was replaced with vapour on surface F_w involving an energy expenditure $(F_w\sigma_{vw} - F_w\sigma_{lw})$. The work spent to ensure separation of liquid from the surface is called the *work of adhesion*. On the whole the total work required to form new surfaces, when a vapour bubble

appears on the flat section of a solid, is:

$$F_l\sigma_{l,v}+F_w(\sigma_{l,w}-\sigma_{vw})$$

According to Fig. 13-1 $\sigma_{vw}-\sigma_{lw}=\sigma_{lv}\cos\theta$ (the subscripts l, v, *boil* are omitted below). The preceding expression therefore acquires the following form:

$$F\sigma\left[1-\frac{F_w}{F}(1-\cos\theta)\right]$$

and

$$L=-\Delta pV+F\sigma\left[1-\frac{F_w}{F}(1-\cos\theta)\right] \qquad (13\text{-}5)$$

For the case illustrated in Fig. 13-1 the ratio of the base surface F_w to the whole surface F of the vapour bubble can be expressed by trigonometric functions of the contact angle θ. The expression obtained, however, applies only to the more general case in which the bubble forms not on a flat section (as shown in Fig. 13-1), but in a recess or on a protruding surface irregularity of arbitrary shape [13.3, 13.4] (Fig. 13-2). Then, the ratio F_w/F characterizes the fraction of the bubble surface on which vapour is in contact with the heating surface. This ratio depends on the shape of the surface irregularity or roughness.

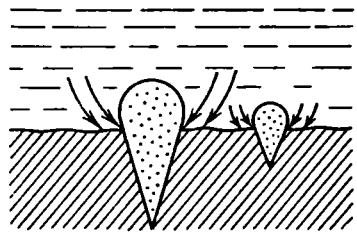

Fig. 13-2. Formation of vapour bubbles on the surface with recesses

It will then be seen that the work of forming the interfaces will be the smaller, the larger the ratio F_w/F and the larger the contact angle θ. Hence it follows that the places where the formation of bubbles is most probable are recesses in the surface and those recesses where local conditions for wetting have deteriorated for some reason. Local deterioration of wetting (and increase in the contact angle θ) may be caused by heterogeneity of the material of the heating surface, foreign inclusions, various contaminations, and, in particular, by adsorption films of oil and grease, mechanical stresses, etc. Under otherwise equal conditions a decrease in pressure leads to a smaller absolute value of the term in the right-hand side of relation (13-5), i.e. results in an increase of the amount of work expended on the formation of a vapour bubble. In consequence, with a reduction in pressure the beginning of the process of vapourization shifts into the region of higher liquid superheat temperatures.

Relationships (13-1) and (13-5) can be used to interrelate the contact angle and the shape of recesses.

(*d*) *Rate of Bubble Growth*

One of the principal characteristics explaining the mechanism of heat transfer in a boiling liquid is the rate of growth of vapour bubbles on the heating surface.

Following the formation of bubbles at starting points of radii $R \geqslant R_c$, the bubbles grow in size at the expense of the heat added to them. The heat is added by conduction from the superheated layer of liquid, surrounding the vapour bubbles, through the vapour-liquid interface F_f and through the surface underneath the bubble F_w. The heat added to a bubble is spent to evaporate the liquid and perform work of expansion.

The rate of bubble growth depends on the intensity of heat transfer by the two components of the heat flux. The intensity of heat transfer in a boiling liquid can be determined with the aid of the *Jacob number* which is obtained by presenting in a dimensionless form the set of differential equations and the conditions of unambiguity describing the heat transfer in boiling liquid. For the dimensionless set of equations indicated Eq. (13-8) was derived. The last dimensionless term in the right-hand side of this equation is the Jacob number

$$\mathrm{Ja} = \frac{c_p \Delta t}{r} \frac{\rho_f}{\rho_v}$$

The Jacob number characterizes the relation between the heat flux expended to superheat a unit volume of liquid and the volumetric heat of vapourization. It depends on pressure and liquid superheat temperature. With rising pressure the Jacob number diminishes since the density of vapour increases substantially, and on the contrary, this number increases with dropping pressure. The Jacob number increases with liquid superheat temperature. Depending on various conditions, heat balance equations are written for the bubble-liquid interface from which analytical relationships are found permitting to determine the radius of a bubble while it grows at the starting point of vapourization. At pressures above atmospheric ($\mathrm{Ja} \leqslant 20$) the vapour bubble grows at the expense of the heat transferred from the heating surface to its root through the adjoining layer of liquid. The variation of bubble radius in time is determined with the aid of relationship [13.5-13.6]

$$R = \sqrt{2\beta \mathrm{Ja}} \sqrt{a\tau} \qquad (13\text{-}6')$$

where τ = vapour bubble residence time on the heating surface;
a = thermal diffusivity of the liquid phase.

The second limiting case occurs at low pressures ($\mathrm{Ja} \geqslant 20$). Heat is transferred to the vapour bubble from the superheated layer of liquid in contact with the interface, then the radius of the vapour bubble is determined by the relationship

$$R = 2\gamma \mathrm{Ja} \sqrt{a\tau} \qquad (13\text{-}6'')$$

In the general case vapour bubbles grow both at the expense of the heat transferred from the heating surface into the root of a bubble and of the heat transferred from the superheated layer of liquid covering the liquid-bubble interface. The variation of bubble radius in time then is expressed by equation [13.7]

$$R = (\gamma \mathrm{Ja} + \sqrt{\gamma^2 \mathrm{Ja}^2 + 2\beta \mathrm{Ja}}) \sqrt{a\tau} \qquad (13\text{-}6)$$

The parameters γ and β depend on geometric factors. For wetting contact angles θ ranging from 40 to 90°, $\gamma = 0.1$ to 0.49. The parameter β was found to be equal to 6 by experiment.

Relationship (13-6) was derived for pressures falling between 0.1×10^5 and 100×10^5 Pa (the Jacob number is between 1000 and 0.1).

It follows from the relationships quoted above that vapour bubbles increase in size with the Jacob number. At low pressures, however, the effect of the Jacob number is more essential than at high pressures, which means that at low pressures vapour bubbles grow at a rate greater than at high pressures. An increase in liquid superheat temperatures results in a greater rate of bubble growth in both cases.

(e) Separation Diameter of a Bubble

The vapour bubble generated on the wall grows to a certain size, characterized by the diameter d_0 at which it separates from the wall.

The dimension of a bubble at the end of its growth on the heat-exchange surface is called the *separation diameter*. Throughout the period of formation and growth a vapour bubble is acted upon mainly by the forces retaining it at the starting points of vapourization. As the bubble grows in size the buoyancy force increases tending to separate the bubble from the starting point. The condition of equilibrium of forces can be used to derive analytical expressions for the separation diameter of a bubble. In the general case the forces acting upon a bubble include the buoyancy forces, surface tension forces, inertial forces and drag forces. The latter two forces are classified as forces of hydrodynamic origin, since they originate with the flow of a mass of liquid provoked by the growth of a bubble [13.8-13.10].

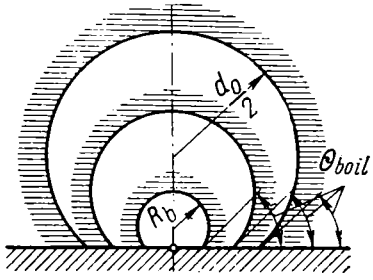

Fig. 13-3. Schematic diagram of growth of a vapour bubble:
d_0—separation diameter of vapour bubble

Under static conditions the separation diameter is determined from the conditions of mechanical equilibrium between the buyoancy, tending to tear the bubble from the heating surface, and the surface tension, retaining it on the hard surface.

The growth of a vapour bubble is illustrated in a simplified form in Fig. 13-3. In reality, it should be borne in mind, even if the dynamic effect is not taken into account, that the shape of the bubble will deviate more and more from the initial spherical shape, as the bubble grows. This is explained by the increasing role of the gravitational field, which tends, as it were, to elongate the bubble in a direction away from the heating surface. At the moment of detachment the bubble is deformed considerably.

Fritz [13.11] theoretically calculated the volumes of vapour bubbles before separation from the surface, under static conditions, and for various contact angles. The results of calculations can be interpolated by a simple formula. If the separation diameter of the bubble d_0 is assumed to be equal to the equivalent diameter $\sqrt[3]{6V_0/\pi}$, where V_0 is the volume of the deformed bubble at the moment of separation, the formula has the following form:

$$d_0 = 0.0208\theta \sqrt{\sigma/g\,(\rho_l - \rho_v)} \qquad (13\text{-}7)$$

(the contact angle θ being measured in degrees).

It follows from Eq. (13-7) that the separation diameter of a bubble depends on the contact angle θ. The wetting of a surface by a boiling liquid deteriorates with increase of the contact angle, and vapour bubbles of a larger size separate from the surface.

When θ becomes larger than $\pi/2$ (i.e. the liquid does not wet the surface), the fraction of the heating surface that is screened by the bases of growing bubbles increases sharply. Liquid is displaced, as it were, from the surface, and the intensity of heat transfer therefore diminishes. Non-wetting liquids include mercury whose angle of contact $\theta = 140°$.

Liquids finding application in cryogenic engineering (hydrogen, oxygen, nitrogen) are characterized by small contact angles θ. It should be noted that

the heat-carrying agents finding usage in power engineering are capable of wetting metallic surfaces.

With forced circulation of the boiling liquid separation of vapour bubbles is further affected by the dynamic head of liquid flow. Due to the hydrodynamic effect of the flow on the bubble the separation diameter becomes smaller than d_0 determined by Eq. (13-7) and is determined by the thickness of the layer of liquid at the wall. The thickness of the boundary layer varies, depending on the Reynolds number and the intensity of vapourization. The separation diameter of the bubble d_0 also depends on these two factors, so that both thickness of the liquid boundary layer and the separation diameter d_0 decrease with increasing Re.

After separation a vapour bubble moves through the layer of liquid. The transfer of heat between the superheated liquid and the surface of a bubble is distinguished by a high intensity. So, the coefficient of heat transfer from water to steam reaches about 200 000 W/m^2·K. As a result, the rising bubble increases in size considerably.

(*f*) *Dependence of Heat Flow on the Temperature Difference (Boiling Curve)*

We were considering above the conditions under which a vapour phase originates and develops at a heated surface, which may at the same time determine the intensity of heat transfer between this surface and the boiling liquid. In a boiling liquid the growing and periodically separating vapour bubbles cause a random turbulization of the liquid-vapour mixture near the heating surface. In addition, the intensity of heat transfer is related with the thermal resistance offered to heat flow by the rather thin layer of liquid (film) remaining, due to the phenomenon of wetting, on the heating surface underneath the region of vapour bubbles and through which heat is transferred by conduction.

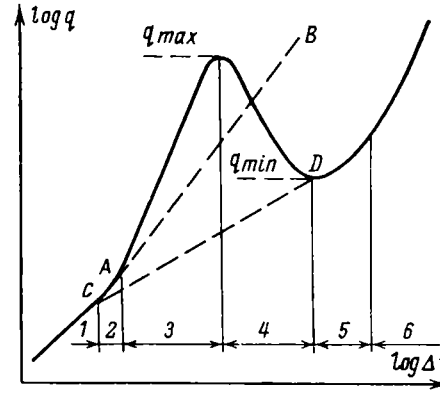

Fig. 13-4. Dependence of α and q on liquid superheat Δt

With an increase in turbulization and thinning of the liquid-vapour sublayer the rate of heat transfer increases. The effect of turbulization however is essential only at slight superheating of the liquid, and the determining factor is the varying thickness of the liquid-vapour sublayer underneath the vapour bubbles. An increase in superheat temperature causes a higher rate of vapourization at each of the starting points and an increase of the number of starting points, the thickness of the liquid sublayer diminishes and the intensity of heat transfer increases.

Let us investigate how the rate of heat transfer depends on liquid superheat (consider the boiling curve).

With increasing temperature difference, the rate of heat flow passes through a maximum (Fig. 13-4) which is preceded by a convection region *1*, corresponding to low liquid superheat temperatures and the region of bulk boiling *3*. Arranged between these two regions is the region of unsteady boiling *2* which is characterized by a small number of starting points of vapourization per unit area.

Having passed its maximum q reduces gradually, as nucleate boiling is replaced by film boiling. The transition region *4* is followed by the region of steady-state film boiling where the radiation heat transfer is comparatively small along section *5*, and it becomes of essential importance along section *6*.

The heat-transfer coefficient also increases with an increase in Δt in the regions of nucleate and film boiling. In the latter case, however, α increases considerably less than the increase in the rate of heat transfer q.

The shown boiling curve does not embrace the all possible modes of boiling as has been proved in the works of V.I. Subbotin and other authors [13.12]. So, with thorough preliminary degassing of the system, and also with boiling under lower pressures delay of heat transfer by convection may take place resulting in high superheating of the liquid (line AB). The upper borderline of these superheats is determined by spontaneous formation of vapour bubbles in the bulk of liquid.

With boiling non-wetting liquids ($\theta > 90°$) film boiling may set in at small superheats (line CD).

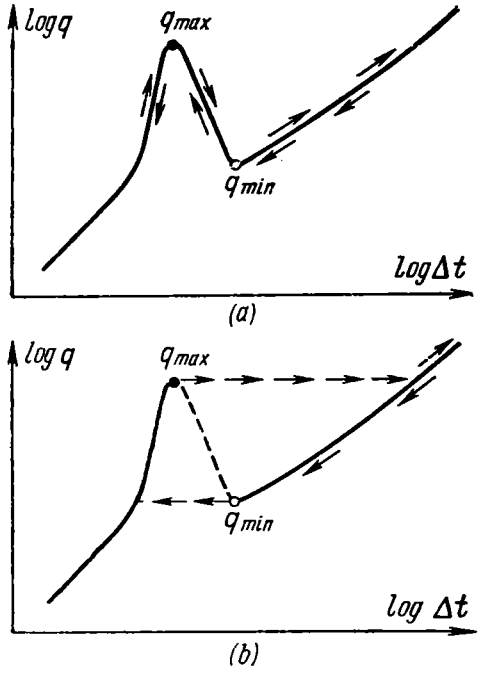

Fig. 13-5. Dependence of q on Δt in a boiling liquid:
a—transition to film boiling with steam heating; *b*—same with electrical heating

(g) *Effect of the Method of Heating on the Heat Transfer Surface*

The boiling curves shown in Figs. 13-4 and 13-5*a* illustrate the gradual change or transition from nucleate to film boiling which is observed when the condensing steam is used for heating purposes. These conditions are characterized by that the temperature of the heat-transfer surface becomes constant ($T_w = \text{const}$). With steam (vapour) heating surface temperature T_w is independent of the process of heat transfer and, consequently, this is also true of the temperature difference $\Delta t = T_w - T_s$. Therefore, the amount of heat transferred from the surface in the transition region diminishes gradually as the intensity of heat transfer deteriorates, traced to the positive replacement of nucleate by film boiling.

With the rate of heat flow being independent of the process ($q_w = \text{const}$) there occurs a jump-like change from nucleate to film boiling and back (Fig. 13-5*b*). The constancy of the rate of heat flow q_w is characteristic of electric heating. If the amount of power supplied is not reduced, the system will come into a state distinguished by a considerable rise in wall temperature. The moment of transition from one kind of boiling to the other is identified experimentally by the sharp increase in the temperature or electrical resistance of the wall. The moment of transition from nucleate to film boiling may be accompanied by destruction (melting) of the heat-transfer surface. If the rate of heat flow, or the heat flux, q_w is reduced as soon as film boiling sets in, the change back to nucleate boiling will occur only at q_{min}. Thus, a kind of thermal hysteresis appears when the heat-transfer surface is heated by electricity.

(h) Dependence of Heat Transfer on Pressure and Thermal and Physical Properties

With increasing pressure the dimensions of a bubble decrease at the moment of formation and separation from the wall; the number of starting points increases as well as the frequency with which the bubbles depart from these points. The effect of pressure on the bubbles and phenomena involved depends on the remoteness of the state being considered and the critical state, since it determines the degree of metastability of liquids, the probability of heterogeneous density fluctuations and also the quantitative variation of the physical properties

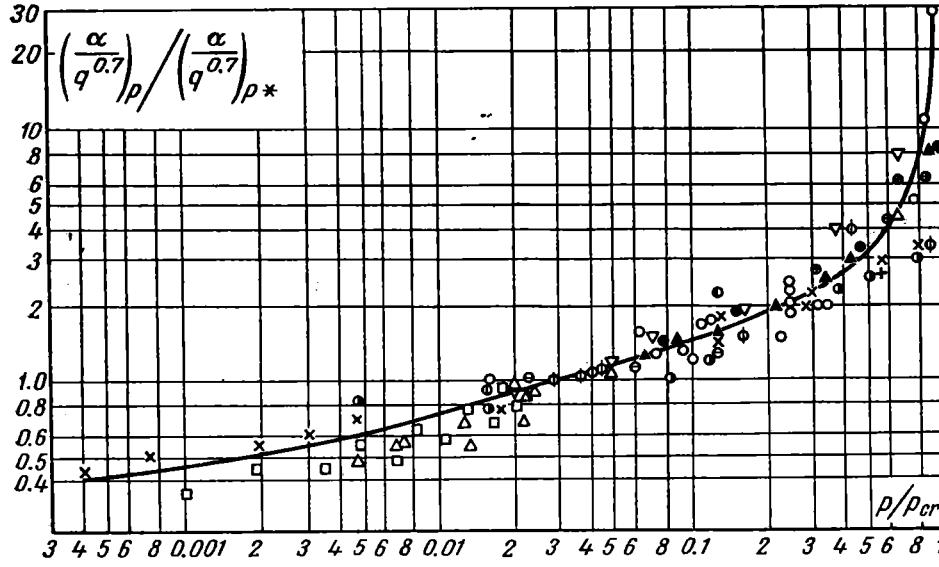

Fig. 13-6. Dependence of α on p for various liquids

of a substance. As the thermodynamic state approaches the critical, the effect of these factors increases and the effect of pressure on heat transfer becomes correspondingly more important.

The dependence of heat transfer on pressure is presented graphically in dimensionless coordinates for a number of liquids in Fig. 13-6. Test data are plotted along the y-axis in terms of the ratio of $\alpha/q^{0.7}$ at the current pressure p and at a pressure $p^* = 0.03\ p_{cr}$. Different liquids have different critical pressure; p^* is the same fraction of p_{cr} and it characterizes in the same way the remoteness from the critical state in respect to pressure selected for all liquids. The quantity 0.03 is chosen arbitrarily with account taken of the fact that maximum experimental data are available. In Fig. 13-6 the ratios of the actual pressure to the critical are plotted along the abscissa [13.13].

The thermal and physical properties of a liquid also change with pressure (temperature) materially.

The rate of heat transfer rises with an increase in thermal conductivity, since the resistance offered to heat flow by the microlayer of liquid underneath the vapour bubbles diminishes and the rate of bubble growth increases.

On the contrary, the rate of heat transfer diminishes with increasing viscosity, since the thickness of the microlayer of liquid increases and the intermixing,

due to separation of the bubbles from the surface, decreases. The height of the layer of liquid topping the heat-transfer surface may affect heat transfer if it is commensurable with the size of vapour bubbles ($H \ll d_0$), as shown graphically in Fig. 13-7.

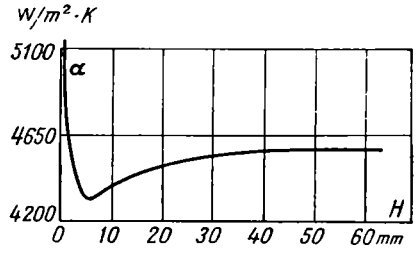

Fig. 13-7. Dependence of heat-transfer coefficient on the level of boiling water (q = 40 000 W/m²; $p = 10^5$ Pa)

(i) Effect of Liquid Subcooling

Considered above was the effect of external pressure and liquid superheat on processes of vaporization and heat transfer. In addition, the factors affecting heat transfer include subcooling of the liquid involved and the rate of circulation. In the first case boiling with liquid subcooling takes place.

By boiling with subcooling (surface boiling) is meant the process of boiling at the heat-transfer surface when at a distance from it the liquid is not heated to the saturation temperature (Fig. 13-11). The vapour bubbles, originating in the boundary layer of the boiling liquid, condense as they get into the bulk of the liquid. Thus boiling at the wall is combined with convection of a single-phase liquid at a distance from it and condensation of vapour at interface between the boiling boundary layer and cold mass of liquid. The intensity of vaporization on the wall depends on the degree of superheat of the liquid; the process of condensation is determined by the temperature difference $\Delta t_s = t_s - t_l$, i.e. by subcooling of the liquid. If superheating determines the intensity of vaporization, subcooling determines the size of the region that is affected by the disturbing action of vaporization. The greater the subcooling of liquid, the narrower is the region where boiling takes place. If the extent of subcooling is low, vapour bubbles separate from the surface and condense in the flow; with appreciable subcooling the bubbles condense not departing from the surface. The processes of heat transfer in surface (local) boiling are of great practical importance, as they make it possible to ensure greater heat fluxes than with free convection in a single-phase liquid. They are used to cool aircraft engines and rockets, and in the equipment used for continuous casting of steel, etc. One of the demerits of surface boiling consists in the possible occurrence of high-frequency pressure fluctuation in a working conduit.

(j) Effect of the Rate of Forced Liquid Circulation

With forced circulation of a two-phase flow supplementary disturbances due to turbulent fluctuation of velocity are superimposed on the disturbances of the boundary layer due to vaporization. Forced circulation also directly affects the mechanism of vaporization. This effect manifests itself in distortion of the natural wetting angle θ, detachment of vapour bubbles from the wall before they grow to the separation diameter characteristic of free-convection boiling. At small circulation velocities the process is only slightly affected hydrodynamically, and heat transfer in tubes is determined by the intensity of vaporization, i.e. by the magnitude of q_w (Fig. 13-8). Heat transfer increases considerably with increase of q_w. With rising w the effect of q_w decreases continuously, and gradually heat transfer coefficient approaches the values proper for free convection in a single-phase liquid (i.e. with no boiling at $\alpha \sim w^{0.8}$).

Figure 13-9 shows the dependence of α on q_w at different rates of circulation [13.14]; it follows that the effect of q_w on α diminishes continuously with rising w.

The dependence of heat transfer on heat flux at various rates of circulation under conditions in bulk boiling is similar to that for boiling in tubes [13.15].

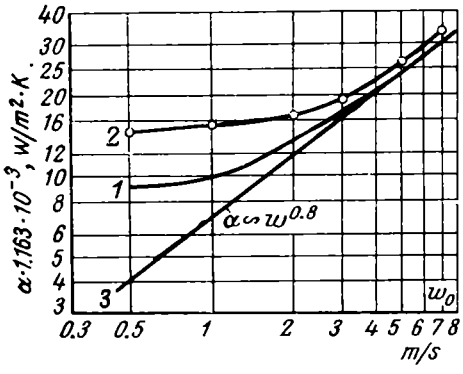

Fig. 13-8. Dependence of α on w with liquid boiling in tubes:
1—q = 233 000 W/m²; *2*—q = 465 200 W/m²; *3*—calculated from $\alpha \backsim w^{0.8}$

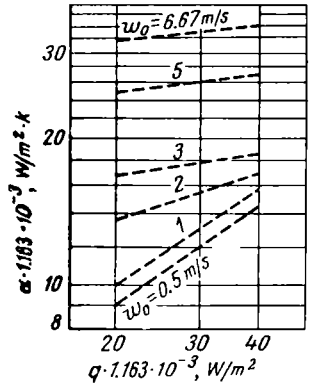

Fig. 13-9. Dependence of α on q for liquid boiling in tubes under forced circulation

Thus, the effect of q_w and w on α is determined by their proportion; as a result three regions can be distinguished: in two limiting cases $\alpha = \alpha(w)$ or $\alpha = \alpha(q_w)$; in the general case $\alpha = \alpha(q_w, w)$ (Fig. 13-17).

(*k*) *Effect of Roughness, Thermal and Physical Properties of a Wall*

Upon boiling of ordinary (high-temperature) liquids the only surface irregularities capable of serving as starting points of vapourization are the recesses and depressions in the heat-transfer surface which are able to retain vapour or gas. Large recesses are easily filled with the liquid and they cease to serve in the capacity of active starting points of vapourization. There exists, therefore, a border of roughness beyond which further roughing of the heat-transfer surface does not cause any change in the intensity of heat transfer. It was shown [13.16] that this phenomenon is observed when dealing with heat-transfer surfaces machined with an accuracy exceeding the 6th or 7th grade of surface finish. The thermal and physical properties of the wall affect the intensity of heat transfer. The thermal and physical properties manifest themselves most distinctly upon boiling of cryogenic (low-temperature) liquids due to their wetting ability ($\theta \to 0$) and it being possible to preclude the effect of the contact angle, which differs for different liquids and various wall materials. The effect of the properties of materials tells itself in the magnitude of the *heat accumulation factor for the wall involved*, equal to $(\sqrt{ac_p\rho})_w$.

With an increase of the heat accumulation factor the intensity of heat transfer rises. However, it is yet impossible to account quantitatively for the effect of surface phenomena on the intensity of heat transfer upon boiling.

13-2. The Structure of Flow for Liquid in Nucleate Boiling in an Infinite Volume

The structure of a two-phase flow depends on the geometric properties of the system. Infinite-volume systems are relatively large reservoirs (vessels) filled with liquid into which are immersed various heat-transfer surfaces, such as single tubes, banks of tubes, etc. heated from the inside. The vapour forming as the liquid boils on their external surfaces is branched off without hindrance from the system. Let us consider a system consisting of a vessel filled with a liquid; the horizontal bottom of the vessel is heated. Temperature variation in the liquid with distance from the heated bottom is shown graphically in Fig. 13-10. The liquid is considerably superheated at the wall. At a distance from the wall the liquid is also somewhat superheated.

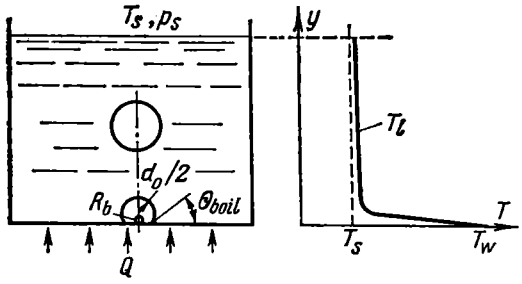

Fig. 13-10. Formation, growth, and separation of a vapour bubble from a starting point:
T_s, p_s—saturation temperature and pressure; T_w—wall temperature; Q—heat flux

With developed boiling there is a considerable number of starting points of vapourization on the heating surface. The concurrent growth of many bubbles and their periodic detachment from the surface cause intensive mixing of the boundary layer of liquid. Bubbles of various size detach from the surface. The rate of rising of large bubbles is greater than that of small ones. Some of the larger bubbles split up into a number of smaller bubbles as they rise from the heating surface, while small bubbles may coalesce and form larger ones. Small bubbles may also coalesce directly on the heating surface before separating from it. Thus the picture of boiling is very complex. The free surface of the liquid becomes intensively agitated.

At any moment of time a boiling liquid contains a definite quantity of vapour in the form of rising bubbles. In consequence the volume of this two-phase mixture swells as it were; the swelling manifests itself in a rise of the mid-position of the free (evaporating) surface. If the mass fraction of vapour contained in the liquid at any moment is denoted by M_v, and the mass fraction of the liquid by M_l, the total volume of the two-phase mixture will be $V_{mix} = (M_v/\rho_v) + (M_l/\rho_l)$. The ratio of the volume of vapour M_v/ρ_v to the volume of the mixture is known as the *volumetric vapour fraction*:

$$\varphi = \frac{M_v/\rho_v}{V_{mix}}$$

The value of φ depends on the shape and size of the heating surface and vessel, the heat flux, and the kind of liquid, and in a number of cases the values of φ may often reach 30 per cent or more. The height of a boiling two-phase layer increases by about the same percentage. In this respect the process is similar to that of steam flow through a layer of liquid.

The rate of rise of large vapour (or gas) bubbles in a quiet liquid is practically independent of the size of bubbles, when their reference dimension exceeds the *capillary constant* $\sqrt{\sigma/g\,(\rho_l - \rho_v)}$. Such bubbles are nearly oblate spheroids (flattened in the direction of rising). Their velocity of rise is usually about se-

veral tens of centimetres per second, and is determined by the formula

$$w_r = 1.18\sqrt{\sigma g(\rho_l - \rho_v)/\rho_l^2}$$

i.e. it depends on the Archimedean buoyant force $g(\rho_l - \rho_v)$, the surface tension σ, and the liquid density ρ_l.

Rising vapour bubbles of a diameter d small in comparison with the capillary constant are spherical, and their rising velocity is determined by the law of viscous resistance

$$w_r = cg(\rho_l - \rho_v)d^2/\mu_l$$

where c depends on the presence of surface-active components in the liquid and ranges from 2/9 to 1/3.

For the system depicted in Fig. 13-10 the total heat flux Q from the heating surface to the boiling liquid equals qF, where F is the area of heating surface, equal in this case to the cross-sectional area of the vessel. All the imparted heat is used for evaporation; therefore, the velocity of vapour flow from the evaporating surface (free surface) can be determined from the heat balance equation

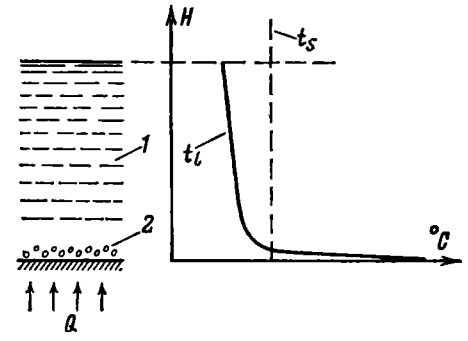

Fig. 13-11. Temperature profile in subcooled liquid:
1—single-phase liquid; *2*—boiling boundary layer

$$w_* = \frac{Q}{r\rho_v F} = \frac{q}{r\rho_v}$$

This rate is called the *reduced rate of evaporation* (boiling). The actual vapour velocity in the two-phase layer must be greater than this value, because the area occupied by the vapour fraction of the mixture in each horizontal cross-section on average is only φF. From the continuity equation it follows that

$$w_v = w_*/\varphi$$

This is an approximate relationship, because it is derived without taking into account the growth of rising bubbles. At high saturation pressures however the error involved is apparently not large.

The actual (or real) velocity of the vapour w_v in the two-phase layer usually exceeds the velocity of individual bubbles in liquids of low fluidity. This is due to the fact that, with developed boiling, liquid moves quite intensively above the heating surface in individual streams or columns entrained by the rising bubbles. The descending motion, compensating for this buoyant or ascending motion of liquid in the central section of the vessel, occurs near the walls, where there are fewer bubbles and the liquid is "heavier" on average. In consequence most of the bubbles rise in the ascending flow of liquid. Their velocity with respect to the walls of the vessel, therefore, is larger than the rising velocity calculated by the formulae for liquids of low fluidity.

The structure of vapour flow is quite different in boiling of a liquid not heated to the saturation temperature (Fig. 13-11). The two-phase state is observed here only near the wall. Surface boiling originates at a temperature difference Δt_{in}, at which the rate of heat flow can be determined based on the conditions of convection of a single-phase liquid. On the other hand, the rate of heat flow can be found from the conditions of liquid boiling q_{boil} (see below). The heat-balance

equation makes it possible to express the unknown as

$$\Delta t_{in} = t_w - t_l = q_{boil}/\alpha$$

wall temperature t_w, then find q_{boil}, at which surface boiling begins.

The heat flux corresponding to the beginning of boiling increases with rising liquid subcooling [13.4].

13-3. Heat Transfer with Liquid in Nucleate Boiling Under Conditions of a Natural Circulation

The heat transfer in boiling liquid is a very complicated process. A great variety of hydrodynamic regimes of flow is observed, depending on the concrete environmental conditions, and the quantitative relationships of transfer corresponding to them are extremely complex. Therefore no strict theory for the process of heat transfer at liquid boiling is as yet available.

Approximate theories permit different approaches to the process of heat transfer. One of the most important directions followed in the theory of heat transfer in boiling liquids consists in attempting to relate quantitatively the characteristics of microboiling (the sizes and representative bubble velocities, bubble separation frequency, number of starting points of vapourization, etc.) and the integral characteristics (q, α) required for engineering calculations.

An approach is possible involving the expression of the heat-transfer coefficient in terms of the quantities determining the conductivity of the liquid sublayer filling the space underneath the bubbles [13.17, 13.6]. The heat-transfer coefficient and the factors effecting it are usually related experimentally, making use of the theory of similarity.

Analysis of the conditions of similarity [13.18] is based on the following initial assumptions. A single-component wetting liquid ($\theta < \pi/2$) is considered at constant physical parameters in conditions of free flow or free convection. The heat flux created by the heating surface is assumed to be imparted to the liquid phase only; boiling is nucleate, and takes place on a horizontal flat wall (Fig. 13-10). The size of the heating surface is large, compared with the size of the vapour bubbles forming. The temperature field in the *liquid phase* is determined by the system of differential equations for convective heat transfer. It includes:

(a) the *equation of energy*

$$\frac{\partial t}{\partial \tau} + \vec{w}\,\mathrm{grad}\,t = a_l \nabla^2 t$$

(b) the *equation of flow*

$$\frac{\partial \vec{w}}{\partial \tau} + (\vec{w}\,\mathrm{grad})\,\vec{w} = -\frac{1}{\rho_l}\,\mathrm{grad}\,p + \nu_l \nabla^2 \vec{w}$$

(c) the *equation of continuity*

$$\mathrm{div}\,\vec{w} = 0$$

To this system are added:

the *equation of motion for the vapour bubble*, describing the conditions of equilibrium between the forces of buoyancy and the hydraulic resistance offered to the rising bubble:

$$\left(\frac{\pi}{6}\right) d^3 g\,(\rho_l - \rho_v) = c\left(\frac{\pi}{4}\right) d^2 \frac{\rho_v u^2}{2}$$

where d = vapour bubble diameter;
$u = w_v - w_l$ = relative velocity of rising bubble;
c = drag coefficient.

The *equation of heat transfer* on the bubble surface, characterizing the quantity of heat flowing to the surface of the bubble due to conduction and expended to ensure evaporation of the liquid inside the bubble, which predetermines its growth in volume:

$$-\int_F \lambda_l \left(\frac{\partial t}{\partial n}\right)_{n=0} dF = r\rho_v \frac{d}{d\tau}\left(\frac{\pi d^3}{6}\right)$$

where n = normal to the external surface of the vapour bubble;
dF = bubble surface element.

In real conditions there are z starting points of evaporation per unit of surface area, with

$$z = f(d_0, R_{cr})$$

For the *conditions of unambiguity* we assume that the temperature of liquid at the free surface is equal to saturation temperature t_s, and that a constant temperature t_w is given for the heating surface.

Analysis of this system of differential equations and conditions of unambiguity by the theory of similarity enables the following similarity equation to be obtained

$$\frac{\alpha d_0}{\lambda_l} = f\left(\frac{\nu_l}{a_l};\ \frac{R_{cr}}{d_0};\ \frac{c_p \Delta t}{d_0}\frac{\rho_l}{\rho_v}\right) \tag{13-8}$$

According to Eq. (13-7), at θ = const, the diameter d_0 of the vapour bubble is proportional to the capillary constant $\delta = \sqrt{\sigma/g(\rho_l - \rho_v)}$. As follows from Eq. (13-4), the value R_c depends on the temperature difference $\Delta t = t_w - t_s$. It is, therefore, more suitable to rearrange the dimensionless terms in Eq. (13-8) by replacing δ for d_0. Let us denote:

$$\mathrm{Nu} = \frac{\alpha\delta}{\lambda_l}$$

$$\mathrm{K}_z = \frac{R_{cr}}{2\delta}\frac{c_p\Delta t}{r}\frac{\rho_l}{\rho_v}$$

$$\mathrm{K}_q = \frac{q\delta}{\rho_v r a_l}\frac{\delta}{l_*}$$

$$\mathrm{Pr} = \frac{\nu_l}{a_l}$$

where

$$l_* = \frac{R_{cr}}{2}\frac{c_p\Delta t}{r}\frac{\rho_l}{\rho_v} = c_p\sigma\rho_l T_s/(r\rho_v)^2 = \frac{1}{2}R_{cr}\,\mathrm{Ja}$$

The value of l_* is the linear reference dimension, measured in units of length and is proportional to the critical radius R_{cr}, ratio of liquid superheat enthalpy to the heat of vaporization and ratio of the vapour and liquid phase densities.

Equation (13-8) then can be presented in the form

$$\mathrm{Nu} = 0.082\mathrm{K}_z^{0.33}\mathrm{K}_q^{0.7}\mathrm{Pr}_f^{-0.45} \tag{13-9}$$

The dimensionless terms K_z and K_q determine respectively the number of acting starting points of vapourization and the frequency with which vapour bubbles depart from these starting points.

With bulk boiling the rate of heat transfer on immersed surfaces is independent of the shape and orientation (arrangement in space) of the heat-transfer surface (provided unimpeded branching-off of the separated vapour bubbles is ensured). This means that acceleration due to gravity must not affect appreciably the rate of heat transfer.

If we assume heat transfer to be self-similar with respect to the acceleration of gravity g, the dimensionless equation is simplified and acquires the following form [13.19]:

$$\mathrm{Nu}_* = f_1(\mathrm{Re}_*, \mathrm{Pr}_l) \tag{13-10}$$

The notation used in Eq. (13-10) is as follows:

$$\mathrm{Nu}_* = \frac{\alpha l_*}{\lambda_l}; \quad \mathrm{Re}_* = \frac{w_* l_*}{\nu_l}; \quad \mathrm{Pr}_l = \frac{\nu_l}{a_l}$$

The main thermal resistance offered to heat flow is the surface film of liquid underneath the vapour bubbles. The thickness of this film increases with increasing ν_l and diminishes with rising intensity (or rate) of vapourization. The Reynolds number Re_* is the measure of the influence produced by these effects.

The heat-transfer coefficient for a saturated liquid in surface boiling is expressed as

$$\alpha = \frac{q}{t_w - t_s}$$

The similarity equation (13-10) has been used to work up the numerous test data on the boiling of various liquids (including liquid metals), and the following formula has been obtained [13.20]:

$$\mathrm{Nu}_* = c\mathrm{Re}_*^n \mathrm{Pr}^{1/3} \tag{13-11}$$

The physical parameters in the dimensionsless terms are based on saturation temperature.

The constants for boiling non-metal liquids are:

$$\text{at } \mathrm{Re}_* \leqslant 0.01, \; c = 0.0625, \; n = 0.5;$$

$$\text{at } \mathrm{Re}_* \geqslant 0.01, \; c = 0.125, \; n = 0.65$$

The relationship holds for the following ranges of the Reynolds and Prandtl numbers, and velocity:

$$\mathrm{Re}_* = 10^{-5}\text{-}10^{+4}; \quad \mathrm{Pr}_l = 0.86\text{-}7.6; \quad w \leqslant 7 \text{ m/s}$$

for a wide range of saturation pressure (to the near-critical pressure).

For boiling liquid metals $\mathrm{Re}_* \geqslant 0.01$; the exponent of the Prandtl number is assumed to be equal to 0.65.

These relationships do not contain the vapour fraction, for its effect on heat transfer is insignificant for pool boiling.

The approximate self-similarity of heat transfer with respect to the acceleration of gravity g (or, what is the same, to the value of the separation diameter d_0) in developed nucleate boiling has been corroborated by a number of experiments carried out both under excessive g and at small values of acceleration of gravity, i.e. under conditions approaching zero gravity or weightlessness. The

fact that practically the same regularities govern both free and forced circulation of bulk boiling liquid is also explained by these considerations. A number of external factors (vibration of the heat-transfer surface, superimposition of electric fields, etc.) affect heat transfer only at small rates of heat flow. But the effect produced by these factors vanishes gradually with increasing q_w [13.17].

With bulk boiling the heat-transfer coefficient of a given liquid depends only on the rate of heat transfer (heat load) and saturation pressure. Empirical dimensional formulae, therefore, are very suitable for practical calculation. These formulae are either based directly on test data, or generalized dimensionless relationships are used to derive them. For water boiling at pressures between 1 and 40 bar ($p/p_{cr} \leqslant 0.18$, Fig. 13-6) the following formulae were obtained [13.21, 13.22]:

$$\alpha = 3.0q^{0.7}p^{0.15} \tag{13-12}$$

$$\alpha = 38.7\Delta t^{2.33}p^{0.5} \tag{13-13}$$

in which q and p must be substituted respectively in W/m² and bar.

13-4. Structure of Two-phase Flow and Heat Transfer With Liquid Boiling in Tubes

(*a*) *Vertical Tube*

A tube or conduit is a restricted system in which the flow of a boiling liquid is accompanied by a continuous increase of the vapour fraction and decrease of the liquid fraction of a two-phase mixture. The hydrodynamic structure of flow changes accordingly, both along the tube and over its cross-section and, consequently, heat transfer also changes.

Three main regions of different structure of flow are observed when liquid is in downward flow in a vertical tube (Fig. 13-12): *I*—a *heating region (economizer section)* to that part of the tube where the wall temperature T_w is equal to saturation temperature T_s; *II*—a boiling region (evaporating section), extending from the cross-section where $T_w = T_s$, $i_l < i_s$, to the cross-section where $T_w > T_s$, $i_{mix} \to i_v$; and III—a region where the wet vapour dries.

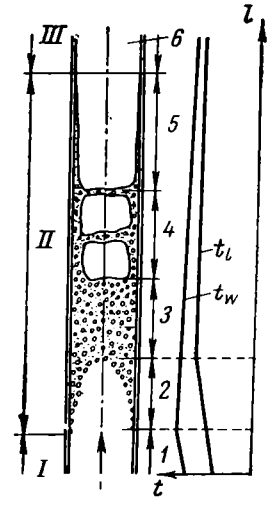

Fig. 13-12. Structure of flow for liquid boiling in a vertical tube:
t_w and t_l—wall and liquid temperature

The *evaporating section* includes the regions of surface boiling and saturation boiling of the liquid. The section of tube where boiling of saturated liquid occurs includes the regions of *emulsion 3*, *slug-type 4* and *annular 5 flows**. Emulsion flow consists of liquid and uniformly distributed fine vapour bubbles. Some of these fine bubbles coalesce with an increase in the vapour fraction and form large slugs commensurable with the diameter of the tube. In the region of

* In respect to the liquid phase it is called the *annular regime*.

slug flow individual large bubbles (slugs) are separated from one another by layers of vapour-liquid emulsion. Further increase in the vapour fraction brings about merging of the slugs and the formation of the annular structure of flow, in which there is a continuous flow of wet vapour at the core, and a thin annular layer of liquid flowing near the wall of the tube. The thickness of the annular layer gradually decreases, owing to evaporation, and the region of annular flow turns into one of drying *6* after complete evaporation of liquid has occurred. A *drying region* (*dispersed regime*) is observed only in long tubes.

An increase in the rate of circulation (see Sec. 12-3) at a given q_w, length of tube, and inlet temperature brings about a reduction in the section of developed boiling and an increase in the length of the economizer section of the tube; and, on the contrary, an increase in q_w at a given rate of circulation results in an increase of the length of the developed boiling section and a reduction of the economizer section of the tube.

(*b*) *Horizontal and Inclined Tubes*

In addition to these changes in flow structure along the tube, two-phase flow in horizontal or slightly inclined tubes undergoes considerable variation in structure over the circumference. Thus, if the rate of circulation and the vapour fraction are not large, separation of the flow is observed into a liquid phase, moving in the bottom section of the tube, and a vapour phase flowing at the top (Fig. 13-13*a*). Further increase of the vapour fraction and of the rate of circulation results in the appearance of a wave-like interface between the two components of the flow, and liquid periodically wets the top of the tube. With a further increase in the vapour fraction and velocity the wave motion at the interface intensifies, and leads to partial displacement of liquid into the vapour space. In consequence, the two-phase flow first becomes nearly slug-type, then annular flow sets in. The annular flow consists of a thin layer of liquid moving at the tube circumference, while its inner space is filled with moving vapour-liquid emulsion (Fig. 13-13*b*). Complete axial symmetry of flow, however, is not observed in this case.

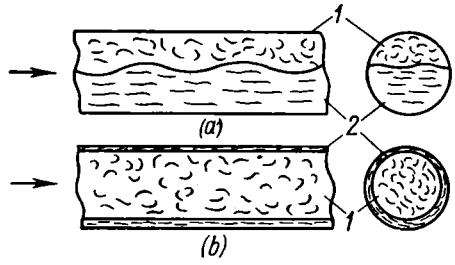

Fig. 13-13. Structure of flow for liquid boiling in a horizontal tube:
a—laminated regime of boiling; *b*—annular regime of boiling; *1*—vapour; *2*—liquid

Let us examine the characteristics employed to describe the flow of two-phase mixtures in tubes and conduits.

The total mass flow rates (kg/s) of liquid and vapour G_{mix} is:

$$G_{mix} = G_l + G_v$$

and is constant and identical for any cross-section.

The ratio of the vapour flow rate to the mass flow rate of the mixture

$$x = \frac{G_v}{G_{mix}} = \frac{G_v}{G_l + G_v} \tag{13-14}$$

is known as the *mass fraction of vapour.*

When boiling liquid moves along a long tube x may change in the general case from zero ($x = 0$, when only liquid flows) to unity ($x = 1$, when only vapour flows). Thus, the mass rates of vapour flow are within the limits of $0 \leqslant x \leqslant 1$.

The volume rates of flow corresponding to the mass flow rates G_l and G_v (m^3/s) are as follows:

$$V_l = G_l/\rho_l;\ V_v = G_v/\rho_v$$

The sum of the two is referred to as the volume rate of mixture flow:

$$V_{mix} = V_l + V_v$$

In contrast to the mass rate of mixture flow the volume rate of flow varies generally along the tube, and can vary from $V_{mix} = G_{mix}/\rho_f$, if there is only liquid in the inlet section of the tube, to $V_{mix} = G_{mix}/\rho_v$, if there is only vapour in the outlet section. With full vapourization of the liquid the volume rate of flow changes ρ_f/ρ_v times. Liquid velocity at the tube inlet ($x = 0$) corresponds to the rate of circulation (12-28). With full evaporation of the liquid the outlet velocity of the vapour is also ρ_f/ρ_v times greater than the rate of circulation. At low pressures, when $\rho_f \gg \rho_v$, velocity increases considerably. When liquid boils inside tubes and conduits, therefore, there is a considerable acceleration of flow as the vapour fraction increases.

The volumetric fraction of vapour is equal to the ratio of the volume flow rate of the vapour to that of the mixture:

$$\beta = \frac{V_v}{V_{mix}} = \frac{V_v}{V_l + V_v} \tag{13-15}$$

The values x and β are interrelated by the expression

$$\frac{x}{1-x} = \frac{\rho_v}{\rho_l}\,\frac{\beta}{1-\beta}$$

from which x can be calculated in terms of β and vice versa. At $x = 0$, $\beta = 0$; at $x = 1$, $\beta = 1$. In the remainder of the region x is always smaller than β, since $\rho_v/\rho_l < 1$.

In a two-phase flow through any section of the tube, the cross-section f is filled partly with vapour (f_v) and partly with liquid (f_l).

The part of the flow cross-section filled with vapour is called the *true vapour fraction*:

$$\varphi = \frac{f_v}{f} = \frac{f_v}{f_l + f_v} \tag{13-16}$$

The value of this fraction is of great importance for calculating boiling nuclear reactors.

The *real velocities* of the liquid and vapour in a given cross-section are related to φ as follows:

$$w_v = \frac{V_v}{f_v} = \frac{V_v}{f\varphi};\quad w_l = \frac{V_l}{f_l} = \frac{V_l}{f(1-\varphi)}$$

The values V_v/f and V_l/f are often referred to as the *reduced velocities* of vapour and liquid flow:

$$w_{vr} = \frac{V_v}{f};\quad w_{rl} = \frac{V_l}{f}$$

The term "reduced" is introduced because the true volume flow rates are here reduced to the full section of the conduit. The reduced velocities are fictitious values. The sum of the reduced velocities of the vapour and liquid charac-

terize the true velocity of the mixture

$$w_{mix} = w_{vr} + w_{rl} = \frac{V_v + V_l}{f}$$

The difference between the true velocities of the two components of the liquid-vapour mixture is called the *slip velocity*:

$$u_{sl} = w_v - w_l$$

If the slip is positive, the vapour moves at a velocity greater than that of liquid. Slip is positive in upflow in vertical tubes and in horizontal tubes. It is negative in downward flow in vertical tubes.

With emulsion and slug-type flow the vapour phase is not sufficiently dispersed, so that slip is slight if large rates of circulation are maintained. In annular flow, however, slip u_{sl} can be of considerable magnitude owing to the separation of the two phases. The relative slip diminishes in all cases with an increase in the rate of circulation. At $u_{sl} = 0$, $w_f = w_v$; $\varphi = \beta$.

The true vapour fraction φ is thus equal to the volumetric fraction of vapour β when the real velocities of vapour and liquid flow coincide.

Let us find the mixture enthalpy i_{mix}. Let the liquid be at the saturation temperature T_s at the tube inlet and possess a saturation enthalpy denoted by i_s. If it is known that a quantity of heat of Q watts is imparted to a section of a tube of length l, then, from the heat balance equation

$$Q = G_{mix}(i_{mix} - i_s)$$

we can determine the enthalpy of the mixture i_{mix} at the section at a distance l from the tube inlet. The heat imparted is all expended on evaporation; therefore the mass flow rate of vapour in this section is determined by the equation

$$Q = rG_v$$

From a comparison between these two equations it follows that

$$\frac{G_v}{G_{mix}} = x = \frac{i_{mix} - i_s}{r} \tag{13-17}$$

It shows that the relative enthalpy of a two-phase flow ($i_{mix} - i_s$), measured in fractions of the heat of evaporation r, is equal to the mass fraction of vapour contained in the given section at boiling liquid in tubes and conduits. This assertion holds, if the liquid is saturated at the tube inlet or the inlet flow is two-phase.

If the liquid at the inlet is subcooled, the parameter $(i_{mix} - i_s)/r$ is negative along the sections where $i_l = i_{mix} < i_s$ (economizer section). Here it represents the relative subcooling enthalpy of the flow in that cross section. It should, however, be mentioned here that in the region of surface boiling, where $i_{mix} < i_s$, notwithstanding the negative value of this parameter, there is already in reality a small fraction of vapour in the flow, owing to the movement of the boiling boundary layer (see Fig. 13-12).

At $i_{mix} = i_s$ the parameter x is formally equal to zero, but the liquid is still subcooled at the core of the flow, although, with large heat fluxes, there is already a boiling boundary layer of considerable size at the surface. With subcooled liquid at the conduit inlet, $x = \frac{i_{mix} - i_s}{r}$ coincides with the vapour fraction only at sections some distance from the inlet section, where $x > 0$, i.e. in the region where the whole liquid attains saturation temperature.

(c) *Dependence of Heat Transfer on the Parameter x*

When liquid boils inside tubes heat transfer is determined by the phase structure of the liquid-vapour mixture. The heat-transfer coefficient increases with the parameter x, reaching maximum values at a weight vapour fraction falling between 0.3 and 0.40 ($\beta \approx 0.98$). Then, it reduces sharply approaching values corresponding to pure vapour (Fig. 13-14), and the vapour drying region sets in, characterized by a minimum intensity of heat transfer. The liquid heating region corresponds to $x \approx -0.2$, then surface boiling and further bulk boiling initiates. Maximum values of the heat-transfer coefficient correspond to the annular regime of bulk boiling during which the lignid film (which offers the

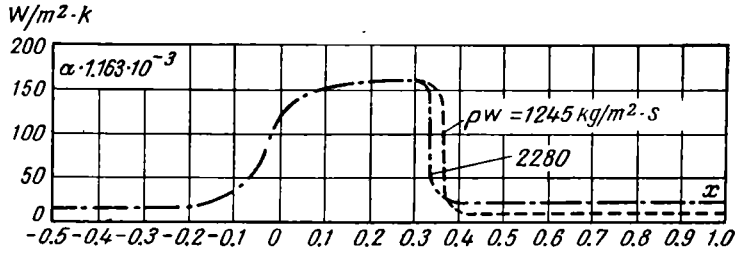

Fig. 13-14. Dependence of heat-transfer coefficient on x:
$p = 1.75 \times 10^7$ Pa; $q = 5 \times 10^6$ W/m²

main thermal resistance to heat flow as in the case of a single-phase liquid in natural circulation) is of a small thickness, and in the core of the flow vapour moves with a high velocity. The external surface of the film is wavy, resulting in that drops of liquid are torn off by the vapour and entrained into the core of the flow. The film becomes thinner with increasing x, rippling and, consequently, detachment of drops terminate.

With full evaporation of the film there is a sharp drop in the intensity of heat transfer (*the dry wall regime*). The value of the vapour fraction at which heat transfer is minimum depends on the velocity, pressure, and physical parameters of both liquid and vapour. The vapour fraction at which heat-transfer coefficients are maximum diminishes with rising velocity; its magnitude is calculated by means of special formulae [13.23].

(d) *Variation of Surface and Liquid Temperature Along the Horizontal Tube*

The temperature of the inside surface of a tube varies along it in full accordance with the intensity of heat transfer (Figs. 13-12 and 13-14). In the liquid preheating region *1*, when a single-phase flow moves inside the tube, wall temperature t_w and liquid temperature t_f rise simultaneously along the tube. Along the surface boiling section *2* wall temperature is practically constant and liquid temperature increases. The regions *3*, *4* and *5* correspond to bulk boiling inside the tube; the temperature t_w does not change; the temperature of the liquid, reaching the saturation temperature, remains practically constant; the difference between the wall temperature and that of the two-phase flow drops to several degrees due to the rising heat-transfer coefficient. As the process develops this temperature difference becomes somewhat smaller, then it increases due to a sharp deterioration of heat transfer. The last case, connected with dete-

rioration of heat transfer, is separately given in Fig. 13-15, where the variation of the heat-transfer coefficient and surface temperature is shown in the region of deteriorated heat transfer [13.23].

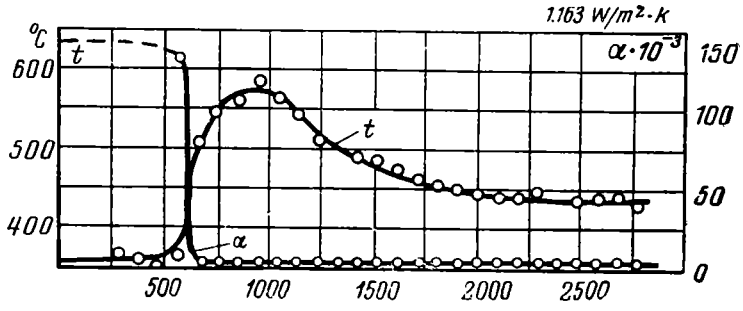

Fig. 13-15. Variation of α and t_w along a vertical tube in the region of deteriorated heat transfer:
$d = 8$ mm; $\rho w = 670$ kg/m²·s; $p = 1.96 \times 10^7$ Pa

(e) *Circumferential Variation of Wall Temperature in a Horizontal Tube*

Figure 13-16 shows the variations in wall temperature for a horizontal tube with water boiling at pressures close to the critical [13.24]. The highest uneven distribution of wall temperature and, consequently, of heat transfer, is most irregular where the two-phase flow is split (curve *1*), and least irregular in the region of annular flow (curve *2*). Annular flow is marked by maximum rates of heat transfer. The conditions of heat transfer where emulsion- and slug-type boiling take place resemble those with bulk boiling. The heat-transfer coefficient for annular flow in tubes may be a little higher than that for bulk boiling.

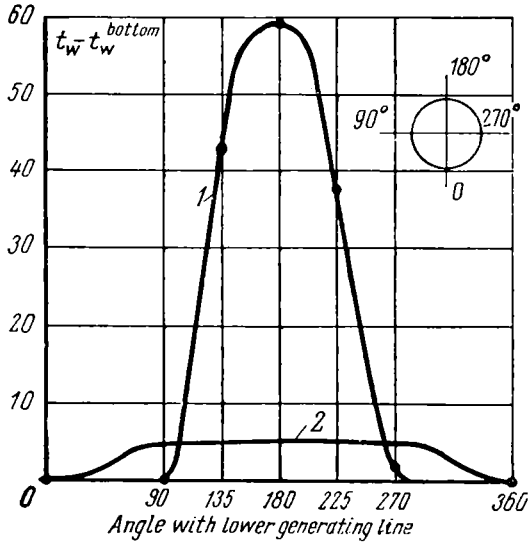

Fig. 13-16. Circumferential variation of wall excess temperature with liquid boiling in a horizontal tube:
1—laminated boiling; *2*—annular boiling; t_w^{bottom} — temperature at lower generating line

13-5. Calculation of Heat Transfer for Liquid in Nucleate Boiling in Tubes under Conditions of Forced Circulation

The intensity of heat transfer is then determined by the interaction of the factors determining the intensity of heat transfer in a boiling liquid (α_q) with the factors effecting hydrodynamically the liquid due to forced circulation (α_w). In the event of forced circulation of a two-phase flow heat transfer is expressed by the functional dependence ($\mathrm{Re} \geqslant \mathrm{Re}_c$):

$$\mathrm{Nu}_* = f(\mathrm{Re}_*, \mathrm{Re}, \mathrm{Pr})$$

where the Reynolds number $\mathrm{Re} = wd/\nu$ is determined by the rate of liquid circulation $w = G_{mix}/f\rho_f$. The interpolation formula [13.20] for heat transfer

in a superheated liquid has the appearance

$$\frac{\alpha}{\alpha_w} = \frac{4\alpha_w + \alpha_q}{5\alpha_w - \alpha_q} \tag{13-18}$$

for α_q/α_w ranging from 0.5 to 2 (Fig. 13-17); when this ratio is less than 0.5, $\alpha = \alpha_w$, and when larger than 2, $\alpha = \alpha_q$; here α_q = heat-transfer coefficient calculated by the formula of developed boiling (when velocity does not effect transfer); α_w = heat-transfer coefficient calculated by the formulae derived

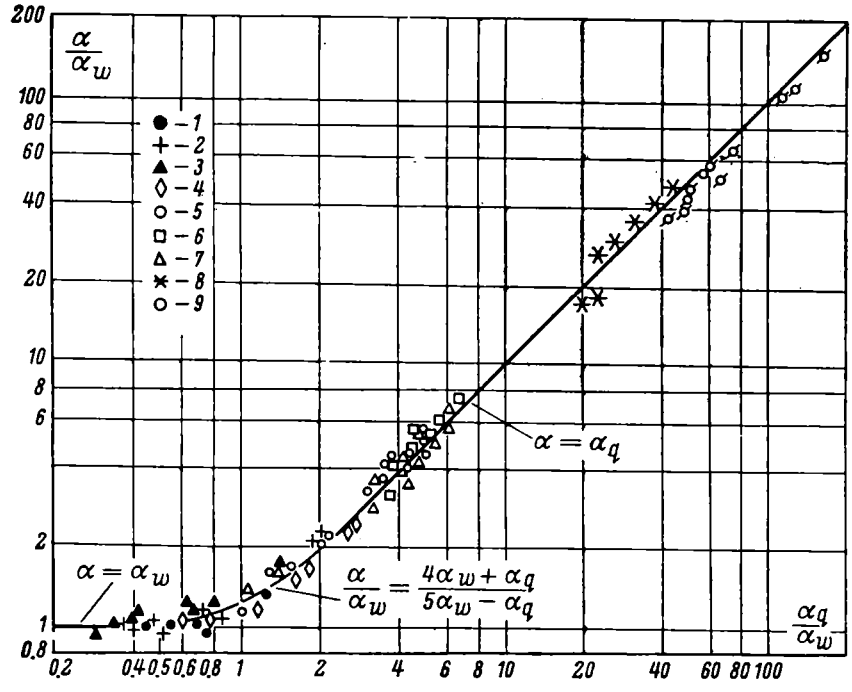

Fig. 13-17. The ratio of heat-transfer coefficients of boiling and non-boiling liquids: *1-7*—water at p = (0.5-86) × 10^5 Pa, w = 0.5-6.67 m/s; *8-9*—ethyl alcohol at p = (10-30) × 10 Pa, w = 0.2-0.81 m/s

for convective heat transfer in a single-phase liquid (when q does not effect heat transfer). It follows from the graph that the region within which $\alpha = f(w, q)$ is very narrow.

The dependence (13-18) holds for volume vapour fractions not exceeding 70 per cent. With high vapour fractions the rate of circulation accounts insufficiently for the amount of heat transferred by convection. That is why heat transfer is calculated allowing for the intensifying effect of true velocity on fluid flow in the core [13.25-13.27].

13-6. Mechanism of Heat Transfer in Film Boiling of Liquid

Film boiling is observed in metals hardening in a liquid medium, in a number of quick-action stills, and in boiling of cryogenic liquids (coolants). It occurs in cooling systems of chemical-fuel and nuclear rocket engines. The absolute value of α becomes considerable at high pressures (Fig. 13-18), which explains why boiler tubes do not burn out, although the temperature difference between

wall and liquid rises appreciably. This makes it possible to utilise the process of film boiling in various steam-generating installations [13.13].

In direct-flow reactors the coolant water is subcooled at the inlet, and emerges as superheated steam. As the water-vapour mixture flows in a reactor of this type, the heat-transfer coefficient varies at the inlet and outlet sections in accordance with the laws governing convective heat transfer in a single-phase flow; along the intermediate section, it varies in accordance with the laws governing heat transfer in nucleate and film boiling. The rate of heat transfer in film boiling is considerably smaller than in nucleate boiling, but owing to larger rates of

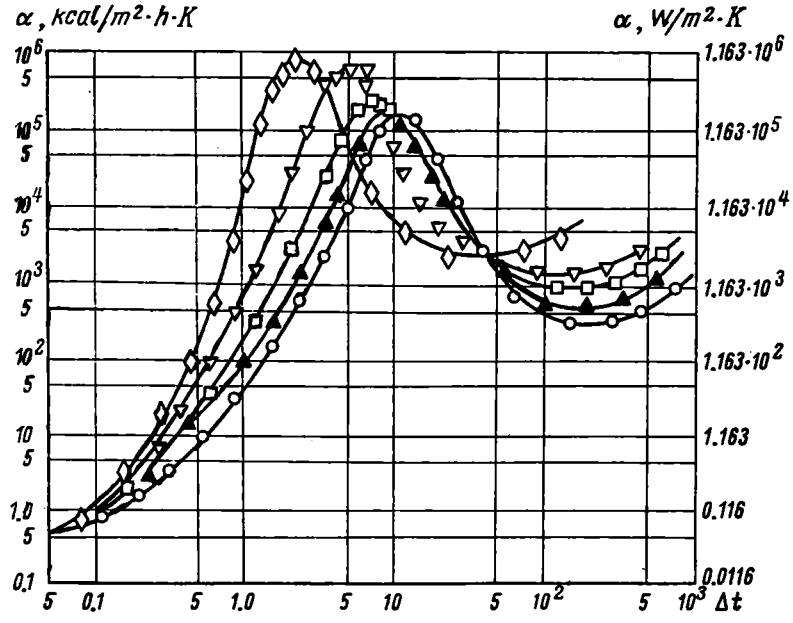

Fig. 13-18. Effect of p on α in boiling water on horizontal chromel wire, $d = 1$ mm: ○—$p = 1$; ▲—2.8; □—4.6; ▽—6.3; ◇—8.1 × 10^5 Pa

vapour flow the surface temperature remains below the melting point of the material and burn-out of the heating surface does not occur. Film boiling, therefore, is of great practical importance.

As was already mentioned above with film boiling the liquid is separated from the heated surface by a layer of vapour. Heat flows to the liquid-vapour interface through a layer of vapour of a low thermal conductivity. In the event of free circulation the heat-transfer coefficient changes slightly with a change in the rate of heat flow (Fig. 13-18). The process of heat transfer is materially affected by pressure and physical properties of the liquid involved as is the case of nucleate boiling.

Since heat is transferred through the vapour layer by thermal radiation as well as by conduction and convection, the heat-transfer coefficient is also influenced by the radiation coefficients of the heat-transfer surface and liquid, and by the emissive capacity of the vapour. The fraction of heat transferred by radiation increases sharply with increasing surface temperature. The two modes of heat transfer—conduction and radiation—are not independent of each other. Their mutual effect shows in a thickening of the vapour film due to the vapour formed by radiation, and brings about a corresponding reduction in the intensity of heat transfer by convection and conduction.

With film boiling of a saturated liquid the heat flow from the heating surface is not only absorbed in vaporization of the layers of liquid at the vapour film border, but a fraction of it is expended, in addition, on superheating the vapour in the film, since the mean temperature of the vapour film is always higher than saturation temperature.

With film boiling of a subcooled liquid the heat passing through the vapour film from the boiling surface is partly transferred further into the bulk of the liquid by convection. The intensity of heat transfer by convection into the bulk of liquid depends on the degree of subcooling and on the rate of liquid circulation. Both factors favourably influence the process of heat transfer.

13-7. Heat Transfer Through Vapour Films in Laminar Flow

Calculation data for heat transfer in film boiling can be obtained theoretically by using an approximate model, like that assumed in the theory of film condensation considered earlier (Sec. 12-2). Also identical is the initial system of equations and conditions of unambiguity.

The solution for the mean heat-transfer coefficient for a vertical wall has the following form:

$$\alpha = 0.943 \sqrt[4]{\frac{\lambda_v^3 r \rho_v (\rho_l - \rho_v) g}{\mu_v \Delta t h}} \qquad (13\text{-}19)$$

For liquid boiling on the outside wall of a horizontal cylinder of diameter d the mean heat-transfer coefficient is expressed by

$$\alpha = 0.728 \sqrt{\frac{\lambda_v^3 r \rho_v (\rho_l - \rho_v) g}{\mu_v \Delta t d}} \qquad (13\text{-}20)$$

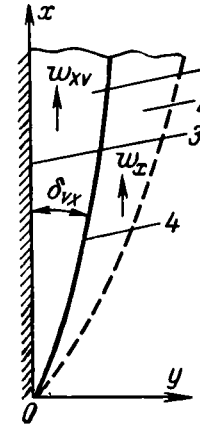

Fig. 13-19. Theoretical approach to film boiling:
1—vapour film; *2*—liquid at saturation temperature t_s; *3*—heated wall at t_w = const; *4*—vapour-liquid interface

Formulae (13-19) and (13-20) coincide formally with the calculation equations derived for heat transfer with vapour in film condensation on a cold wall. The heat-transfer formulae account for the transport of heat across the vapour film only by conduction. The radiant component of the heat-transfer coefficient can be determined analytically, by calculation (Chapter 18).

When account is taken of the action of the inertia forces in the vapour film and of the tangential stresses at the liquid-vapour interface, together with the vapour layer (Fig. 13-19) the boundary layer of liquid is considered. Therefore, in addition to the initial set of differential equations of energy and flow for the vapour film, we must add a similar system of equations for the boundary layer of liquid. Then, the boundary condition for the liquid-vapour interface acquires the following appearance:

$$w_{xv} = w_{xl}; \quad t = t_s; \quad \mu_v \frac{\partial w_{xv}}{\partial y} = \mu_f \frac{\partial w_{xl}}{\partial y} \qquad (13\text{-}21)$$

the continuity condition for mass transfer in the direction normal to the interface is expressed by:

$$\rho_v \left(\frac{d\delta_v}{dx} w_{xv} - w_{yv} \right) = \rho_l \left(\frac{d\delta_l}{dx} w_{xl} - w_{yl} \right) \qquad (13\text{-}22)$$

at $y \to \infty$, $w_{xl} = 0$; $t = t_s$, δ—thickness of the film.

For so exact a calculation the superheat of the vapour film must also be taken into account.

The results of the numerical solutions of the distribution of velocity and temperature in the vapour film [13.38] are plotted in Figs. 13-20 and 13-21 (Pr = 1). From these curves it follows that the nature of temperature and velo-

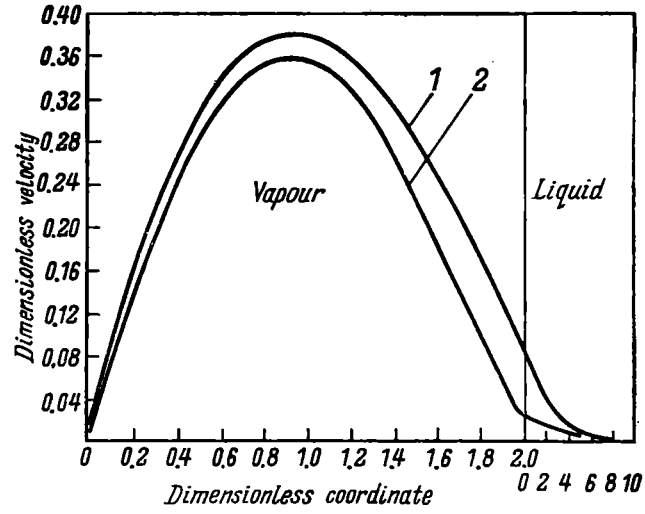

Fig. 13-20. Velocity distribution across vapour and boundary layers of liquid with film boiling:
1—at $(\mu_v\rho_v/\mu_l\rho_l)^{0.5} = 0.1$; *2*—at $(\mu_v\rho_v/\mu_l\rho_l)^{0.5} = 0.01$; $Pr_v = 1$

city distribution is similar to that in the boundary layer of a single-phase liquid in free convection. At low temperature differences the temperature distribution is practically linear.

In distinction to film condensation, the velocity distribution or profile is here influenced by the parameter $\rho_v\mu_v/\rho_l\mu_l$. Velocity becomes zero at the liquid-vapour interface only with a very small value of the parameter $\rho_v\mu_v/\rho_l\mu_l$. The velocity of vapour film flow increases with an increase in this parameter, because it corresponds to a smaller frictional force acting upon the vapour from the side of the liquid at the vapour-liquid interface. This explains why in film boiling the heat-transfer coefficient increases somewhat with a rise in the parameter $(\rho_v\mu_v/\rho_l\mu_l)^{0.5}$

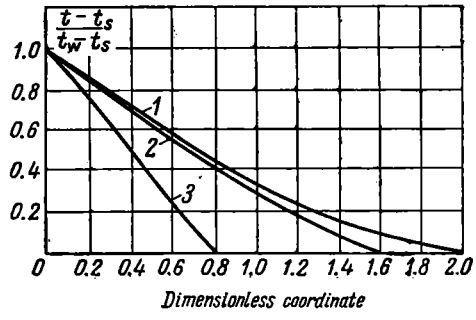

Fig. 13-21. Temperature distribution in vapour film thickness with $Pr_v = 1$:
1—$\frac{c_{pv}\Delta t}{r} = 10$; *2*—$\frac{c_{pv}\Delta t}{r} = 3$; *3*—$\frac{c_{pv}\Delta t}{r} = 0.18$ (the curves are plotted on different scale along the x-axis; for curve *1* film thickness is 2 units, for curve *2*—1.6 units, and for curve *3*—0.8 unit)

With liquid in film boiling heat transfer depends only on subcooling of the liquid in respect to saturation temperature. The effect of subcooling is slight at small values of $c_{pv}\Delta t/r$ and, on the contrary, great, if the vapour film is heavily superheated. To maintain film boiling of water, the minimum calculated value of the parameter $c_{pv}\Delta t/r$ must be equal to or exceed 0.01. In the extreme case, in which considerable subcooling of the liquid takes place and the rate of mass flow

through the liquid-vapour interface is small, the calculated relationship for heat transfer with film boiling reduces into the relationship governing heat transfer without boiling, i.e. heat transfer by convection in a single-phase liquid. For the case of film boiling with forced circulation (convection) the effect of velocity is taken into account [13.28].

13-8. Heat Transfer Through Vapour Films in Turbulent Flow

With a vertical wall turbulent flow of a vapour film is more probable than laminar flow.

It has already been noted (Figs. 13-20, 13-21) that the nature of the velocity and temperature distribution in the boundary layer during boiling is similar to the corresponding distribution profiles in the boundary layer typical of free convection in a single-phase liquid. In film boiling heat transfer can, therefore, be represented by a relationship of the form used to describe natural-convection heat transfer in single-phase liquids. With turbulent flow of the vapour film heat transfer is described by the following relationship [13.5]:

$$\mathrm{Nu}_m = c\mathrm{Ar}_m^{1/3} \tag{13-23}$$

As applied to film boiling, the force that determines the nature of vapour flow in the film is equal to $g(\rho_f - \rho_v)$. The Archimedes number, therefore, will be expressed by the relation

$$\mathrm{Ar} = \mathrm{Ga}\frac{\rho_f - \rho_v}{\rho_f}; \quad \mathrm{Ga} = gl^3/\nu_v^2$$

The physical properties in formula (13-23) refer to the mean temperature of the vapour film, which is indicated by the subscript "m".

The constant in formula (13-23) is equal to 0.25 due to the more intensive heat transfer in a boiling liquid, compared with heat transfer by convection in a single-phase liquid for which the constant equals 0.15.

The process of heat transfer through a film in turbulent flow is self-similar with respect to the geometry and size of the heat-transfer surface. Hence, it drops from formula (13-23).

13-9. Boiling Crises

(*a*) *First Boiling Crisis*

By *boiling crises at heat transfer* are meant processes of boiling, involving a radical change in the mechanism of heat transfer. They are observed when nucleate boiling begins to turn into film boiling or at the beginning of a reverse transformation from film to nucleate boiling.

The heat flux to be removed from the heating surface of many of the heat exchangers used in modern power and rocket engineering is of a fixed magnitude, and is often practically independent of the temperature of the heat-transfer surface. Thus the rate of heat flow to the outside surface of waterwall tubes located in the furnace of a steam boiler is mainly determined by the radiation of heat from the furnace space. The incident radiant heat flux is practically independent of the temperature of the tube surface, until this temperature is much below that of the glowing combustion products in the furnace. A similar phenomenon takes place in the passages of rocket engines. In the fuel elements of the active section of a nuclear reactor there is continuous liberation of heat,

due to the nuclear reaction. The heat flux from the surface of these elements, therefore, is also of a given magnitude. The same is also true of the heat liberation in metal rods or other surfaces by the passage of an electric current.

The change from nucleate to film boiling can be taken to be a crisis phenomenon since at the moment the change of the regime of boiling takes place a sudden sharp drop of the rate, or intensity, of heat transfer is observed and a corresponding increase in the temperature of the heated surface (Fig. 13-4). The increase in surface temperature is sometimes so great that the boiling crisis is accompanied by destruction (melting or burnout) of the heat-transfer surface. After the rate of heat flow becomes equal to $q_{max\ cr}$ even with a slight increase of the heat flux the layer of vapour bubbles turns into a continuous vapour film which forces the liquid away from the heat-transfer surface. As a result the heat-transfer mechanism undergoes a radical change, i.e. a boiling crisis sets in.

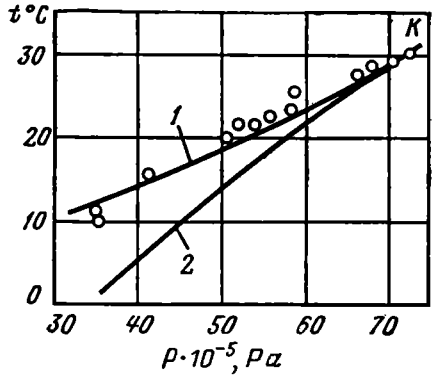

Fig. 13-22. Maximum liquid superheat as a function of pressure:
1—maximum superheat line; *2*—saturation line; *K*—critical point; ○—temperature t_w at the moment of a crisis

The boiling crisis usually occurs before the surface temperature approaches the maximum superheat temperature for the liquid involved*, at which spontaneous vapour-phase nuclei may appear. This can be explained by the fact that in the presence of ready starting points of vapourization phase mass transfer is disturbed and a corresponding disturbance of steady-state nucleate boiling takes place. With rising pressure, however, phase mass transfer at the wall improves, since the density of vapour increases, the separation diameter diminishes and, consequently, the volumetric intensity of vapourization decreases. With a deficiency in ready starting points nucleate boiling continues until vapourization nuclei of spontaneous origin appear. Therefore, with an increase in pressure, the surface temperature at which a boiling crisis sets in approaches gradually the maximum superheat temperature. Shown in Fig. 13-22 are the lines of maximum superheat *1* and saturation *2* at different pressures $p/p_{cr} =$ $= 0.5$ to 1.0 for boiling carbon dioxide [13.29]. Points indicate the experimentally-determined surface temperatures at which a boiling crisis sets in. These temperatures almost coincide with the maximum liquid superheat temperatures.

The maximum heat flux in nucleate boiling is called the *critical heat flux* and denoted by $q_{max\ cr}$.

Experiments with bulk boiling of a saturated liquid have shown the critical heat flux to depend on the kind of liquid, pressure, the state of the heating surface, wetting conditions, the presence of admixtures in the liquid, and surface-active additives.

If the heating surface exceeds the size of individual vapour bubbles, then the shape and size of the heating surface have little effect on the magnitude of critical heat flux.

* The liquid superheat temperature is determined by the thermodynamic limit of the metastable state depending on pressure.

In practice the magnitude of acceleration of mass forces has no effect in the field.

The temperature difference observed at the moment heat flux reaches its critical or peak value (i.e. the temperature difference corresponding to the maximum point of the boiling curve shown in Fig. 13-4) *is known as the critical temperature difference* Δt_{cr}. At the burn-out point the heat-transfer coefficient is equal to

$$\alpha_{max\ cr} = q_{max\ cr}/\Delta t_{max\ cr}$$

The peak heat fluxes and critical temperature differences for water and a number of cryogenic liquids at atmospheric pressure are given in Table 13-1.

TABLE 13-1

Critical Heat Fluxes and Temperature Differences (First Boiling Crisis)

Critical parameter	Fluid			
	H_2O	O_2	N_2	H_2
$q_{max\ cr} \times 10^{-3}$, W/m²	1200	150	100	30
Δt_{cr1}, K	25-30	11	11	2

The effect of pressure on the magnitudes of $q_{max\ cr}$, Δt_{cr1}, and α_{cr1}, in free-convection [13.21] of bulk boiling of water is illustrated graphically in

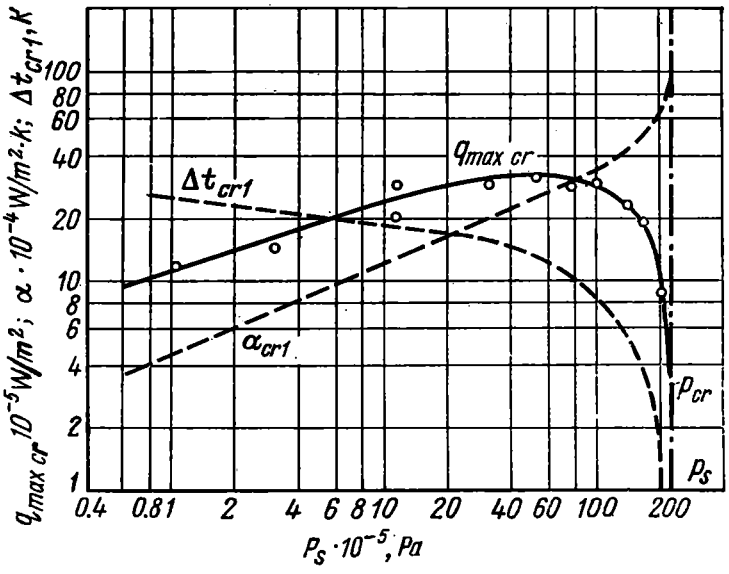

Fig. 13-23. Dependence of q_{cr}, Δt_{cr}, α_{cr} on pressure for water in pool boiling under conditions of free circulation

Fig. 13-23. The critical heat flux is maximum at pressures between 60 and 80 bar, and is equal to about $(0.3\text{-}0.4)p_{cr}$. The picture for other liquids is similar.

The critical heat flux is determined based on the hydrodynamic theory of boiling crises suggested by S.S. Kutateladze [13.30], visualizing a boiling crisis as a process distinguished by its purely hydrodynamic origin. The crisis arises

from the loss of the dynamic stability of a two-phase flow due to the liquid being thrown by the vapour from the heat-transfer surface.

The conditions of stability for a boundary two-phase flow are determined by the interaction of the kinetic energy of the vapour phase, forces of gravity, present in the two-phase flow, and surface tension forces. The order of magnitude of the vapour dynamic head is determined by the product $w_{cr.v}^2\rho_v$; the order of magnitude of gravitational forces—by $g\delta(\rho_f - \rho_v)$, where δ is the mean thickness of the appearing vapour layer, related to surface tension by the capillary constant, since it is assumed that

$$\delta = \sqrt{\sigma/g(\rho_f - \rho_v)}$$

A boiling crisis is equiprobable to appear at any spot on the heat-transfer surface, hence

$$\frac{w_{cr.v}^2\rho_v}{g\delta(\rho_f - \rho_v)} = \text{const}$$

Upon introduction of the capillary constant and extraction of the square root the above relationship reduces to

$$\frac{w_{cr.v}\sqrt{\rho_v}}{\sqrt[4]{g\sigma(\rho_f - \rho_v)}} = k$$

Making use of the reduced rate of vapourization and the rate of heat flow $w_{cr.v} = q_{max\ cr}/\rho_v r$, we obtain the following calculation formula for the critical heat flux:

$$q_{max\ cr} = kr\sqrt{\rho_v}\sqrt[4]{g\sigma(\rho_f - \rho_v)} \qquad (13\text{-}24)$$

where $q_{max\ cr}$ is measured in W/m^2.

Formula (13-24) describes the experimental data on critical heat fluxes for non-metallic heat-carrying agents in pool boiling and free circulation of a low viscosity liquid.

In the regions of high vacuum and near-critical pressure ρ_v, r and σ tend to zero and $q_{max\ cr} \to 0$ (Fig. 13-23). The constant $k = 0.13$ to 0.16. The quantity k is called the stability criterion; it characterizes the ratio of the energy of flowing vapour to the energy required to accelerate the particles of the liquid, thrown from the wall, to the velocity of the main flow. For a two-phase flow in forced circulation the magnitude of k depends on velocity [13.30].

The increase in $q_{max\ cr}$, W/m^2, due to liquid subcooling, can be found from the experimental relationship [13.31]:

$$q_{max\ cr,\ s} = q_{max\ cr}\left[1 + 0.065\left(\frac{\rho_l}{\rho_v}\right)^{0.8}\frac{c_p\Delta t_s}{r}\right] \qquad (13\text{-}25)$$

which holds for $p = (1\text{-}20)\cdot 10^5$ Pa; $\frac{c_p\Delta t_s}{r} \leqslant 0.6$; $\frac{\rho_l}{\rho_v} = 45\text{-}1650$, where $\Delta t_s = t_s - t_l$, subcooling of liquid in respect to t_s.

With forced-circulation boiling of liquid in tubes and conduits the critical heat flux depends on the rate of circulation, the extent of subcooling, and the vapour fraction in the two-phase mixture; it increases with increasing rate of circulation (Fig. 13-24). Figure 13-25 shows the effect of x on $q_{max\ cr}$ at different velocities of the liquid at the tube inlet. From the chart it follows that the critical heat flux diminishes with rising x at positive values of this parameter. This drop in $q_{max\ cr}$ points to the effect of the vapourization. An increase of $q_{max\ cr}$ with increasing absolute values of x in the negative region indicates

the effect of liquid subcooling. Thus, the change from nucleate to film boiling is possible both at negative and positive values of the parameter x. One of the methods of increasing $q_{max\ cr}$ is to practise higher degrees of subcooling. The maximum reached values can be evaluated by theoretical methods [13.32].

The curves in Fig. 13-26 correlate the critical heat fluxes for water in pool boiling and in forced-circulation boiling in tubes. The data for forced circulation boiling are based on $x = 0$ at the cross-section corresponding to the burnout point. It follows from the chart that the maximum value of $q_{max\ cr}$ for water boiling in tubes corresponds to lower pressures (of about 40 bar) than with pool boiling. The data presented show, as well, that at pressures below 70 bar velocity does not affect peak heat flux (at $x = 0$). In general, the relation $q_{max\ cr} = f(p, w, x)$ is intricate, because its constituent variables mutually influence one another.

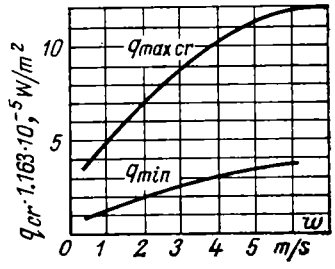

Fig. 13-24. Dependence of $q_{max\ cr}$ and q_{min} on velocity w for isopropyl alcohol flowing in a tube

At relatively high pressures and small rates of circulation, the critical heat flux $q_{max\ cr}$ with boiling in tubes may be equal to, or even smaller than, the critical heat flux observed in pool boiling. The critical heat fluxes are independent of the relative length of the tube [13.33] provided the length is larger than eight to ten diameters. If the length of the tube is less than that, the critical heat flux diminishes with increase in relative length. This phenomenon is explained by the abrupt change

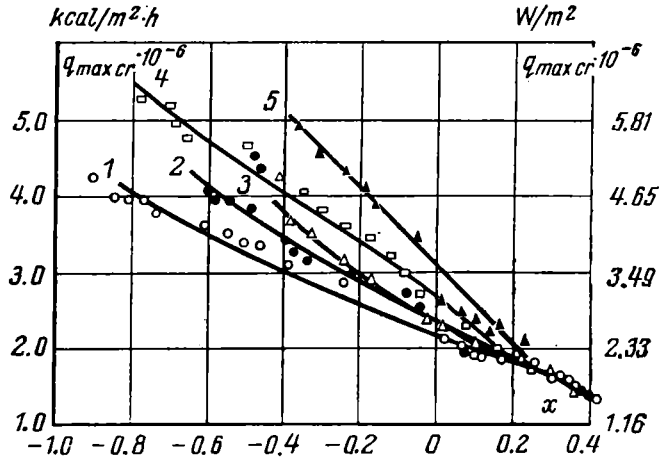

Fig. 13-25. Dependence of $q_{max\ cr}$ on x with water boiling inside tubes ($p = 1.67 \times 10^7$ Pa):
1—$\rho w = 750$; *2*—$\rho w = 880$; *3*—$\rho w = 1100$; *4*—$\rho w = 1435$; *5*—$\rho w = 2190$ kg/m²·s

occurring in the intensity of heat conduction along the initial section of the tube. The thickness and roughness of the wall do not affect the critical heat flux, $q_{max\ cr}$.

Apart from these factors, $q_{max\ cr}$ can be affected by stream fluctuation along the upstream section, irregular distribution of the heat flux over the length and perimeter of the tube, and the method used to heat the heat-transfer surface.

It follows from the foregoing that the critical heat flux is affected by many various factors, the basic ones being pressure p, velocity w, temperature difference Δt_s, x, the physical properties of the liquid, and the state of the surface.

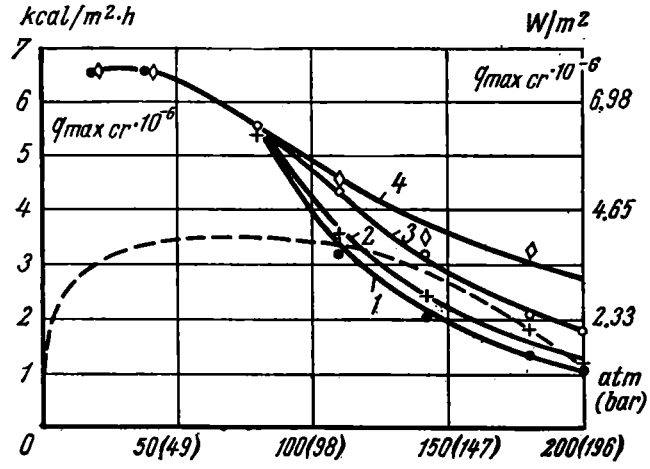

Fig. 13-26. Dependence of $q_{max\ cr}$ on pressure with pool boiling of liquid in tubes ($x = 0$): *1*, *2*, *3*, *4* correspond to $w = 1$, 2, 5, 8 m/s

(*b*) *Second Boiling Crisis*

When the given heat flux for the heating surface is independent of the conditions of heat transfer, a reverse transition from film boiling to nucleate takes place at a minimum heat flux q_{min}, as shown in Fig. 13-5. This transition, too, is of critical nature: the vapour film is suddenly destroyed, and the surface temperature drops abruptly. The minimum rate of heat transfer in nucleate boiling is referred to as the *minimum heat flux* and denoted by q_{min}. The temperature difference corresponding to the minimum on the boiling curve is denoted by Δt_{min}.

The second peak heat flux q_{min} is considerably smaller than $q_{max\ cr}$ with pool boiling of a saturated liquid. The second peak heat fluxes have been shown by experiment to depend on the kind of liquid, the size of the heat transfer surface [13.34], pressure, acceleration of gravity, surface roughness, and a number of other factors. For water boiling at atmospheric pressure on the surface of horizontal tubes and cylinders q is equal to (2-5) × 10^4 W/m². Depending on the tube diameter d, its magnitude varies in accordance with the approximate relationship $q_{min} \sim d^{1/4}$.

Experimentally measured values of q_{min} and Δt_{cr2} for other liquids are given in Table 13-2.

TABLE 13-2

Critical Heat Fluxes and Temperature Differences (Second Boiling Crisis)

Critical parameter	Boiling liquid			
	Carbon tetrachloride	H-pectane	Isopropyl alcohol	Liquid nitrogen
$q_{min} \times 10^{-4}$, W/m²	1.3	1.0	3.5	0.6
Δt_{min}, K	80	60	110	30

The reduced critical rate of vapourization $w_{cr2} = q_{min}/\rho_v r$ is assumed to be proportional to the rising velocity of large deformed vapour bubbles, i.e.

$$\frac{q_{min}}{\rho_v r} = c \sqrt[4]{\sigma g (\rho_l - \rho_v)/\rho_l^2} \tag{13-26}$$

where the constant $c = 0.11$ to 0.14.

(c) *Equilibrium Rate of Heat Flow Flux*

With saturated liquids in pool boiling on the surface of horizontal tubes with electrical heating there is a heat flux at which stable coexistence is possible of film boiling on one section and nucleate boiling on the other. This heat flux is referred to as the *equilibrium flux* q_{eq} [13.34]. If the rate of heat flow is increased somewhat after the equilibrium heat flux has set in, the boundary between the two regimes of boiling begins to shift in the direction of film boiling, and boiling will become of film type over the entire surface after a certain period. If the rate of heat flow is diminished relative to the equilibrium heat flux, the process is reversed and nucleate boiling sets in over the entire surface.

The equilibrium heat flux is numerically equal to about 1/5 of the maximum rate of heat flow so that $q_{max\ cr} > q_{eq} > q_{min}$. Equilibrium heat fluxes are of interest when analysing the stability of boiling regimes.

(d) *Crises of the 1st and 2nd Kind*

The boiling crises associated with a sharp change in the rate of heat transfer occurring when nucleate boiling changes into film boiling and vice versa, film into nucleate boiling (Fig. 13-4), are referred to as *crises of the 1st kind.* When crises of this kind occur, as one mode of boiling changes into another, there is a radical change of the mechanism of heat transfer and of its intensity. Characteristic of boiling crises of the 1st kind is the presence of critical and minimum heat fluxes.

Boiling crises of the 2nd kind differ from these of the 1st kind in their origin. They characterize the deterioration of heat transfer occurring at the moment the annular film of liquid which is in the annulus pattern of flow dries on the wall of the conduit (Fig. 13-14). Characteristic of crises of this kind is the *boundary vapour fraction* by which is meant the vapour fraction at which heat transfer undergoes a crisis of the second kind.

A 2nd kind crisis may occur at any rate of heat flow as soon as the vapour fraction reaches some limiting value associated with the drying of the liquid film on the wall of the conduit. With high-velocity liquid-vapour flow the moment at which a boiling crisis sets in in the core of the flow depends on the intensity of mass transfer at the expense of the mechanical carryover of drops of liquid from the film into the core of the flow [13.35-13.37].

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CHAPTER

14

HEAT AND MASS TRANSFER IN TWO-COMPONENT SYSTEMS

14-1. Fundamental Concepts and Laws

Many processes of heat transfer encountered in nature and engineering are accompanied by processes of transfer of the mass of one component into the other*. This, for instance, is the case observed in the condensation of vapour from a vapour-gas mixture and the evaporation of liquid into a vapour-gas flow. The evaporated liquid is distributed throughout the vapour-gas flow by diffusion; the process is accompanied by a change in the nature of flow and a variation in heat-transfer intensity, and this, in turn, influences the process of diffusion.

By diffusion is meant the spontaneous process of spreading or scattering of matter in a binary medium or two-component system under the influence of concentration. In a mixture homogeneous in respect to temperature and pressure, diffusion is directed towards equalizing the concentration in the system and is accompanied by transfer of mass from the region of higher concentration to the region of lower concentration.

From here on we shall be concerned mainly with process of heat and mass transfer in gaseous two-component (binary) media, for problems of this kind are of a great practical interest. The subscripts "1" and "2" will pertain to the first and second components.

By analogy with heat transfer, diffusion (mass transfer) may be either molecular (microscopic) or molar (macroscopic). In gases molecular diffusion is due to the thermal motion of molecules.

Diffusion is characterized by the flow of the mass of a component, i.e. by the quantity of mass passing per unit time through the given surface in a direction normal to the surface. Let us denote the flow of mass by J; its unit of measurement is kg/s.

The rate of mass flow j is defined as the flow of mass through a unit area of the given surface:

$$j = \frac{dJ}{dF} \tag{14-1}$$

hence

$$J = \int_F j\, dF$$

or, at $j = \text{const}$,

$$J = jF$$

The rate of mass flow is a vector.

---

* According to W. Gibbs by components are meant substances, the smallest number of which is sufficient to form all phases, i.e. the homogeneous parts of a given system. A system, consisting of a layer of liquid and a mixture of water vapour and nitrogen which is in contact with the layer of liquid, for instance, has two components: water (water vapour) and nitrogen. We assume that components do not react chemically with one another.

With stationary macroscopic two-component system, homogeneous as regards temperature and pressure, the rate of mass flow of one of the components, due to molecular diffusion, is determined by *Fick's law*:

$$j_{cdi} = -D\frac{\partial \rho_i}{\partial n} \tag{14-2}$$

or

$$j_{cdi} = -\rho D\frac{\partial m_i}{\partial n} \tag{14-3}$$

where ρ_i = local concentration of the given substance (component), equal to the ratio of the mass of the component to the volume of the mixture, kg/m³;

$m_i = \rho_i/\rho$ = relative mass concentration of the ith component;

ρ = mixture density;

D = coefficient of molecular diffusion of one component in respect to the other, m²/s; or, usually for short, the coefficient of diffusion;

n = the direction normal to the surface of a similar concentration of the component;

$\frac{\partial \rho_i}{\partial n}, \frac{\partial m_i}{\partial n}$ = concentration gradient (relative concentration) that is always directed to the side of rising concentration.

The concentration gradient is the motive force determining the transfer of matter. In heat conduction it is the temperature gradient* that is the motive force.

The minus sign of Eq. (14-3) indicates that the mass is being transferred, in accordance with Fick's law, in the direction of diminishing concentration. The process described by Fick's law is known as *concentration diffusion*.

As follows from the kinetic theory of gases, the coefficient of diffusion increases with rising temperature and diminishes with increasing pressure. It also depends to some extent on the proportions of the mixture, but this dependence is slight, if the concentration of the considered component is small, and is usually ignored in engineering calculations.

The coefficient of diffusion is identical for the two mutually diffusing components of a two-component mixture.

The unit of concentration is that of density. Making use of the Clapeyron equation for ideal gases, Eq. (14-1) may be written as follows:

$$j_{cdi} = -D_{pi}\frac{\partial p_i}{\partial n} \tag{14-4}$$

where p_i is the local partial pressure of the given component, N/m²; n is the direction normal to the surface of equal partial pressures of the given component; $D_{pi} = \frac{D}{RT}$ is the coefficient of molecular diffusion (unit of measurement—s) reduced to the gradient of partial pressure of the considered component.

Unlike D, the coefficient D_p is different for the components of the given binary mixture. Proceeding from the equality

$$D = D_{p1}R_1T = D_{p2}R_2T$$

* In general Fick's law must be expressed in terms of the chemical potential gradient $\nabla\mu$ [14.1].

we can write

$$\frac{D_{p1}}{D_{p2}} = \frac{R_2}{R_1} = \frac{M_1}{M_2}$$

where M is the molecular weight; and the subscripts 1 and 2 refer to the first and second component of the mixture.

If the temperature of the mixture is variable, *thermal diffusion* sets in (the Soret effect). From the kinetic theory of gases [14.2] it follows that if the masses of molecules of two components are different, thermal diffusion causes the heavier molecules to pass to the colder regions; but if the masses of the molecules are nearly equal, the larger molecules will tend to pass into the colder regions. The direction of thermal diffusion can change under definite conditions. In an ionized gas, for instance, the heavier molecules (or ions) tend to pass to the hotter regions.

Thermal diffusion results in the formation of a concentration gradient. The process of concentration or ordinary diffusion, which tends to even out the composition of a mixture, interferes with the formation of a concentration gradient. As a result a steady state may set in with time, i.e. the opposing effects of thermal diffusion and concentration diffusion will be balanced.

The transfer of heat by molecular diffusion results in the so-called *thermal effusion effect* (Dufour effect) which consists in the appearance of a temperature difference resulting from the mixing of two gases, which *are initially at one and the same temperature*, by means of diffusion. The thermal effusion effect is a phenomenon contrary to thermal diffusion. With steady-state mixing of gases by diffusion (hydrogen and nitrogen, for instance) a temperature difference of several degrees sets in.

The direction of the temperature gradient appearing as a result of the thermal effusion effect is such that the thermal diffusion due to it is opposite to the diffusion that causes the appearance of this gradient.

If a mixture is subject to a full pressure gradient, a process of diffusion due to non-uniform pressure may set in, known as *pressure diffusion*. Under it heavy molecules tend to pass to the region of higher pressure, and light molecules to that of lower pressure. Like thermal diffusion, pressure diffusion is accompanied by ordinary diffusion caused by the difference in concentration. Diffusion due to non-uniform pressure takes place, for instance, in a gas rotating about an axis; the heavy molecules then tend to pass to the region farthest from the axis.

With account taken of ordinary, thermal and pressure diffusion, the rate of transfer for the mass of the ith component, due to molecular diffusion is described by the following equation:

$$\vec{j}_{mdi} = -\rho \left(D \nabla m_i + \frac{D_T}{T} \nabla t + \frac{D_p}{p} \nabla p \right) \qquad (14\text{-}5)$$

where $D_T = k_T D$ = coefficient of thermal diffusion, m²/s;
$D_P = k_p D$ = coefficient of pressure diffusion, m²/s;
p = mixture pressure.

The first term in Eq. (14-5) accounts for ordinary diffusion, the second, for thermal diffusion and the third, for pressure diffusion.

The components of a flow of mass are normal to their corresponding isopotential surfaces, i.e. to equal concentration surfaces isothermal (t = const) and isobaric (p = const) surfaces. At grad $p = 0$ and grad $t = 0$ Eq. (14-5) becomes the Fick law.

The coefficient $k_T = D_T/D$ is referred to as the thermodiffusion ratio; it is dimensionless; $k_p = D_p/D$ is called the pressure diffusion ratio.

For a gas mixture k_T is usually less than 0.1. Due to the small value of k_T, an appreciable rate of mass transfer will take place only at large temperature gradients; thermal diffusion is especially slight, if the concentration of one of the components is small.

For a binary mixture k_b is determined from equation

$$k_b = \frac{\rho_1 \rho_2}{\rho^2} \frac{M_2 - M_1}{M} \tag{14-6}$$

where M_1, M_2, M—molecular weights of the first and second mixture components and of the mixture.

Pressure diffusion takes place at considerable pressure drops which are rarely encountered in the processes of heat transfer. Where the molecular weights M_1 and M_2 are equal, there is no pressure diffusion.

As follows from the thermodynamics of irreversible processes, pressure diffusion should be accompanied by a corresponding thermal effect, consisting in the appearance of a temperature difference.

Thus, the total mass transfer of any component by means of molecular diffusion is the result of ordinary diffusion, thermal diffusion, and pressure diffusion*.

Below we shall first of all take into account effects associated with concentration diffusion.

In a moving medium mass is transferred not only by molecular diffusion, but also by convection. The displacement of any volume of a mixture of a density ρ at a velocity $\vec{w}$ is accompanied by a transfer of mass, and the rate of transfer is determined by the equation

$$\vec{j}_c = \rho \vec{w} \tag{14-7}$$

or, for a definite component of the mixture, by

$$\vec{j}_{ci} = \rho_i \vec{w} \tag{14-8}$$

The resulting total rate of mass transfer due to molecular transfer and convection is determined by the equation

$$\vec{j}_i = \vec{j}_{mdi} + \vec{j}_{ci} \tag{14-9}$$

If the rate of mass flow of any component is j_i, then an enthalpy $j_i i_i$ is transferred with its mass (i_i being the specific enthalpy of the ith component in joules per kilogram). In the general case an enthalpy $\Sigma j_i i_i$ is transported through an imaginary reference surface in the mixture. There may also be a resultant flow of enthalpy even through a plane so located in the mixture that there is no transmission of the resultant mass flux through it.

The rate of heat flow in a mixture is thus described by the equation:

$$\vec{q} = -\gamma \nabla t + \sum \vec{j}_i i_i \tag{14-10}$$

* The total mass flux may include, in the general case, a component appearing, when different external forces act upon the components. An illustration is the diffusion of electrically charged particles in a partly ionized gas under the action of an electric or magnetic field. Separation or effusion of the mixture, caused by the process of internal friction, may also take place. Molecules of larger mass move in the direction of lower velocity [14.3].

which can also be presented in the following form:

$$\vec{q} = -\lambda \nabla t + \rho \vec{w} i + \sum \vec{j}_{mdi} i_i \qquad (14\text{-}10')$$

The first term in the right-hand side of Eq. (14-10) accounts for the transport of heat by conduction, the second—by convection and the third—by molecular diffusion.

According to Eq. (4-2),

$$\vec{q} = -\lambda \nabla t + \rho \vec{w} i$$

in a moving single-component fluid.

Thus, an additional diffusion component of the heat flow appears in the mixture.

14-2. Differential Equations of Heat and Mass Transfer

To determine the heat flow it is necessary to know the temperature and velocity fields, and the field of mass flux.

The energy equation (4-10), previously derived for a single-phase, single-component fluid, does not account for heat transfer by diffusion, therefore we must derive an energy equation for a binary mixture of interdiffusing components.

(a) *Equation of Energy*

We proceed from the assumption that there are no heat sources. Heat of friction is ignored, and physical parameters are assumed invariable.

Consider a stationary elementary volume (Fig. 14-1) with sides dx, dy and dz, in the moving binary mixture, and write the equation of heat balance for this element, assuming that all imparted heat is consumed to change the enthalpy of the considered volume (the work of expansion being zero).

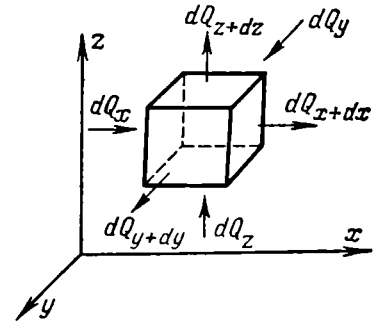

Fig. 14-1. Deriving the differential equation of energy for simultaneous heat and mass transfer processes

These assumptions permit us to use the previously derived heat balance-equation (4-6):

$$\rho \frac{\partial i}{\partial \tau} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) \equiv -\operatorname{div} \vec{q}$$

According to Eq. (14-10')

$$q_x = -\lambda \frac{\partial t}{\partial x} + \rho w_x i + \sum j_{xi} i_i$$

$$q_y = -\lambda \frac{\partial t}{\partial y} + \rho w_y i + \sum j_{yi} i_i$$

$$q_z = -\lambda \frac{\partial t}{\partial z} + \rho w_z i + \sum j_{zi} i_i$$

Here the subscript '*md*' is omitted.

From these last equations we obtain:

$$\frac{\partial q_x}{\partial x} = -\lambda \frac{\partial^2 t}{\partial x^2} + \rho \left(w_x \frac{\partial i}{\partial x} + i \frac{\partial w_x}{\partial x} \right) + \frac{\partial}{\partial x} \sum j_{xi} i_i$$

$$\frac{\partial q_x}{\partial y} = -\lambda \frac{\partial^2 t}{\partial y^2} + \rho \left(w_y \frac{\partial i}{\partial y} + i \frac{\partial w_y}{\partial y} \right) + \frac{\partial}{\partial y} \sum j_{yi} i_i$$

$$\frac{\partial q_z}{\partial z} = -\lambda \frac{\partial^2 t}{\partial z^2} + \rho \left(w_z \frac{\partial i}{\partial z} + i \frac{\partial w_z}{\partial z} \right) + \frac{\partial}{\partial z} \sum j_{zi} i_i$$

Summing them up, and taking into account that div $\vec{w} = 0$ for an incompressible fluid, we obtain:

$$-\operatorname{div} \vec{q} = \lambda \nabla^2 t - \rho \left(w_x \frac{\partial i}{\partial x} + w_y \frac{\partial i}{\partial y} + w_z \frac{\partial i}{\partial z} \right) -$$

$$- \left(\frac{\partial}{\partial x} \sum j_{xi} i_i + \frac{\partial}{\partial y} \sum j_{yi} i_i + \frac{\partial}{\partial z} \sum j_{zi} i_i \right)$$

Substituting the value of $-\operatorname{div} \vec{q}$ in Eq. (4-6) let us write the differential equation of energy in the following form:

$$\frac{\partial i}{\partial \tau} = \frac{\lambda}{\rho} \nabla^2 t - \left(w_x \frac{\partial i}{\partial x} + w_y \frac{\partial i}{\partial y} + w_z \frac{\partial i}{\partial z} \right) - \frac{1}{\rho} \operatorname{div} \sum \vec{j}_i i_i \qquad (14\text{-}11)$$

The left side of Eq. (14-11) describes the local change in the specific enthalpy due to conduction, convection and molecular diffusion. The first term of the right side of the equation accounts for conduction, the second, for convection, and the third, for molecular diffusion.

Eq. (14-11) can be presented in an abbreviated form:

$$\rho \frac{di}{d\tau} = \lambda \nabla^2 t - \operatorname{div} \sum \vec{j}_i i_i \qquad (14\text{-}12)$$

The value of $\vec{j}_i$ must be substituted in Eq. (14-12). For the sake of simplicity, and, taking into consideration that the intensities of thermal and pressure diffusion are small, we assume that molecular mass transfer is effected only by concentration diffusion. Then,

$$\vec{j}_i = -\rho D \nabla m_i$$

For the two-component mixture considered $m_1 + m_2 = 1$ and, consequently, $\partial m_1 / \partial n = -\partial m_2 / \partial n$, whence it follows that $\vec{j}_1 = -\vec{j}_2$ and

$$\sum \vec{j}_i i_i = \vec{j}_1 (i_1 - i_2) = -\rho D (i_1 - i_2) \nabla m_1$$

Substituting the value of $\Sigma \vec{j}_i i_i$ in Eq. (14-12), we get

$$\rho \frac{di}{d\tau} = \lambda \nabla^2 t + \operatorname{div} [(i_1 - i_2) \rho D \nabla m_1] \qquad (14\text{-}13)$$

Using expression $di = c_p dT$, Eq. (14-13) or (14-12) can be presented only in terms of temperatures.

From Eq. (14-13) it follows that if $i_1 = i_2$ there is no heat transfer by diffusion, and therefore the equation of energy (14-13) with $di = c_p dT$ acquires the form of the previously derived equation (4-10).

In a moving mixture the temperature field depends on the velocity components w_x, w_y, and w_z, and on the mass fraction m. The field of the mass fractions is described by the differential equation of mass transfer (equation of diffusion).

(*b*) *Mass Transfer Equation*

We can now derive the differential equation describing the distribution of any component in a moving mixture, assuming that the fluid is incompressible and that it has no inner mass sources. Thermal and pressure diffusion are assumed to be negligible.

Considering a stationary elementary parallelepiped (Fig. 14-2) dx, dy and dz in the mixture and, assuming D and ρ to be constant, we can write the equation of mass balance for the parallelepiped.

The mass of the ith component flowing into the parallelepiped per elementary period of time $d\tau$ along the x-axis is $dM_{x,\,i} = j_{x,\,i}\,dydz\,d\tau$, kg, and the mass leaving the element is $dM_{x+dx,\,i} = j_{x+dx,\,i}\,dydz\,d\tau$, kg.

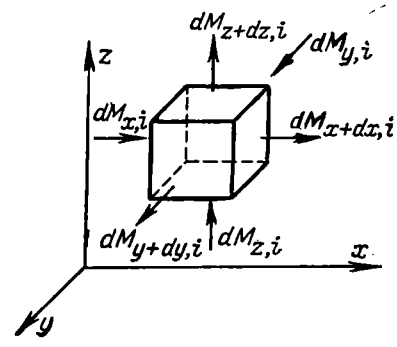

Fig. 14-2. Deriving the differential equation of mass transfer

The difference between the quantities of the mass of the ith component flowing into and out of the element along the Ox-axis is determined from the expression

$$dM_{x,\,i} - dM_{x+dx,\,i} = -\frac{\partial j_{x,\,i}}{\partial x}\,dx\,dy\,dz\,d\tau = -\frac{\partial j_{x,\,i}}{\partial x}\,dV\,d\tau$$

Similarly, for the other axes we have:

$$dM_{y,\,i} - dM_{y+dy,\,i} = -\frac{\partial j_{y,\,i}}{\partial y}\,dV\,d\tau$$

$$dM_{z,\,i} - dM_{z+dz,\,i} = -\frac{\partial j_{z,\,i}}{\partial z}\,dV\,d\tau$$

Summing up along the three axes, we find that the change in the mass of the ith component is:

$$dM_i = -\left(\frac{\partial j_{x,\,i}}{\partial x} + \frac{\partial j_{y,\,i}}{\partial y} + \frac{\partial j_{z,\,i}}{\partial z}\right) dV\,d\tau$$

Since

$$dM_i = -\frac{\partial \rho_i}{\partial \tau}\,dV\,d\tau = \frac{\partial\,(\rho m_i)}{\partial \tau}\,dV\,d\tau = \rho\,\frac{\partial m_i}{\partial \tau}\,dV\,d\tau$$

then,

$$\rho\,\frac{\partial m_i}{\partial \tau} = -\left(\frac{\partial j_{x,\,i}}{\partial x} + \frac{\partial j_{y,\,i}}{\partial y} + \frac{\partial j_{z,\,i}}{\partial z}\right) \equiv -\operatorname{div} \vec{j}_i \qquad (14\text{-}14)$$

Assuming the mass of the ith component to be transferred only by ordinary diffusion and convection, we obtain:

$$j_{x,\,i} = -\rho D\,\frac{\partial m_i}{\partial x} + \rho m_i w_x$$

$$j_{y,\,i} = -\rho D\,\frac{\partial m_i}{\partial y} + \rho m_i w_y$$

$$j_{z,\,i} = -\rho D\,\frac{\partial m_i}{\partial z} + \rho m_i w_z$$

and

$$\frac{\partial j_{x,\,i}}{\partial x} = -\rho D\,\frac{\partial^2 m_i}{\partial x^2} + \rho\left(m_i\,\frac{\partial w_x}{\partial x} + w_x\,\frac{\partial m_i}{\partial x}\right)$$

$$\frac{\partial j_{y,i}}{\partial y} = -\rho D \frac{\partial^2 m_i}{\partial y^2} + \rho \left(m_i \frac{\partial w_y}{\partial y} + w_y \frac{\partial m_i}{\partial y} \right)$$

$$\frac{\partial j_{z,i}}{\partial z} = -\rho D \frac{\partial^2 m_i}{\partial z^2} + \rho \left(m_i \frac{\partial w_z}{\partial z} + w_z \frac{\partial m_i}{\partial z} \right)$$

Summing up these equalities and substituting them in Eq. (14-14), we obtain the following equation:

$$\rho \frac{\partial m_i}{\partial \tau} = \rho D \nabla^2 m_i - \rho \left(w_x \frac{\partial m_i}{\partial x} + w_y \frac{\partial m_i}{\partial y} + w_z \frac{\partial m_i}{\partial z} \right) - \rho m_i \operatorname{div} \vec{w}$$

At $\rho = \text{const}$ the last term in the right-hand side is equal to zero. Then

$$\frac{\partial m_i}{\partial \tau} + w_x \frac{\partial m_i}{\partial x} + w_y \frac{\partial m_i}{\partial y} + w_z \frac{\partial m_i}{\partial z} = D \nabla^2 m_i \qquad (14\text{-}15)$$

or, in an abbreviated form

$$\frac{\partial m_i}{\partial \tau} = D \nabla^2 m_i \qquad (14\text{-}15')$$

Equation (14-15′) is precisely the unknown differential equation of mass transfer describing the distribution of the ith component in a moving mass. The equation of heat transfer (14-15) describes the conservation of the mass of the ith component.

When $w_x = w_y = w_z = 0$, the equation of mass transfer acquires the following form:

$$\frac{\partial m_i}{\partial \tau} = D \nabla^2 m_i \qquad (14\text{-}16)$$

This equation, known as Fick's equation, takes only heat transfer by ordinary diffusion into account. It is similar to the differential equation of conduction (1-29) at $g_v = 0$. If we use similar notations for temperature and mass fraction, the two equations do not differ in appearance.

Velocity components w_x, w_y, and w_z are present both in the equation of diffusion and energy. We must therefore add to these equations of flow written for the mixture as a whole. In addition, the equation of continuity for the mixture must also be added.

To formulate the boundary value problem of heat and mass transfer, the conditions of unambiguity should be added to the system of differential equations of energy, mass transfer, flow and continuity. They consist of geometric, physical, boundary, and time conditions (see Sec. 4-3). The statement of the boundary conditions for mass transfer has a number of peculiarities; to become familiar with them, we shall have to consider the processes of heat and mass transfer into or from a two-component mixture.

14-3. Heat and Mass Transfer

In a moving single-component medium heat is transferred by conduction and convection; the process is known as convective heat transfer. By analogy, the process of simultaneous molecular and molar transport of matter in a moving multi-component medium is called *convective mass transfer*.

The processes of heat and mass transfer are of practical interest in evaporation, condensation, sorption, desorption, sublimation, etc. The system is then heterogeneous. The surface of the liquid (or solid) phase plays a part similar to that of a solid wall in heat-transfer processes without accompanying diffusion.

Similar to heat transfer, the exchange of mass between a fluid or hard surface and the surrounding medium by convection is called *mass transfer.*

Heat and mass transfer develop concurrently in the processes mentioned. Practical calculations of heat transfer are based on the Newton law

$$q_w = \alpha (t_w - t_0)$$

here q_w is in J/m²·s.

Practical calculations of mass transfer are made by equation

$$j_{iw} = \beta (\rho_{iw} - \rho_{i0}) \qquad (14\text{-}17)$$

or

$$j_{iw} = \rho\beta (m_{iw} - m_{i0}) \qquad (14\text{-}18)$$

where j_{iw} = rate of mass flow, kg/m²·s;

β = mass-transfer coefficient, reduced to the concentration gradient (difference) of the diffusing substance, m/s.

The subscripts "w" and "0" indicate that the concentration of the diffusion substance is taken directly on the phase interface and at a distance from it.

Using the law of ideal gases, Eq. (14-17) or (14-18) can be presented as follows:

$$j_{iw} = \beta_p (p_{iw} - p_{i0}) \qquad (14\text{-}19)$$

where β_p = mass-transfer coefficient reduced to the difference in partial pressures $\Delta p = p_{iw} - p_{i0}$.

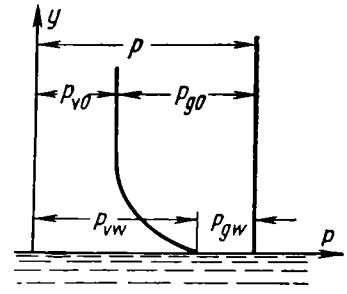

Fig. 14-3. Distribution of vapour and gas concentration at the surface of evaporation

The mass-transfer coefficients β and β_p are bound by the relationship

$$\beta = \beta_p RT \qquad (14\text{-}20)$$

Let us consider the process in which a liquid evaporates into a vapour-gas medium, assuming that the total pressure remains constant over the entire volume of the mixture, and that temperature differences are negligible. Thermal and pressure diffusion can then be ignored. The process of evaporation develops in the absence of any foreign external motion exciters.

Vapour concentration changes from $m_{v\,w}$ on the surface of the evaporating liquid to m_{v0} at a distance from the interface (Fig. 14-3). Since $m_v + m_g = 1$, then

$$\frac{\partial m_v}{\partial y} = -\frac{\partial m_g}{\partial y} \qquad (a)$$

The gas component must, consequently, diffuse in the opposite direction to the vapour. Vapour is free to diffuse into the vapour-gas medium, but the surface of liquid is an insurmountable obstacle for the gas component. As a result the quantity of gas accumulating near the liquid surface must increase continuously. In a steady-state case the partial pressure profiles for vapour and gas must be constant in time; the shifting of the gas towards the surface of evaporation must therefore be compensated by convective flow of the vapour-gas mixture away from the liquid, into the vapour-gas medium. This flow is called the convective or *Stefan's flow*; we denote its velocity by w_{sf}.

Total vapour flow will be equal to the sum of the molecular and convective mass flow:

$$j_{vw} = -\rho D \left(\frac{\partial m_v}{\partial y}\right)_w + \rho m_{vw} w_{sfw} \tag{14-21}$$

Total gas flow at the liquid surface is zero:

$$j_{gw} = -\rho D \left(\frac{\partial m_g}{\partial y}\right)_w + \rho m_{gw} w_{sfw} = 0$$

Taking into account equations (a), from the last equation we get:

$$w_{sfw} = -\frac{D}{m_{gw}} \left(\frac{\partial m_v}{\partial y}\right)_w \tag{14-22}$$

Substituting the value of w_{sfw} in Eq. (14-21), we have:

$$j_{vw} = -\rho D \left(\frac{m_v + m_g}{m_g}\right)_w \left(\frac{\partial m_v}{\partial y}\right)_w = -\rho D \frac{1}{m_{gw}} \left(\frac{\partial m_v}{\partial y}\right)_w \tag{14-23}$$

This equation was first derived by Stefan. It differs from the diffusion law (14-4), relating to conditions of unobstructed diffusion of both components, by the supplementary factor $\frac{1}{m_{gw}}$ which accounts for convective (Stefan's) mass flow resulting from the impermeability of the surface of evaporation for the gas component. Hence it follows that convective (Stefan's) mass flow also appears in the absence of forced or free convection of heat.

The rate of mass flow at the surface of evaporation is determined by means of Eq. (14-17). This mass flow can also be found from Eq. (14-23). Equating the right sides of Eqs. (14-17) and (14-23), we obtain:

$$\beta (m_{vw} - m_{w0}) = -D \frac{1}{m_{gw}} \left(\frac{\partial m_v}{\partial y}\right)_w \tag{14-24}$$

or

$$\beta = -D \frac{1}{m_{gw}} \frac{(\partial m_v / \partial y)_w}{m_{vw} - m_{v0}} \tag{14-25}$$

The process of liquid evaporation into a vapour-gas mixture considered corresponds to the case of a semi-permeable surface, i.e. a surface permeable for one (active) component (vapour) and impermeable for the other (inert) component (gas). A semi-permeable surface is also observed when vapour condenses from a vapour-gas mixture.

With a fully permeable interface both components of the mixture pass through it. The surface is also fully permeable with condensation of the two components of a binary vapour system. The same effect may occur during the evaporation of certain solution.

Let us proceed from the assumption that in the case of heat and mass exchange

$$\alpha = \frac{\lambda}{t_w - t_0} (\nabla t)_w \tag{14-26}$$

This definition of the heat-transfer coefficient does not differ from that previously used.

The total quantity of heat q_w removed from, or imparted to, the liquid and vapour-gas mixture is equal to the sum of q_α, the heat transferred by convection, and q_β, that transported as enthalpy by the diffusing mass.

With a semi-permeable surface the convective mass flow compensates under steady-state conditions for the oppositely directed molecular gas flow, and only a transverse flow of vapour actually appears. Then

$$q_w = q_\alpha + q_\beta = -\lambda (\nabla t)_w + j_{vw} i_{vw} \tag{14-27}$$

14-4. Triple Analogy

Let us compare the equations of diffusion, energy and flow, describing the fields of concentration, temperature, and velocity for independent processes of mass, heat and momentum transport. Let us present the previously derived equations, making some simplifying assumptions.

The equations of mass exchange (disregarding the processes of thermal and pressure diffusion) is

$$\frac{dm_i}{d\tau} = D\nabla^2 m_i \tag{a}$$

The equation of energy (disregarding the diffusion component of the heat flow) is

$$\frac{dt}{d\tau} = a\nabla^2 t \tag{b}$$

The equation of flow (disregarding the mass forces, and for free flow) is

$$\frac{d\vec{w}}{d\tau} = \nu\nabla^2 \vec{w} \tag{c}$$

Equations (a), (b), and (c) are mathematically similar. They include three physical parameters: D, a and ν, each characterizing respectively the transport of mass, heat, and impulse. The units of the parameters D, a, and ν are identical (m^2/s). With $D = a = \nu$ the calculation fields of concentration, temperature, and velocity will be similar, provided the conditions of unambiguity are similar. In particular, the concentration and temperature fields will be similar, if $D = a$, or $\frac{D}{a} = 1$. The ratio $\frac{D}{a}$ is known as the *Lewis-Semenov* number and is denoted by Le.

For a heat transfer not complicated by simultaneous mass transfer ("pure" heat transfer), and disregarding the mass forces, it was shown earlier that

$$\overline{\mathrm{Nu}} = \varphi\,(\mathrm{Re},\ \mathrm{Pr}) \tag{d}$$

Proceeding from the analogy between the processes of heat and mass transfer, we can write:

$$\overline{\mathrm{Nu}}_D = \psi\,(\mathrm{Re},\ \mathrm{Pr})_D \tag{e}$$

where

$$\overline{\mathrm{Nu}}_D = \frac{\overline{\beta} l}{D}$$

is the Nusselt number for diffusion; and

$$\mathrm{Pr}_D = \frac{\nu}{D}$$

is the Prandtl number for diffusion. These numbers are analogs of the numbers $\overline{\mathrm{Nu}}$ and Pr.

On the analogy between heat and mass transfer the functions φ and ψ must be similar. If the like reference dimensionless terms are equal, the Nu and Nu_D numbers will also be numerically equal. It is possible, for instance, to conduct investigations of heat transfer and to use the derived dimensionless formulae to calculate mass transfer, replacing the Nu and Pr numbers respectively with the Nu_D and Pr_D numbers. If, for instance, the equation

$$\overline{Nu} = a\mathrm{Re}^n\mathrm{Pr}^m$$

is derived to calculate heat transfer, mass transfer developing under similar conditions can be calculated by means of equation

$$\overline{Nu_D} = a\mathrm{Re}^n\mathrm{Pr}_D^m$$

where a, n, and m are identical values.

The analogy between these processes is often used in calculations. The analogy, however, is, strictly speaking, only approximate. In the general case the equations of diffusion (14-15), energy (14-13), and flow (14-18) are not similar. The equations of heat transfer (4-22) and mass transfer (14-25) are also different. The physical parameters which are essential to mass and energy transfer may vary in a different manner. The boundary conditions are also different. As a result the analogy between heat transfer and mass transfer breaks down.

An account must be taken of the dependence of heat and mass transfer on supplementary dimensionless variables reflecting the peculiar nature of the concurrently developing processes of heat and mass transfer. These variables can be obtained, analyzing the boundary value of heat and mass transfer with the aid of the methods described in Chapter 5.

If mass transfer proceeds at a low rate, its effect on heat transfer can often be ignored with an accuracy sufficient for practical applications.

14-5. Diffusion Boundary Layer

By analogy with the notions of the hydrodynamic and thermal boundary layers the notion of the diffusion boundary layer can be introduced. The concentration of the active component within the boundaries of this layer varies from $m_{i,\,w}$ on the phase-interface to $m_{i,\,0}$ on the external border of the layer (Fig. 14-4). The condition $\partial m_i/\partial y \neq 0$ is valid outside the diffusion boundary layer, and the following conditions are satisfied outside the diffusion boundary layer and on its external border

$$m_i = m_{i0}; \quad \partial m_i/\partial y = 0$$

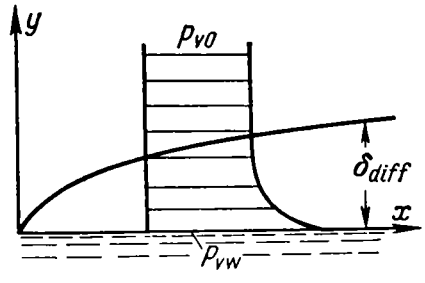

Fig. 14-4. Diffusion boundary layer

A diffusion boundary layer may form during processes of evaporation, sublimation, blowing of a substance through a porous wall, condensation of vapour from a gas-vapour mixture, etc.

The differential equation of mass transfer can be simplified for the diffusion boundary layer. With an infinite plate exposed to flow the concentration field for the diffusion boundary layer can be described by the following equation:

$$\rho\left(w_x\frac{\partial m_i}{\partial x} + w_y\frac{\partial m_i}{\partial y}\right) = -\frac{\partial j_{y,\,i}}{\partial y} \qquad (14\text{-}28)$$

The differential equation of the diffusion boundary layer (14-28) is similar to the equations of the thermal and hydrodynamic boundary layers (4-28), (4-30) and it is valid under identical conditions. Hence, under analogous unambiguity conditions the solutions of these equations must be the same.

In Eq. (14-28) the term $j_{y,\,i}$ represents the cross component of the mass rate of flow of the ith component of a mixture. For the turbulent boundary layer

$$j_{y,\,i} = -\rho D \frac{\partial m_i}{\partial y} - \rho \varepsilon_j \frac{\partial m_i}{\partial y} = -\rho (D + \varepsilon_j) \frac{\partial m_i}{\partial y} \qquad (14\text{-}29)$$

where ε_j is the coefficient of eddy mass transfer (coefficient of turbulent diffusion) of substance. For turbulent flow m_i, w_x, w_y, $j_{y,\,i}$ are time-averaged quantities (see Sec. 4-5).

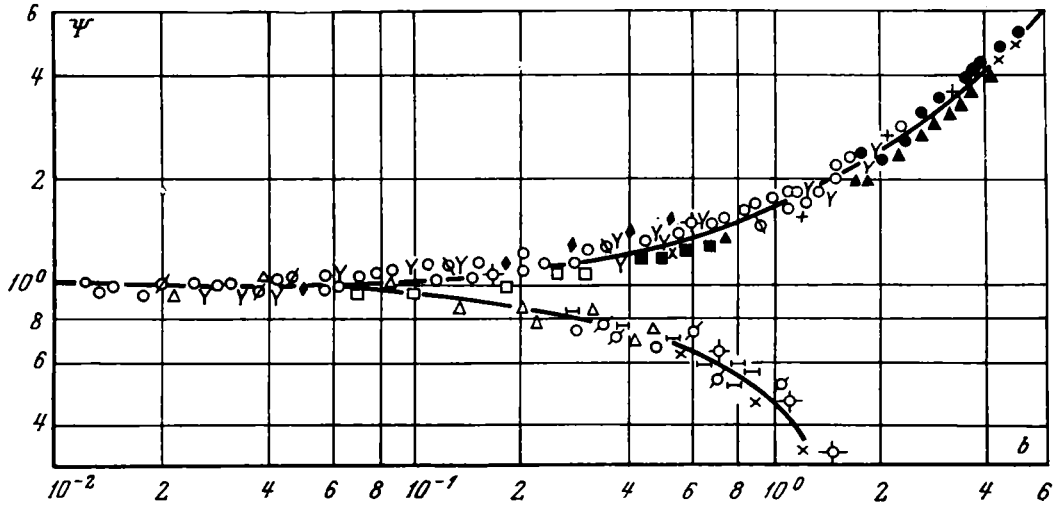

Fig. 14-5. Effect of cross flow of substance on heat transfer

The normal component of velocity turns into zero on the surface of an impermeable wall. In the presence of mass transfer the cross component of velocity $w_y = j_y/\rho$ is not equal to zero directly at the phase interface.

The presence of a transverse mass flow, in respect to the main flow, results in that temperature distribution and the magnitude of the heat-transfer coefficient may differ from these that are true for heat transfer not accompanied by mass transfer.

It follows from the theory of the boundary layer that when the transverse mass flow of substance is directed away from the interface (evaporation, sublimation, desorption, injection of a gas through a porous plate) the thickness of the boundary layer increases, and the derivatives $\partial w_x/\partial y$ and $\partial t/\partial y$ diminish, resulting in a smaller heat-transfer coefficient.

If the transverse mass flow is directed towards the interface (condensation, sorption, evacuation of a gas), the thickness of the boundary layer decreases and the derivatives $\partial w_x/\partial y$ and $\partial t/\partial y$ increase. As a result, the heat-transfer coefficient increases with the rate of transverse mass flow.

Mass-transfer coefficients depend qualitatively on the direction and magnitude of the transverse mass flow like heat-transfer coefficients.

This influence of the transverse mass flow of substance manifests itself both with a laminar and turbulent boundary layer.

The effect of the transverse mass flow on heat transfer is illustrated in Fig. 14-5 [14.4-14.5]; here $\Psi = \mathrm{St}/\mathrm{St}_0$, where St_0 is the Stanton number in the absence of

mass transfer (Chapter 7); $b = (j_{iw}/\rho w_0)\ 1/\mathrm{St}_0$ is the permeability factor proportional to the rate of transverse mass flow on the wall (interface), j_{iw}; w_0 = flow velocity outside the boundary layer.

The upper curve in Fig. 14-5 corresponds to the processes of evacuation or condensation, and the lower—to injection or evaporation.

At $b \lesssim 0.1$ heat transfer is practically independent of the transverse flow of substance.

In conclusion it will be noted that in the event of a very intensive injection the boundary layer equations may not hold. The validity condition for these equations is $b < 10$, with the boundary layer being still comparatively thin.

14-6. Heat and Mass Transfer With Vapour Condensing From a Vapour-gas Medium

The presence of a non-condensing gas in a vapour interferes with the flow of the vapour to the surface of condensation. As a result the rate of condensation reduces.

Let us assume the wall to be impermeable. Its temperature t_w is lower than the temperature of the bulk of the gas-vapour mixture t_{v0}. A film of the forming condensate flows down the wall (Fig. 14-6).

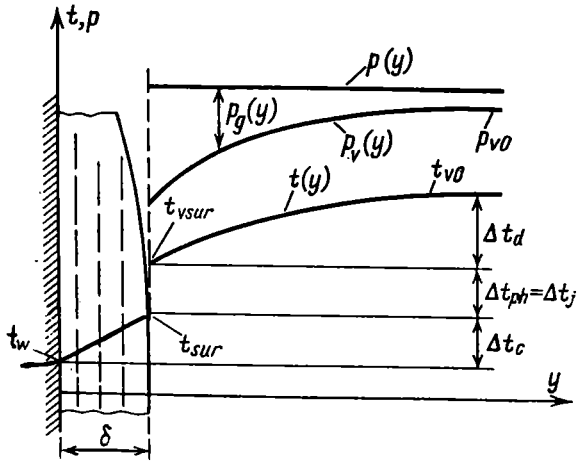

Fig. 14-6. Distribution of concentrations and temperature with vapour condensing from a gas-vapour mixture

The total amount of heat transferred to the film surface is

$$q_{sur} = \alpha (t_{v0} - t_{vs}) + j_{vs} i_{vs} \tag{14-30}$$

where α is the coefficient of heat transfer from the gas-vapour mixture to the condensate film.

Vapour reaching the interface condenses on it. Condensation is accompanied by liberation of the latent heat of phase transition $rj_{vs} = (i_{vs} - i_{ls})\, j_{vs}$. The latent heat of evaporation and the heat transferred by convection are both imparted to the solid wall on which the condensed phase is located Transport of heat through the moving condensate film is governed by the regularities of heat convection within the liquid (already described in Chapter 12). Also transferred to the solid wall is some heat of condensate subcooling in respect to t_{sur}, since temperature changes across the thickness of the film from t_{sur} to t_w (Fig. 14-6). The heat of subcooling is usually small and can be ignored in many calculations. For steady-state processes the rate of heat flow can then be taken as constant throughout the condensate layer (for a plain layer).

Ignoring subcooling of condensate, the rate of heat flow on the wall can be described by the following equation:

$$q_w = \alpha (t_{v0} - t_{vs}) + r j_{vs} \tag{14-31}$$

and often another equation is used for finding q_w

$$q_w = \alpha_{mix} (t_{v0} - t_w) \tag{14-32}$$

Here α_{mix}—heat-transfer coefficient related to the difference between the temperatures of the wall and vapour (vapour-gas mixture) at a distance from the surface of condensation. It is a complex value, accounting for different thermal resistances.

The total thermal resistance $\sum R = 1/\alpha_{mix}$ can be broken up into the thermal resistance of the condensate R_c, thermal resistance of phase transition R_{ph} and the thermal resistance offered to heat (vapour) flow to the surface of condensation (diffusive thermal resistance) R_D. Corresponding to these thermal resistances are the temperature differences Δt_c, Δt_{ph}, and Δt_D (Fig. 14-6), and

$$t_{v0} - t_w = \Delta t_c + \Delta t_{ph} + \Delta t_D$$

Equation (14-32) can be written as follows:

$$q_w = \frac{t_{v0} - t_w}{\sum R} = \frac{t_{v0} - t_w}{R_c + R_{ph} + R_D} \tag{14-33}$$

Here

$$\alpha_{mix} = \frac{1}{\sum R} = \frac{1}{R_c + R_{ph} + R_D} \tag{14-34}$$

In many problems $R_{ph} \ll R_c$ and $R_{ph} \ll R_D$, which makes it possible to ignore the thermal resistance of phase transition (to assume $t_{vs} = t_s$). Ignoring the temperature jump Δt_{ph}, the temperature of the surface of condensation t_s can be considered as the vapour saturation temperature at the saturation pressure p_{ps}. Then,

$$R_c \equiv \frac{1}{\alpha_c} = \frac{t_s - t_w}{q_w} = \frac{t_s(p_{vs}) - t_w}{q_w} \tag{14-35}$$

The values of R_c or α_c can be determined by the formulae given in Chapter 12. The diffusive thermal resistance

$$R_D = \frac{t_{v0} - t_{vs}}{q_w} = \frac{t_{v0} - t_{vs}}{\alpha(t_{v0} - t_{vs}) + r j_{vs}} = \frac{t_{s0} - t_{vs}}{\alpha(t_{v0} - t_{vs}) + r\beta(\rho_{v0} - \rho_{vs})} \tag{14-36}$$

The local heat-transfer coefficient α must be determined with account taken of the accompanying process of mass transfer (see Sec. 14-5).

It follows from Eq. (14-34) that the local heat-transfer coefficient α_{mix} depends on the intensity of the interrelated processes of heat and mass transfer in the gas-vapour mixture and in the film. To calculate the coefficient α_{mix}, first the mass-transfer coefficient β must be determined.

For vapour film and nucleate condensation from a vapour-gas mixture on a horizontal tube, with the mixture being practically stagnant at a large distance from the tube, the mean mass-transfer coefficient can be determined from formula [14.6]:

$$\overline{\mathrm{Nu}}_D = \frac{f/m_{gs}}{1 - P^{-f}} \psi \tag{14-37}$$

where $f = \ln \frac{m_{gs}}{m_{g0}} \Big/ \ln \frac{R_0}{R_T}$; $P = R_0/R_T$; $\mathrm{Nu}_D = \frac{\beta R_T}{D}$; R_T = radius of tube (surface of condensation); R_0 = radius at which the partial pressure of the gas m_{g0} is taken at a distance. The subscript s corresponds to the surface of condensate.

Formula (14-37) was derived for conditions at which the parameters of the vapour-gas mixture are homogeneous at a fixed value of $R_0 \gg R_T$. The formula accounts both for the Stefan flow and natural convection. At $\mathrm{Gr} = \frac{g\beta\Delta t_D (2R_T)^3}{\nu^2} \geqslant 8$

$\psi = 0.66\mathrm{Gr}^{0.2}$; at $\mathrm{Gr} \lesssim 8$ ψ can be assumed equal to unity (there is no effect of natural convection). The physical parameters of the vapour-gas mixture substituted in Eq. (14-37) are taken by the temperature t_{v0}.

Of practical concern is the process of heat and mass transfer with film condensation from a moving vapour-gas mixture, both on horizontal single tubes and on tubes in a bank. Heat and mass transfer have been investigated in a number of works already mentioned.

A vapour-gas mixture [14.7] was ducted to a single horizontal tube and a horizontal bank of tubes from the top. The pressure of the vapour-gas mixture p was varied from 0.0627 to 0.089 MN/m²; the initial volumetric content of air in steam was $\varepsilon_{g0} = p_{g0}/p = 0.01$ to 0.56.

It was found that the mean mass-transfer coefficient for a definite row of tubes could be calculated at Re = 350 to 4800 by the equation

$$\overline{\mathrm{Nu}}_D = c\mathrm{Re}^{0.5}\varepsilon_{g0}^{-0.6}\pi_D^{-1/3} \quad (14\text{-}38)$$

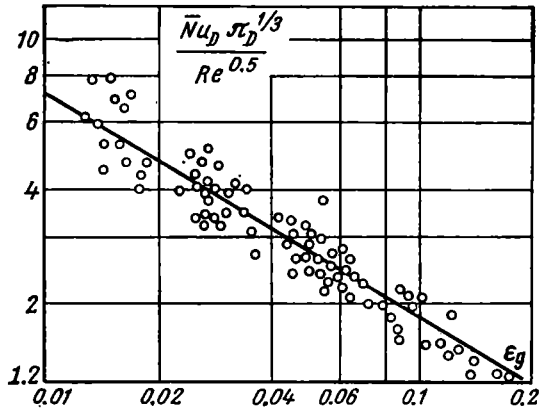

Fig. 14-7. Mass transfer for vapour condensing from a moving vapour-gas mixture on a single tube ($p \approx 0.09 \cdot 10^5$ and $0.8 \cdot 10^5$ Pa)

Here $c = 0.47$ for a single tube, while for the first row of a bank $c = 0.53$, and for the third and following rows $c = 0.82$.

Formula (14-38) for a single tube is correlated with test results in Fig. 14-7.

With Re = 40 to 350 the following formula was obtained for tubes of the fifth row:

$$\overline{\mathrm{Nu}}_D = 0.52\mathrm{Re}^{0.5}\varepsilon_{g0}^{-0.7}\pi_D^{-1/3} \quad (14\text{-}39)$$

Here and in formula (14-38) $\pi_D = (p_{v0} - p_{vs})/p$.

The Reynolds number in formulae (14-38) and (14-39) was calculated for the inlet (tube or row of tubes) velocity of the mixture, the reference dimension was the outside tube diameter; the physical parameters of the mixture are determined by the state of the vapour gas mixture upstream of the tube or row of tubes. The viscosity coefficient of the mixture was calculated from equation

$$\mu_{mix} = \frac{(1-\varepsilon_{g0})\mu_v + 1.6\varepsilon_{g0}\mu_g}{1+0.61\varepsilon_{g0}}$$

where μ_v and μ_g are respectively the viscosity coefficients of vapour and air.

Formulae (14-38) and (14-39) were obtained in tests on a bank of staggered tubes, the relative transverse pitch of which was 1.475, and the longitudinal pitch 1.275. These formulae may also be used in practice for a first approximation with banks of tubes of other pitches.

According to [14.7], the quantity of heat transferred by convection from the mixture to the surface of the condensate film, and the heat due to subcooling of the condensate, are fairly small compared with the latent heat of phase transition, and can be ignored.

The value of t_s, on which p_{vs} used in the calculations is based, is determined by successive approximations, making use of the equation:

$$r\beta_p = (p_{v0} - p_{vs}) = \alpha_c (t_s - t_w)$$

14-7. Heat and Mass Transfer With Liquid Evaporating Into a Vapour-gas Medium

The evaporation of liquid from a free surface arises from the thermal motion of the molecules of the liquid. Molecules possessing kinetic energy of a magnitude sufficient to overcome the intermolecular forces, break away from the surface layer of the liquid into the surrounding medium. As a result of collision with one another or with molecules of the gas, a fraction of the detached molecules are deflected back to the surface of liquid, where they are either deflected or absorbed by the liquid, while another fraction is distributed through the surrounding medium by molecular and molar diffusion and is lost by the liquid.

By analogy with the condensation coefficient the concept of the evaporation coefficient may be introduced. It is equal to the ratio of the number of irrevocably separating vapour molecules to the number of molecules emitted by the liquid.

The two coefficients are usually considered equal and the temperature jump is mostly ignored, and no consideration is given to the thermal resistance offered to heat flow, due to phase transition. The pressure of the vapour component in the non-rarefied layer of the gas-vapour mixture at the surface of the liquid is assumed to be equal to saturation pressure at the liquid surface temperature.

If the gas is not saturated with vapour at a distance from the liquid, a flow of substance originates, always directed from the surface of evaporation, and the flow of heat may then be directed both from the liquid to the gas and from the gas to the liquid; the direction of heat flow will depend on whether the temperature of the evaporation surface, t_s, is smaller or greater than t_{vg}, the temperature of the gas-vapour mixture.

The quantity of heat consumed by evaporation is rj_{vs}, W/m^2, where r is the latent heat of evaporation. The liquid cools, if the quantity of heat imparted to it is smaller than the heat absorbed by evaporation and heats if greater.

For the case of steady-state evaporation from the surface of a flowing down film, when $t_w > t_s$ (no boiling takes place) and $t_s > t_{vg}$, the calculation may be carried out following the method outlined in the preceding paragraph. Naturally, then the change in the direction of heat flow must somehow be taken into account.

Let us assume that a certain volume of liquid evaporates from an open vessel. The walls of the vessel are covered with heat insulation. The layer of liquid is thin, so that temperature variation within it can be ignored. The liquid is topped by a flowing vapour-gas mixture, which does not become saturated, because its rate of flow is very large. There are no inner heat sources in the liquid, and thermal radiation can be neglected.

Let us further assume that the liquid temperature t_s is larger at the beginning of the process than that of the vapour-gas mixture t_{vg} at a distance from the liquid. The temperature of the liquid will decrease, owing to heat transfer and evaporation, and the process of evaporation will be transient. At a certain moment of time the temperature of the liquid will become equal to that of the mixture. Heat transfer will then cease, in accordance with equation $q_s = \alpha (t_s - t_{vg})$. Evaporation, however, will continue, and lead to a further drop in liquid temperature, which will become less than that of the mixture; this will result, in turn, in a flow of heat beginning from the mixture to the liquid. Evaporation slows with falling liquid temperature, because $p_{vs}(t_{vs})$ and $\Delta p = p_{vs} - p_{v0}$ diminish, but the rate of heat transfer increases. These variations will continue until at a certain liquid temperature t_m dynamic equilib-

rium is established between supply of heat through convection and removal of heat by evaporation and subsequent diffusion.

Further evaporation takes place at $t_s = t_m$, at the expense of heat drawn from the vapour-gas mixture by heat transfer; t_m is the limit below which cooling of the liquid is impossible. Then

$$\alpha (t_{vg} - t_m) = r\beta_p (p_{vs} - p_{v0}) \tag{14-40}$$

The process of evaporation in which all the heat transferred from the vapour-gas mixture is consumed in evaporation of the liquid, and is returned to the

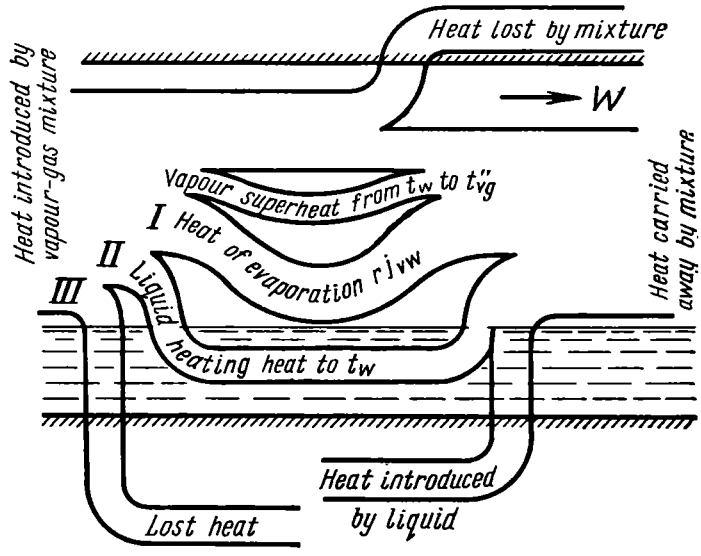

Fig. 14-8. Heat balance diagram for liquid evaporating in a vapour-gas flow

mixture with the vapour formed, is known as *adiabatic evaporation*. The temperature t_m is called the temperature of liquid in adiabatic evaporation, or wet-bulb temperature.

From Eq. (14-40) it follows that with adiabatic evaporation

$$\alpha \equiv \frac{q_w}{t_{vg} - t_m} = \frac{rj_{vs}}{t_{vg} - t_m} \tag{14-41}$$

and

$$t_m = t_{vg} - \frac{rj_{vs}}{\alpha} \tag{14-42}$$

Non-adiabatic processes of evaporation are often encountered in practice. Let us therefore consider a steady-state process of non-adiabatic evaporation of liquid into a vapour-gas mixture flowing above it. We have flow in a conduit whose lower section is filled with evaporating liquid.

Let the inlet temperature of the flow be t'_{vg}. The vapour fraction of the mixture increases during flow owing to evaporation. The temperature of the flow changes as well; we denote the outlet temperature by t''_{vg}. In the general case the outlet temperature of the vapour-gas flow t''_{vg} may be larger or smaller than t'_{vg}.

The evaporated liquid may be made up by more liquid in equal quantity, but at another temperature which we denote by t'_l.

The heat balance diagram of the process considered is presented in Fig. 14-8, in which it is assumed that $t'_l < t_s$.

The quantity of heat transferred from the vapour-gas mixture to the liquid is q_s, J/m²·s (this quantity of heat is represented in Fig. 14-8 by components *I*, *II*, and *III*). The heat q_s is consumed in the general case to evaporate the liquid (rj_{vs}, component *I*), and a fraction of it may be transferred to the liquid phase by conduction and convection (components *II* and *III*; we denote this fraction by q_l). Then,

$$q_s = -\lambda \left(\frac{\partial t}{\partial y}\right)_s = \alpha \Delta t = rj_{vs} + q_l = rj_{vs} - \lambda_l \left(\frac{\partial t_l}{\partial y}\right)_s \qquad (14\text{-}43)$$

where λ_l and $(\partial t_l/\partial y)_s$ are respectively the thermal conductivity of the liquid and its temperature gradient at the interface.

An amount of heat $q_l = -\lambda_l (\partial t_l/\partial y)_s$ is consumed to heat the make-up liquid from t_l' to $t_l'' = t_s$; a fraction of it may be lost to the surroundings through the walls confining the liquid. Then,

$$-\lambda_l (\partial t_l/\partial y)_s = j_{vs} c_{pl} (t_l'' - t_l') + q_T \qquad (14\text{-}44)$$

where q_T is the heat lost by radiation to the surroundings.

But if $t_l' > t_s$, the sign of q_l changes and heat flows, in addition, from the liquid to the surface of evaporation. Evaporation is then due not only to the heat coming from the vapour-gas medium, but also to the heat brought in by the make-up liquid, and coming from outside through the confining walls.

In consideration of that, the following formula can be used to calculate the heat transfer:

$$q_s = \alpha (t_{vg} - t_s) = rj_{vs} \pm j_{vs} c_{pl} (t_s - t_l') \pm q_T \qquad (14\text{-}45)$$

Here the sign is "plus" if $t_l' < t_s$, and "minus" at $t_l' > t_s$.

Evaporation is also possible through a porous plate separating the vapour-gas and liquid components. The temperature distribution through such a wall was derived in Sec. 2-12.

With evaporation through a porous wall

$$q_l = -\lambda_{eq} (\partial t/\partial y)_s$$

where λ_{eq} is the equivalent thermal conductivity of the wall and the liquid filling its pores; t is the temperature of the wall.

Equation (14-45) also applies to evaporation from a porous wall.

The vapour flow toward the surface of evaporation changes the temperature and velocity fields, and this leads to a change in the intensity of heat transfer. As has already been said, theoretical work has shown the thickness of the thermal and hydrodynamic boundary layers to increase during evaporation, sublimation, and injection of a substance through a porous wall; as the layers thicken, the temperature gradient at the evaporating surface falls and there is a consequent drop in the heat-transfer coefficient. The greater the density of the vapour flow toward the surface the larger the drop in the rate of heat transfer.

The coefficients of heat and mass transfer depend on the shape and dimensions of the surface of evaporation, the pattern of flow of the vapour-gas mixture (free or forced, laminar or turbulent), the physical properties of the liquid and gas components, and the concentration of the two components in the vapour-gas mixture.

The process becomes complicated due to other reasons. In illustration, let us consider adiabatic evaporation from a porous body (Fig. 14-9). The heat of evaporation generally comes to the liquid directly (Q_l) and through the porou

body (Q_s). The transport of heat Q_s will depend on the thermal conductivity of the body. The calculated heat-transfer surface F_{cal} (represented by a dotted line in Fig. 14-9) is not equal to the actual heat-transfer surface, corresponding to the outer boderline of the solid and liquid (represented by a thick line in the illustration). The difference shall be the greater, the lower is the level of liquid.

As liquid evaporates from a porous body the actual heat- and mass-transfer surfaces differ, since the liquid evaporates from the entire surface, and this difference depends on how deep is the penetration of the liquid. According to published data [14.8], evaporation also takes place partly from the very thin layers of liquid, adjoining the meniscus, due to the capillary effects of the second kind. With a considerably deep level of liquid the transport of vapour to the calculation surface is determined in many ways by the resistance of the capillaries. With very small narrow capillaries, the flow through them is governed by the laws of flow applicable to rarefied media.

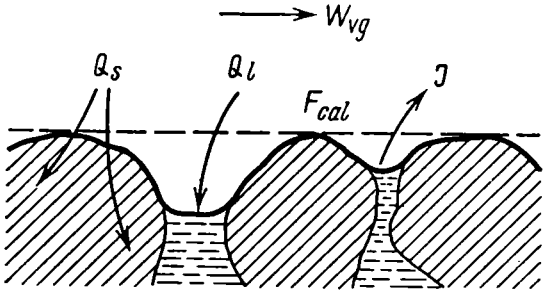

Fig. 14-9. Evaporation from a porous solid

With evaporation from a free surface of a great volume of liquid peculiarities are also observed. The vapour-gas flow usually disturbs the surface layer of the liquid, resulting in a possible wave formation and carry over of drops of liquid into the gas flow.

The mentioned reasons make it rather difficult to derive sufficiently generalized relationships and sometimes they preclude correlation of test data, obtained under different conditions.

With water evaporating from a porous metal plate into a longitudinal air-vapour flow heat and mass transfer was investigated experimentally under conditions close to adiabatic [14.9-14.10].

According to this research, the mean heat-transfer coefficient can be determined, in the presence of a turbulent boundary layer, by the equation:

$$\overline{\mathrm{Nu}}_l = 4.55 \times 10^{-3} \mathrm{Re}_l^{0.8} \mathrm{K}^{0.4} \qquad (14\text{-}46)$$

where $\mathrm{K} = r/c_p \Delta t$ is the criterion representing the phase transition; here $\Delta t = t_{vg} - t_w$ is the temperature head.

The linear dimension in Eq. (14-46) is the length of the plate along the flow, measured from the beginning of the section of evaporation. The reference temperature is taken as the temperature of the mixture at a distance from the plate; mixture parameters are introduced into the dimensionless terms of similarity.

The Reynolds number Re_l was varied in the tests [14.9-14.10] from 1.3×10^5 to 1.65×10^6. with corresponding variations in the velocity and temperature of vapour-air flow from 9 m/s to 115 m/s and from 12 °C to 140 °C respectively. The mean temperature difference was varied from 6 K to 80 K. The heat-transfer coefficient was deduced to the full surface of the plate F_{cal}. The volume of pores in the plate amounted to 40 per cent.

According to these test data [14.9-14.10] and other research the heat- and mass-transfer coefficients decrease with an increase in the vapour fraction in the mixture.

The following equation was obtained to calculate mass transfer [14.10]:

$$\overline{\mathrm{Nu}}_{Dl} = 2 \times 10^{-4} \mathrm{Re}_l^{0.8} \pi_D^{-0.5} \varepsilon_{v0}^{-0.5} \qquad (14\text{-}47)$$

where

$$\varepsilon_{v0} = 1 - \varepsilon_{g0} = \frac{p_{v0}}{p}\ ;\quad \pi_D = \frac{p_{vw} - p_{v0}}{p}$$

p_{vw} is the partial pressure of the water vapour at the surface of evaporation, and p_{v0} its partial pressure at a distance from the evaporating surface; p is the total pressure of the vapour-air mixture. The remaining criteria of similarity are those of the preceding equations. The value of p_{vw} was taken from steam tables as the saturation pressure with $t_s = t_w$.

The magnitude of ε_{v0} was varied during the tests from 0.00365 to 0.0168.

From the research of the Moscow Power Institute, the ratio $\mathrm{Nu}/\overline{\mathrm{Nu}}_D$ is not constant, and is a function of specific criteria accounting for the peculiarities of the two concurrent processes of heat transfer and mass transfer: $(\overline{\mathrm{Nu}}_l/\overline{\mathrm{Nu}}_{Dl}) = 0.7\pi_D\mathrm{K}$. Comparison of Eqs. (14-46) and (14-47) shows that α/β or α/j_{vw} are independent of velocity.

The temperature of the porous wall depends on a great many different factors. In the general case it must be calculated by means of Eq. (2-130).

With adiabatic evaporation the calculation is simplified. The temperature of the evaporating surface can then be calculated by Eq. (14-42):

$$t_w = t_m = t_{vg} - \frac{rj_{vw}}{\alpha}$$

From the dimensionless equations previously obtained it follows that

$$\frac{rj_{vw}}{\alpha} = \frac{r\beta_p \Delta p}{\alpha} = \frac{rD_{pv}p}{230\lambda}\left(\frac{\Delta p}{p_s}\right)^{5/6}$$

where $\Delta p = p_{vw} - p_{v0}$.

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CHAPTER

15

HEAT AND MASS TRANSFER WITH CHEMICAL TRANSFORMATIONS

15-1. Basic Data on Chemical Transformations

Processes of heat transfer which are accompanied by chemical reactions take place in the combustion chambers of various engines (jet engines, gas turbines, etc.), in apparatus of the chemical industry, in MGD plants, at hypersonic flight speeds in dense atmospheres, and in other cases.

Chemical reactions may be accompanied by the generation or absorption of energy of various kinds in the form of heat, electricity, light, mechanical power, etc. Here we shall only consider chemical reactions, involving either the absorption of heat (endothermic reactions) or the generation of heat (exothermic reactions) in which as with phase transitions of the first order, the chemical transformations are indissolubly linked with a process of heat transfer. Chemical reactions are often accompanied with phase changes, which make the phenomena considered more intricate.

Chemical reactions can take place either on the surface of a solid or in a fluid medium to which the solid is exposed. In the first case, the reaction is referred to as *heterogeneous*, and in the second case, as *homogeneous*.

When reactions develop at a distance from a solid (beyond the boundary layer), they may have no effect on heat transfer and heat exchange between the medium and wall in the boundary layer can be calculated by the usual methods. In the opposite case account must be taken of the emission or absorption of heat in the boundary layer or directly on the wall.

In what follows we shall mainly consider processes developing in gaseous mixtures and on walls exposed to them. Such a selection of processes to be considered is explained not only by the fact that they are more exhaustedly covered, but also because such problems are of a considerable practical interest.

Modern engineering is characterized by a rise in heat loads, temperatures, velocities of heat carriers and other parameters. At high temperatures working processes are inevitably accompanied by chemical transformations; thus, for example, as already mentioned in Sec. 11-3, as a result of aerodynamic heating at hypersonic flight speeds, air can acquire such a high temperature that it may begin to dissociate. At much higher temperatures actual ionization may take place*.

The regions of the main chemical reactions in high-temperature air are shown in Fig. 15-1 according to pressure and temperature.

These effects can sometimes take place at comparatively low temperatures. In the temperature range between 0 and 120 °C the effective conductivity of nitrogen dioxide is known to be very high in consequence with the reversible reaction $2NO_2 \rightleftarrows N_2O_4$. At temperatures above 120 °C an equilibrium mixture contains mainly NO_2, and at temperatures below 0 °C, mainly N_2O_4. The unusually high apparent thermal conductivity of NO_2 is a consequence of molecules

* Since an ionized gas is a conductor of electricity, electromagnetic fields and forces may develop.

diffusing from high-temperature to low-temperature regions, where they recombine, releasing the corresponding heat of reaction.

Let us consider certain fundamentals of *thermochemistry* applying to the processes with which we are concerned.

As follows from the first law of thermodynamics the *heat of chemical reaction* ΔH for a reaction developing at constant pressure and temperature will be the difference between the enthalpies of the initial and final states (assuming that the only work performed is that of expansion) and independent of the way the reaction develops

$$Q_{p=\text{const}} = H_2 - H_1 = \Delta H$$

For our purposes we shall assume that the reactions considered occur at constant pressure of the mixture (which usually applies to processes of heat and mass-

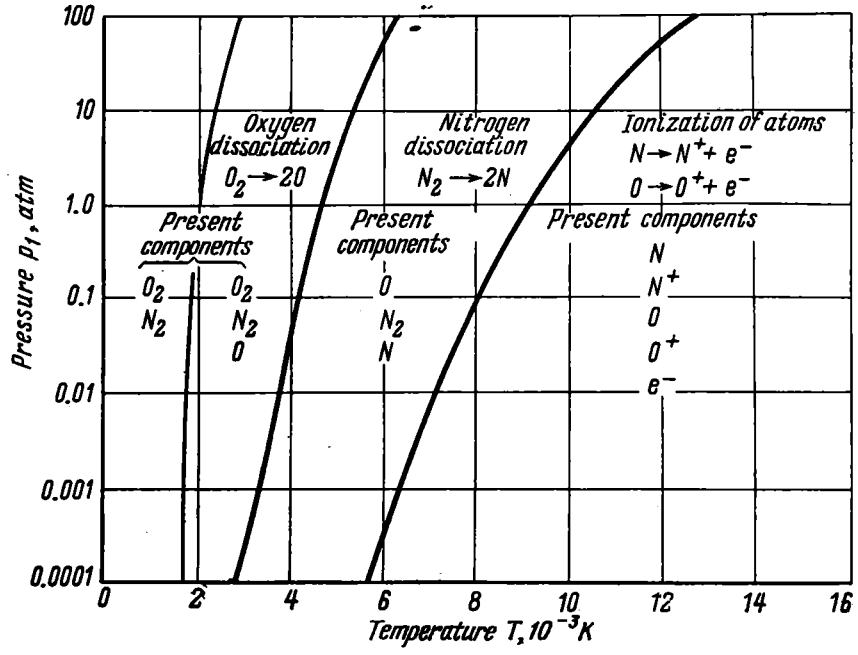

Fig. 15-1. Regions of air dissociation and ionization depending on pressure and temperature

transfer) and at constant temperature (which is ensured for local values in steady-state conditions). Then for any chemical reaction described by a stoichiometric equation of the following form

$$\sum_{i=1}^{i=m} a_i A_i \rightleftarrows \sum_{i=1}^{i=n} b_i B_i$$

the heat of reaction, equal in absolute terms to the quantity of heat liberated ΔH, J/kg mol, taken with opposite sign, can be expressed by the equation

$$-\Delta H = \sum_{i=1}^{i=m} a_i H_{A_i} - \sum_{i=1}^{i=n} b_i H_{B_i} \qquad (15\text{-}1)$$

where A_i, B_i = chemical symbols respectively of reagents and reaction products, e.g. C, O_2, CO_2, etc.;

a_i, b_i = stoichiometric coefficients, representing respectively the number of moles of the reagents and reaction products;
$H_{A_i} H_{B_i}$ = molar enthalpies of the reagents and reaction products respectively.

When the reaction is endothermic, the heat of the reaction is considered positive and when heat is liberated, negative. This system of signs for the heat of reaction is called thermodynamic; an opposite system of signs (or notation of reactions) is occasionally known as thermochemical system used.

The heat of reaction depends on the temperature at which the process occurs, but for most chemical reactions the dependence is weak. When the number and type of molecules are not changed by the reaction the variation in the heat of the reaction is particularly small. Handbooks usually give standard values for the heat of reaction (at 0.09806 MN/m² and 298.15 K). We shall assume that the heat of chemical reaction is constant, i.e. does not change for the specific reactions in a definite example of heat transfer.

In thermochemical calculations the concept of *heat of formation* is used which is the heat developed in the chemical reaction during the formation of a given substance (or component of a mixture) from starting simple substances. For many substances (or components) no reaction of formation from the starting substance can in practice occur, and the heat of formation is taken in the general case as an auxiliary quantity that is of importance for calculating the heat of chemical reaction. The heats of formation of many chemicals are given in handbooks.

In calculations with chemical transformations the enthalpy is used where the formation heat of the given component from initial substances is used. The total specific enthalpy h_i of an ith component is

$$h_i = \int c_p \, dT + h_i^0 \tag{15-2}$$

where h_i^0 = specific heat of formation of the ith component, J/kg i.

Just like the heat of reaction ΔH, h_i^0 can be either positive or negative. If heat is added while the ith component is being formed, then $h_i^0 > 0$, and if heat is removed, $h_i^0 < 0$.

In cases when phase transformations as well as chemical reactions take place, the enthalpy should also account of the heat of phase transition.

The enthalpy h of a mixture of gases is determined by the additivity law:

$$h = \sum_{i=1}^{i=n} m_i h_i \tag{15-3}$$

where $m = \rho_i/\rho$ = the relative mass concentration of the ith component of the mixture;
ρ_i = the mass concentration of the ith component, kg/m³;
ρ = the density of the mixture, kg/m³.

Let us relate the heat of reaction, determined by equation (15-1), to the unit mass of the reagents or reaction products. By definition

$$h_{A_i} = \frac{H_{A_i}}{M_{A_i}}$$

and

$$h_{B_i} = \frac{H_{B_i}}{M_{B_i}}$$

where M_{A_i} and M_{B_i} = molecular weights of substances A_i and B_i, h_{A_i} is in J/kg A_i and h_{B_i}—in J/kg B_i.

Then it follows from equation (15-1) that specific heat liberation is determined by the following equations:

$$\Delta h_{A_1} = \frac{-\Delta H}{M_{A_1}} = a_1 h_1 + \frac{1}{M_{A_1}} \left(\sum_{i=2}^{i=m} M_{A_i} a_i h_{A_i} - \sum_{i=1}^{i=n} M_{B_i} b_i h_{B_i} \right) \qquad (15\text{-}4)$$

$$\Delta h_{B_1} = \frac{-\Delta H}{M_{B_1}} = -b_1 h_1 + \frac{1}{M_{B_1}} \left(\sum_{i=1}^{i=m} M_{A_i} a_i h_{A_i} - \sum_{i=2}^{i=n} M_{B_i} b_i h_{B_i} \right) \qquad (15\text{-}5)$$

Heat and mass transfer depend on the rate of development of the chemical transformations as well as on their heat. By the rate of a reaction is meant the number of molecules of a given kind reacting per unit time. To make it possible to compare the rates of different reactions, they are usually determined as the number of molecules or mols of the substance reacting per unit time per unit of phase volume for a homogeneous reaction, or per unit of interface area for a heterogeneous reaction.

Let us briefly consider certain proportions of *chemical kinetics*. By the kinetics of a reaction is meant the dependence of its rate on the concentration of the reagents, the temperature, and certain other factors.

Reactions may be simple or complex. Complex reactions result from the simultaneous development of several simple ones. For example, the homogeneous gaseous reaction $2NO + O_2 = N_2O_4$ consists of the following elementary reactions (or stages): $2NO = (NO)_2$ and $(NO)_2 + O_2 = N_2O_4$. The dimer of nitric oxide $(NO)_2$ is an intermediate substance.

The stage products (intermediate substances) are rapidly used up in other elementary reactions and are present in the reacting mixture in very small quantities. From the standpoint of chemical kinetics the elucidation of the mechanism of a reaction amounts to finding the nature of its intermediate products and stages.

The rate of a simple reversible reaction $\omega = \vec{\omega} - \overleftarrow{\omega}$, where $\vec{\omega}$ and $\overleftarrow{\omega}$ are the rates of the reaction in the direct and reverse directions.

When $\omega \neq 0$, the process is in chemical disequilibrium, but when $\omega = 0$, it is in equilibrium. Chemical equilibrium is defined as the state of a reacting system in which the chemical reaction proceeds simultaneously at the same absolute rate in both directions ($\vec{\omega} = \overleftarrow{\omega}$), with the consequence that the composition of the mixture remains constant. All stages of a complex reversible reaction are reversible, and with equilibrium the equality $\vec{w}_s = \overleftarrow{\omega}_s$ holds for each chemical stage s.

Equilibrium occasionally exists for all stages except one. The kinetics of the reaction is then in fact determined by the unbalanced stage, which is referred to as the limiting one.

In ideal gas mixtures the rates of single-stage homogeneous reactions obey the law of mass action, according to which the rate of a reaction $\Sigma a_i A_i \rightarrow \Sigma b_i B_i$ depends on the concentration of the reagents A_i and is determined by the expres-

sion

$$\vec{\omega} = \vec{k}\rho_{A_1}^{a_1}\rho_{A_2}^{a_2} \ldots \tag{15-6}$$

where ρ_{A_i} = the concentration of substance A_i, i.e. the number of molecules per unit volume, or a proportional value, measured in kg/m³;

$\vec{k}$ = the reaction rate constant of specific reaction rate.

The rate of a reversible reaction will be

$$\overleftarrow{\omega} = \overleftarrow{k}\rho_{B_1}^{b_1}\rho_{B_2}^{b_2} \tag{15-6'}$$

The rate of a direct reaction diminishes as it develops, if the initial concentrations are fixed, while the rate of a reversible reaction increases. When the rates of the two reactions become equal, i.e. when $\vec{\omega} = \overleftarrow{\omega}$, a state of chemical equilibrium sets in. In that case

$$\frac{\rho_{B_1}^{b_1}\rho_{B_2}^{b_2} \cdots}{\rho_{A_1}^{a_1}\rho_{A_2}^{a_2} \cdots} = \frac{\vec{k}}{\overleftarrow{k}} = K$$

Here K = the equilibrium constant for the given reaction and temperature.

The kinetics of a complex reaction can be described by applying the law of mass action to each of its stages.

The rate constant and, consequently, the rate of a reaction increase with rising temperature.

It follows from the foregoing that the rates of chemical reactions and, consequently, the rates (or absorption) of heat liberation depend on the concentration of the reagents and the temperature. The concentration fields and temperature depend on the processes of heat and mass transfer accompanying the chemical transformations as well as on the course of the reaction. Thus, in the general case, chemical transformations and heat and mass transfer are closely connected and are interdependent.

15-2. The Basic Equations of Heat and Mass Transfer With Chemical Transformations

The enthalpy determined by Eqs. (15-2) and (15-3) enables heat and mass transfer to be calculated by means of many of the relationships already derived.

The heat flux density in a diffusing mixture is described by equations (14-10) and (14-10'):

$$\vec{q} = -\lambda\nabla t + \rho\vec{w}i + \sum\vec{j}_{m\,di}\,i_i$$

For chemical reactions Eq. (14-10') is written as

$$\vec{q} = -\lambda\nabla t + \rho\vec{w}h + \sum\vec{j}_{m\,di}\,h_i \tag{15-7}$$

in which the first term on the right-hand side accounts for heat transport by conduction, the second, for heat transfer by convection, and the third by molecular diffusion. The equation makes it unnecessary to take special account of the liberation or absorption of heat through the chemical reactions.

The relationship (14-10) was used in Chapter 14 to derive the differential equation of energy:

$$\rho\frac{di}{d\tau} = \lambda\nabla^2 t - \operatorname{div}\left[(i_1 - i_2)\vec{j}_1\right]$$

which will also hold for chemical transformations, if the enthalpies i_i and i are replaced by enthalpies h_i and h, determined respectively by equations (15-2) and (15-3). Then, the energy equation for chemical reactions can be written as follows:

$$\rho\frac{dh}{d\tau} = \lambda\nabla^2 t - \operatorname{div}\left[(h_1 - h_2)\vec{j}_1\right] \tag{15-8}$$

The similarity of differential equations (14-12) and (15-8) follows from the similarity of Eqs. (14-10) and (15-7).

According to Eq. (15-8), local variation of enthalpy with time is caused by conduction, convection, and molecular diffusion. Assuming that the last occurs only through concentration diffusion, i.e.

$$\vec{j}_1 = -\rho D\nabla m_1$$

equation (15-8) can be written as follows:

$$\rho\frac{dh}{d\tau} = \lambda\nabla^2 t + \operatorname{div}\left[(h_1 - h_2)\rho D\nabla m_1\right] \tag{15-8'}$$

Of course, Eq. (15-8), like Eq. (14-12), is true for a binary mixture of interdiffusing components. Deriving the equation, no account was taken of possible inner heat sources, not stemming from chemical reactions, physical parameters were assumed to be constant, and no allowance was made for frictional heat.

Let us transform the right-hand side of this equation, first performing certain subsidiary calculations.

By definition the enthalpy of a mixture $h = \sum m_i h_i$. Hence

$$dh = \sum h_i dm_i + \sum m_i dh_i$$

where $dh_i = di_i$, since $dh_i^0 = 0$ (the heat of formation of each component being assumed to be fixed). Assuming that $di_i = c_{pi}dT$, then,

$$dh = \sum h_i dm_i + \sum m_i di_i$$

or

$$dh = \sum h_i dm_i + \sum m_i c_{pi}\, dT$$

Denoting c_p by $\sum m_i c_{pi}$, where c_p= mean capacity of the mixture, we can write

$$dh = c_p dT + \sum h_i dm_i$$

from which it follows that

$$dT = \frac{dh}{c_p} - \frac{\sum h_i\, dm_i}{c_p}$$

Hence, for a binary mixture the derivative in respect to y, for instance, will be

$$\frac{\partial T}{\partial y} = \frac{\partial t}{\partial y} = \frac{1}{c_p}\frac{\partial h}{\partial y} - \frac{1}{c_p}\sum h_i \frac{\partial m_i}{\partial y} =$$

$$= \frac{1}{c_p}\frac{\partial h}{\partial y} - \frac{1}{c_p}\left(h_1\frac{\partial m_1}{\partial y} + h_2\frac{\partial m_1}{\partial y}\right) = \frac{1}{c_p}\frac{\partial h}{\partial y} - \frac{1}{c_p}\frac{\partial m_1}{\partial y}(h_1 - h_2)$$

Here use is made of the fact that $m_1 + m_2 = 1$ and $j_1 = -j_2$.

The value of $\partial t/\partial y$ obtained is used to transform the right-hand side of equation (15-8′). For brevity's sake we shall transform only the expression

$$\lambda \frac{\partial^2 t}{\partial y^2} + \frac{\partial}{\partial y}\left[\rho D (h_1 - h_2) \frac{\partial m_1}{\partial y}\right]$$

The transformations for the Ox- and Oz-axes are performed in the same way.

Substituting the value of $\partial t/\partial y$ in the preceding expression, we get:

$$\lambda \frac{\partial^2 t}{\partial y^2} + \frac{\partial}{\partial y}\left[\rho D (h_1 - h_2) \frac{\partial m_1}{\partial y}\right] = \lambda \frac{\partial}{\partial y}\left[\frac{1}{c_p}\frac{\partial h}{\partial y} - \frac{1}{c_p}\frac{\partial m_1}{\partial y}(h_1 - h_2)\right] +$$

$$+ \frac{\partial}{\partial y}\left[\rho D (h_1 - h_2) \frac{\partial m_1}{\partial y}\right] = \frac{\lambda}{c_p}\frac{\partial^2 h}{\partial y^2} + \frac{\partial}{\partial y}\left[\left(1 - \frac{\lambda}{\rho c_p D}\right)(h_1 - h_2)\rho D \frac{\partial m_1}{\partial y}\right] =$$

$$= \frac{\lambda}{c_p}\frac{\partial^2 h}{\partial y^2} + (1 - \mathrm{Le}^{-1})\frac{\partial}{\partial y}\left[(h_1 - h_2)\rho D \frac{\partial m_1}{\partial y}\right]$$

where $\mathrm{Le} = \dfrac{\mathrm{Pr}}{\mathrm{Pr}_D} = \dfrac{\rho c_p D}{\lambda} = \dfrac{D}{a}$ is the Lewis-Semenov number.

Allowing for the transformations made, the differential equation of energy can be written as follows:

$$\frac{dh}{d\tau} = a\nabla^2 h + (1 - \mathrm{Le}^{-1})\,\mathrm{div}\,[(h_1 - h_2) D \nabla m_1] \qquad (15\text{-}8'')$$

If Le = 1, the last term of the right-hand side of equation (15-8″) is equal to zero and there is consequently no transport of heat by molecular diffusion so that the equation acquires a form similar to the energy equation (4-10) derived for a homogeneous fluid, in the absence of inner heat sources with total mixture enthalpy h playing the part of temperature.

This means that at $\mathrm{Pr} = \mathrm{Pr}_D$ the solutions of equation (4-10) are also true for processes involving chemical transformations, when the conditions of unambiguity are similar.

When h_i and h are replaced in Eq. (15-8″) by i_i and i, the formula becomes the energy equation for processes in which heat and mass transfer take place without chemical reactions. It is clear that the solutions of (Eq. 4-10) in this case will also be applicable at $\mathrm{Pr} = \mathrm{Pr}_D$. The Lewis-Semenov number for gas mixtures is usually close to unity.

According to Eq. (15-8″), the field of enthalpy h depends on the distribution of the rate of the mixture and field of concentrations. The rate of the mixture is present in the full derivative $dh/d\tau$:

$$\frac{dh}{d\tau} = \frac{\partial h}{\partial \tau} + w_x \frac{\partial h}{\partial x} + w_y \frac{\partial h}{\partial y} + w_z \frac{\partial h}{\partial z}$$

The effect of the field of concentrations is accounted for by the second term of the right-hand side of the equation; it will be recalled that account was taken only of concentration diffusion in deriving Eq. (15-8″), as with Eq. (15-8′).

To account for the effect of the velocity and concentration fields, it is necessary to add the equations of motion and mass transfer to the energy equation (15-8′) or (15-8′). The equation of motion contains a new dependent variable, pressure, so that it becomes necessary to add yet another equation. Such an equation may be the equation of continuity (discontinuity).

The equations of motion and continuity do not differ in mathematical expression from those for a homogeneous medium (Sec. 4-3). As for the mass-transfer equation (14-15), an additional term j_{vi}, kg/m^3·s must be introduced to account for the source of mass of the ith component through the chemical transformations.

The quantity j_{vi} is the volumetric resultant of the rate of reaction, being, in the general case, a function of time and the coordinates.

Taking the foregoing into account the mass-transfer equation can be written as follows:

$$\frac{dm_i}{d\tau} = D\nabla^2 m_i + \frac{j_{vi}}{\rho} \tag{15-9}$$

The total mass of the components involved in the reaction does not, of course, change (nuclear reactions not being considered).

To determine j_{vi} the equations of chemical kinetics must also be added to the equations of energy, mass transfer, motion, and continuity, and the necessity of using them complicates the problem even further. The difficulties mentioned in previous chapters are aggravated by non-linearity in chemical kinetics.

There are, however, particular cases for which the problem is simplified, namely:

(a) when the homogeneous reactions are very slow and the rates of mass transfer very high;

(b) when the homogeneous reactions are very fast and the rates of mass transfer very low;

(c) when the Lewis-Semenov number is unity.

In the first case (a) there is little time for the reaction to change the composition of the mixture appreciably, and the problem resolves formally into the calculation of heat and mass transfer leaving chemical transformations out of account, a course referred to as *frozen.*

In the second case (b), when the rates of the reactions are higher than the rates of diffusion and convection, according to Eq. (15-9) the composition of the mixture is mainly determined by the term accounting for the source of mass of a definite component. It can be assumed that chemical equilibrium then sets in and the composition of the mixture becomes a function of temperature only (and also of pressure in the general case). The effect of chemical reactions manifested itself solely through the physical properties of the mixture, which are represented in equations of energy, motion, and continuity similar to those for a homogeneous medium. There is no need to integrate the equation of mass transfer, and the course is referred to as *equilibrium* or *balanced.*

In the third case (c), when Le = 1, the mathematical problem is similar to that for heat transfer in the absence of mass transfer. As follows from the energy equation (15-8″) the enthalpy field is independent of molecular diffusion when Le = 1.

Thus, in the first case, solutions for heat and mass transfer (including experimental solutions) can ignore chemical transformations while solutions can be used in the second and third cases that are applicable to a homogeneous medium (which were considered in earlier chapters). In all cases, of course, total enthalpies should replace temperatures in the corresponding equations.

The physical properties of the gas mixture are assumed constant for the sake of simplicity and clarity. Actually, the physical parameters present in the differential equations may depend on the chemical reactions, since the latter bring about a change in mixture composition, and, consequently, in its properties.

To account for the effect of chemical reactions, this kind of heat transfer is described by Newton's law, presented in the following form:

$$q_w = \frac{\alpha}{c_p}(h_0 - h_w) \tag{15-10}$$

In this formula h_0 and h_w are the enthalpies of the gas mixture, respectively, at a distance from the gas-solid (liquid) interface and on it; h_0 and h_w are calculated by Eqs. (15-2) and (15-3), i.e. accounting for the heat of formation; c_p is the specific heat capacity of the gas mixture at constant pressure, determined according to the principle of additivity from the relation $c_p = \sum m_i c_{pi}$. The selection of the parameters on which calculation of c_p is based is not predetermined by Eq. (15-10). The specific heat capacity c_p is often calculated from the parameters of the mixture at a distance from the wall.

The substitution of enthalpies for temperature in Newton law makes it possible to take account of the fundamental effect of chemical reactions on heat transfer. With Eq. (15-10) the values of the local heat-transfer coefficient can be taken as a first approximation from the formulae ignoring chemical reactions. In the presence of chemical transformations, the values of the local heat-transfer coefficients may change, of course, since there is a corresponding change in the fields of temperature rate, and concentration; the effect of these factors is not however as important as the influence of the thermal effects of the reactions. Equation (15-10) seems to give better results when any of the three particular cases previously mentioned are satisfied.

At the present time problems involving heat transfer for a solid exposed to a fluid flow in which chemical reactions occur are being investigated, mainly problems concerning equilibrium flow of a dissociating gas past a chemically inactive (non-catalytic) wall. Calculation and theoretical research show that with account taken of variation of physical properties the heat-transfer coefficients may differ from the heat-transfer coefficient obtained in the presence of a laminar boundary layer on the plate and constant physical properties by up to 30 per cent, and with a turbulent boundary layer—up to 50 per cent. In both cases α is calculated by Eq. (15-10). The difference observed is the greater, the more the enthalpy ratio, h_0/h_w or the density ratio ρ_w/ρ_0 differs from unity.

In distinction from α determined by Eq. (15-10), the effect of dissociation on the rate of heat flow may be material.

The transport of heat accounted for by Eq. (15-10) is due to conduction, convection, and molecular diffusion. In complex cases of heat transfer, remaining valid, while Eq. (15-10) fails to give a full evaluation of the heat flux passing through the wall. This point is treated in greater detail in Sec. 15-3.

If chemical reactions occur during flow of a gaseous mixture at a high velocity, account has to be taken of the transport of heat by dissipation of mechanical energy. Newton's law is then written as follows:

$$q_w = \frac{\alpha}{c_p}(h_r - h_w) \tag{15-11}$$

where h_r can be referred to as the full restoration enthalpy. Its value is determined by the equation

$$h_r = h + r\frac{w_0^2}{2} \tag{15-12}$$

and is calculated by its own temperature.

Judging from the data available, the restoration factors differ little from those for a homogeneous gas (Chapter 11).

Equation (15-11) presents Newton's law in its most generalized form, compared with the previously used formulations of the law.

15-3. Heat Transfer Between a Gaseous Mixture and Its Phase Interface

The method of calculating heat flux between a gaseous mixture and its phase interface may alter somewhat, depending on the conditions in which the reactions occur. The heat-transfer calculations depend in many respects on the place where the chemical reactions occur.

Homogeneous reactions can take place in gaseous mixture without wall components, the material of the wall remaining chemically neutral and serving only as a heat conductor. In other cases evaporated material from the wall may participate in the homogeneous reactions, with evaporation occurring from both

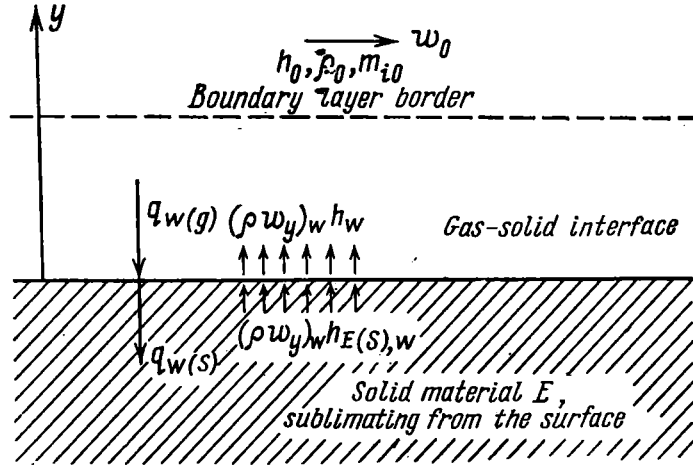

Fig. 15-2. Heat transfer between reacting gas layer and sublimating (evaporating) surface

the solid phase (sublimation) and previously melted solids. The chemical reactions may also involve a heat-carrying agent blown into the gas through a porous wall. A reaction between the gaseous mixture and the solid phase (wall) may also proceed on the surface of the latter (a heterogeneous reaction). Several of these processes of chemical and phase transformations may often occur simultaneously.

Irrespective of the conditions in which reactions occur certain general methods are retained when dealing with problems in heat and mass transfer, to account for the factors affecting the resultant flow of heat to the wall. The following example will help to clarify these factors and to take them into account.

As shown above, account should also be taken of mass transfer, in considering heat transfer involving chemical transformations in the general case. Many propositions considered in Chapter 14 will therefore be used.

Let us consider heat transfer between a reacting boundary layer and the evaporating (sublimable) surface of a solid. Outside the boundary layer the parameters of the gas (mixture density ρ_0; tangential velocity $w_x = w_0$, and concentration of constituents m_{i0}) are constant. For simplicity's sake let us assume that the Prandtl number for the gaseous mixture and the corresponding restoration factor are equal to unity. Thermal radiation is considered negligible. Let us also assume that molecular mass transfer occurs only through concentration diffusion, and that the process is a steady-state one.

Let us write down the heat-balance equation for the gas-wall interface (Fig. 15-2):

$$q_{w(s)} = q_{w(g)} + (\rho w_y)_w h_{E(s)w} - (\rho w_y) h_w \qquad (15\text{-}13)$$

Here $q_{w(s)}$ = rate of heat flow in the solid at the gas-wall interface;
$q_{w(g)}$ = the same, but in the gas;
$(\rho w_y)_w$ = rate of mass flow at the interface boundary;
h_w = gas mixture enthalpy at the interface;
$h_{E(s)w}$ = enthalpy of surface material which is a solid phase.

The first term in the right-hand side of Eq. (15-13) represents the rate of heat transfer by convection (with account taken of conduction, convection and molecular diffusion) from the reacting boundary layer to the interface boundary. The second term describes the reverse flow of heat coming to the phase interface from the solid phase and conditioned by melting, chemical reactions or evaporation. The third term represents the enthalpy transported from the phase interphase into the boundary layer at the expense of the mass flowing from the wall surface. If the transport of mass through a solid non-melting porous material was considered, the second term would represent the quantity of heat carried to the phase interface by the fluid infiltrating through the pores.

According to Eq. (15-10), the term $q_{w(g)}$ can be described by the expression

$$q_{w(g)} = \frac{\alpha}{c_p}(h_0 - h_w$$

With account taken of evaporation $(E_{(s)} \rightarrow E_{(g)})$,

$$h_{E(s)} + r_e = h_{E(g)}$$

where r_e = specific heat of evaporation of the surface material, denoted by the chemical symbol E.

Taking into consideration the last relationships, equation (15-13) can be presented in the following form:

$$q_{w(s)} = \frac{\alpha}{c_p}\left[(h_0 - h_w) - \frac{r_s + h_w - h_{E(g)w}}{\mathrm{St}}\right] \qquad (15\text{-}14)$$

Here

$$\mathrm{St} = \frac{\alpha}{(\rho w_y)_w c_p}$$

Eq. (15-14) can also be presented in the other form, taking into account that $h = \sum m_i h_i$ and, consequently

$$h_0 - h_w = \sum m_{i0}h_{i0} - \sum m_{iw}h_{iw} = \sum (m_{i0}h_{i0} - m_{iw}h_{iw}) =$$
$$= \sum (m_{i0}h_{i0} - h_{iw}) + \sum h_{iw}(m_{i0} - m_{iw}) \qquad (15\text{-}15)$$

It is assumed that the component E is only present in the boundary layer. Then substitution of (15-15) into (15-14) gives:

$$q_{w(s)} = \frac{\alpha}{c_p}\left[\sum m_{i0}(h_{i0} - h_{iw}) - \frac{r_e}{\mathrm{St}} + \right.$$
$$\left. + \sum_{i \neq E} h_{iw}\left(m_{i0} - m_{iw} - \frac{m_{iw}}{\mathrm{St}}\right) + h_{E(g)w}\left(\frac{1 - m_{E,m}}{\mathrm{St}} - m_{Ew}\right)\right] \qquad (15\text{-}16)$$

So, the calculation of the heat leaving into the wall amounts to calculation of the parameters of the gas mixture at the phase interface and at a distance from it.

The rate of heat flow $q_{w(s)}$ is sometimes determined from the equation

$$q_{w(s)} = \frac{\alpha}{c_p}(h_{eff} - h_w) \qquad (15\text{-}17)$$

The value of h_{eff} should be selected so that, for instance, in the example considered, it satisfies the values of $q_{w(s)}$ calculated with the aid of Eq. (15-16). In practice h_{eff} is determined by experiment. The magnitude of the effective total enthalpy may vary substantially, depending on the conditions of heat and mass transfer.

In the general case, calculation of $q_{w(s)}$ from Eq. (15-10) only would result in errors; account must be taken of the completeness of transport to the wall of the heat of chemical reaction or the degree of transformation of chemical energy into heat.

Problems, involving determination of local heat-transfer coefficients in the presence of chemical reactions, are not considered here. As has been noted in Sec. 15-2, use can be made in the first approximation of the heat-transfer coefficients derived for cases without chemical transformations. The methods of determination of heat-transfer coefficients in the presence of chemical reactions are treated in special literature [15.1, 15.2, 15.3].

The effects of heat liberation in phase transitions have been considered in the sections of the book devoted to condensation and boiling of pure substances. These processes become more complex in the presence of phase transitions in binary heterogeneous systems, but the fact that in the latter case phase transitions proceeding only at the phase interfaces were considered somewhat simplified the problem.

The processes of heat and mass transfer at chemical and phase transformations can be considered as a more general case, compared to previously considered processes, and this problem, not withstanding its intricacy and general character, does not cover the diversity of heat- and mass-transfer processes. In particular, the processes in study can be considerably complicated upon application of electromagnetic fields, which takes place in modern engineering practice. The processes of heat transport by conduction, convection and molecular diffusion are often accompanied by processes of thermal radiation especially at high temperatures.

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4 RADIATION HEAT TRANSFER

CHAPTER
16
BASIC CONCEPTS AND LAWS OF THERMAL RADIATION

16-1. Description of the Process

Processes of thermal radiation have found wide application in heat engineering, in nuclear engineering, rocketry, metallurgy, drying, chemical technology, illumination engineering, helium technology, etc.

Thermal radiation is a process of propagation of the internal energy of an emitting body by the agency of electromagnetic waves. By electromagnetic waves are meant the electromagnetic disturbances originating in the emitting or radiating body, and propagating in vacuum at the speed of light, equal to 3×10^8 m/s. When absorbed by other solids, the electromagnetic waves are again converted into thermal-motion energy of molecules. The waves are originated by charged particles of substance, i.e. by electrons and ions forming its constituents. The oscillation of ions corresponds to low-frequency radiation; that due to the motion of electrons may be of a higher frequency, if the electrons are atomic and molecular constituents, and are retained in a state of equilibrium by considerable forces.

Metals have many free electrons. Therefore, we cannot speak here of oscillation about centres of equilibrium. The moving electrons are subjected to irregular deceleration, owing to which the radiation of metals is of the nature of pulsed radiation, and their waves are of various frequencies, including low-frequency waves. In addition to wave properties, radiation also possesses corpuscular properties, which consist in radiant energy being emitted and absorbed not continuously, but in individual discrete portions—light quanta or photons. The emitted photon is a particle of matter possessing energy, momentum, and electromagnetic mass. Thermal radiation can, therefore, be considered as a photon gas.

The passage of photons through a substance is a process of absorption and subsequent emission of the energy of photons by the atoms and molecules of that substance.

Radiation is, thus, of a dual nature, since it possesses the continuity properties of the field of electromagnetic waves and the properties of discreteness typical of photons. The synthesis of the two properties is the concept that energy and pulses are concentrated in the photons, and the probability of their being found in any definite part in space—in the waves. Accordingly, radiation is characterized by wavelength (λ) or oscillation frequency ($\nu = c/\lambda$). All kinds of electro-

magnetic radiation are of the same origin and differ only in wavelength (Table 16-1).

TABLI 16-1

Classification of Electromagnetic Radiation by Wavelength

Kind of radiation	Wavelength, mm	Kind of radiation	Wavelength, mm
Cosmic rays	0.05×10^{-9}	Visible	$(0.4\text{-}0.8) \times 10^{-3}$
Gamma radiation	$(0.5\text{-}0.10) \times 10^{-9}$	Thermal (infrared) radiation	$0.8 \times 10^{-3}\text{-}0.8$
X-rays	$1 \times 10^{-9}\text{-}0.2 \times 10^{-5}$		
Ultraviolet radiation	$2 \times 10^{-5}\text{-}0.4 \times 10^{-3}$	Radio waves	> 0.2

The quantitative difference in the length of electromagnetic waves results in that the properties common to phenomena are manifested differently at different wavelengths. Thus, quantum or corpuscular properties are most pronounced in short-wave radiation; on the contrary, wave properties are most distinctly observed in radio waves.

Most solid and fluid substances have a continuous radiation spectrum, i.e. they emit energy of all wavelengths from zero to infinity. Examples of solids possessing a continuous radiation spectrum are non-conductors and semi-conductors of electricity, metals with an oxidized rough surface. Metals with a polished surface, gases and vapours are characterized by a selective (discontinuous) radiation spectrum. Radiation intensity depends on the nature of the substance, its temperature, wavelength, state of the emitting surface, and with gases, also on the thickness of the layer and pressure. Solids and fluids possess considerable absorptivity and emissive capacity. Consequently, only thin surface layers participate in processes of radiation heat transfer: for non-conductors of heat layer thickness is about 1 mm, for conductors of heat—1 micron. Thus thermal radiation associated with non-conductors and conductors of heat then can be regarded as a surface phenomenon. Semi-transparent bodies (molten quartz, glass, optical ceramics, etc. gases and vapours) are characterized by the space nature of radiation in which all particles of the substance take part. The radiation from all bodies depends on temperature. The radiant energy of an emitter increases with rising temperature, since its internal energy increases. A change in the temperature of a body brings about not only a change in the absolute magnitude of this energy, but also in the radiation spectrum. A rise in temperature is accompanied by an increase in the intensity of short-wave radiation, while the intensity of long-wave emission diminishes. Thermal radiation depends considerably more on temperature than on heat conduction and convection. At high temperatures, therefore, thermal radiation may serve as the main mode of heat transfer.

16-2. Kinds of Radiation Fluxes; Radiant Flux Vector

Hemispherical radiation. A body emits energy in the form of a continuous or discontinuous spectrum with respect to wavelengths.

The radiant energy emitted from an arbitrary area per unit time in all directions of a hemisphere within a narrow wavelength band, from λ to $\lambda + d\lambda$, is referred to as *monochromatic, spectral* or *homogeneous radiation* (Q_λ). The total radiation from the surface of a body in all directions of a hemisphere and of all wavelengths of the spectrum is called *integral* or *total radiation* (Q).

The total flux emitted per unit area in all directions of a hemisphere is called the *integral density of hemispherical radiation*, or is referred to as the *emissive power*

$$E = \frac{dQ}{dF} \tag{16-1}$$

where dQ is the radiant flux emitted by surface element dF, W.

The radiant flux from the entire surface can be presented as

$$Q = \int_F E\,dF \tag{16-2}$$

If the intensity of integral radiation is the same for all surface elements of an emitter, then relationship (16-2) reduces into

$$Q = EF \tag{16-3}$$

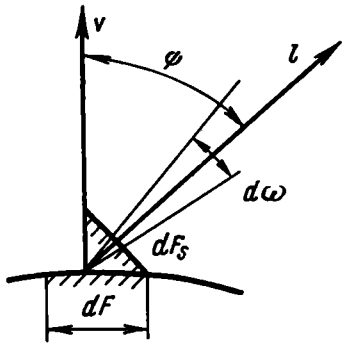

Fig. 16-1. Determining radiosity

The ratio of the radiant flux density emitted in the infinitely small range of wavelength to the surface of a body is called the *spectral density of radiation*:

$$E_\lambda = \frac{dE}{d\lambda} \tag{16-4}$$

Relationships (16-1) through (16-3) are also true for monochromatic radiation.

The radiant flux may vary in certain directions of radiation. The amount of energy emitted in the direction l, determined by angle ψ formed with the normal to surface n (Fig. 16-1) from unit elementary area per unit time within the limits of the unit elementary solid angle $d\omega$, is called the *angular intensity of radiation*. By definition, the brightness of spectral and integral radiation is described by the following expressions:

$$I_{\psi\lambda} = \frac{d^2Q_{\psi\lambda}}{dF\,d\omega} = \frac{dE_{\psi\lambda}}{d\omega} \tag{16-5}$$

$$I_\psi = \frac{d^2Q_\psi}{dF\,d\omega} = \frac{dE_\psi}{d\omega} \tag{16-6}$$

From the defining expressions it follows that

$$dE_{\psi\lambda} = I_{\psi\lambda}d\omega \tag{16-7}$$

$$dE_\psi = I_\psi\,d\omega \tag{16-8}$$

The concept of *radiation intensity* (radiosity, or radiance) is of great theoretical importance.

The intensity of radiation is defined as the quantity of radiant energy emitted in the direction of angle ψ per unit time from an elementary area within the limits of a unit elementary solid angle, reduced to the projection of this area on the plane which is orthogonal to the direction of radiation (Fig. 16-1):

$$I_\lambda = \frac{d^2Q_{\psi\lambda}}{dF_s\,d\omega} = \frac{d^2Q_{\psi\lambda}}{dF\cos\psi\,d\omega} = \frac{I_{\psi\lambda}}{\cos\psi} \tag{16-9}$$

$$I = \frac{d^2Q_\psi}{dF_s\,d\omega} = \frac{d^2Q_\psi}{dF\cos\psi\,d\omega} = \frac{I_\psi}{\cos\psi} \tag{16-10}$$

from which

$$d^2Q_{\psi\lambda} = I_\lambda \, d\omega \cos\psi \, dF \tag{16-11}$$

$$d^2Q_\psi = I \, d\omega \cos\psi \, dF \tag{16-12}$$

where I_λ and I = respectively spectral and total radiation intensity
ψ = angle between the normal to the plane and the direction of radiation.

In the general case the spectral intensity depends on the coordinates of point M, direction, wavelength and time.

The integral intensity characterizes the distribution of radiant energy, total for all wavelengths, in all possible directions at the given point and chosen moment of time.

The directional distribution of radiation intensity may differ greatly and distribution may be uniform in a special case.

Radiation characterized by an intensity which is the same in all directions is referred to as *isotropic*. If radiation is emitted from the surface of a solid, it is called *ideally diffused radiation* [16.1]. The concept of radiation intensity (radiosity) may refer to individual kinds of radiation, considered below. We can speak, therefore, of the intensity of emitted radiation, radiation leaving a surface, incident and of other kinds of radiation.

Total and monochromatic radiation fluxes are related by the following expressions:

$$Q = \int_0^\infty Q_\lambda \, d\lambda; \quad E = \int_0^\infty E_\lambda \, d\lambda; \quad I = \int_0^\infty I_\lambda \, d\lambda \tag{16-13}$$

The kind of radiation which is determined by the nature of a given body and its temperature is called *emitted radiation* (Q, E).

There is usually thermal radiation heat transfer between bodies. The radiation from other bodies that is incident upon a given body is partly absorbed by the body, partly reflected by it, while a fraction of the incident radiation passes through the body. The quantity of incident radiation is denoted by Q_{in} or E_{in}. The fraction of incident radiation that is absorbed by a given body is referred to as the *absorbed radiant flux*, denoted by Q_{ab}, E_{ab}. The absorbed radiant energy again converts into internal energy.

The absorbed radiant energy E_{ab}, measured in W/m^2, is

$$E_{ab} = AE_{in} \tag{16-14}$$

where A is the *total absorptivity* of the given body.

Bodies that absorb all incident radiant energy are called *black bodies* ($A = 1$). Such a body is seen by the eye as a black body, hence the name. If a surface absorbs all incident beams, except light beams, it does not appear black, although as regards radiation properties it approaches a black body, since it possesses a high absorptivity (for instance, the absorptivity of ice and snow $A = 0.95$ to 0.98). Relationship (16-14) may refer to monochromatic radiation just like the subsequent relationships. In the general case spectral absorptivity A_λ may vary with wavelength in a different way, and in a private case it may be independent of wavelength.

Bodies whose spectral absorptivity is independent of wavelength are referred to as *grey bodies* (Fig. 16-2). For them $A_\lambda = \text{const} \leqslant 1$, since grey bodies do not absorb all the incident radiant energy. A fraction of the incident energy will be reflected or pass through the mass of these bodies.

The fraction of incident energy which is reflected back to the bodies surrounding it is called the *reflected radiation*. The reflected radiation flux E_r, W/m², is equal to

$$E_r = RE_{in} \qquad (16\text{-}15)$$

where R is the *total reflectivity of the body*. When processes involving reflection from a surface obey the laws of geometrical optics and $R = 1$, the surface of the body is called a reflecting, or specular, surface, with ideally diffuse reflection the surface is called *absolutely white*. With ideally diffuse (isotropic) reflection radiant energy is reflected uniformly in all directions, irrespective of the direction of incident radiation.

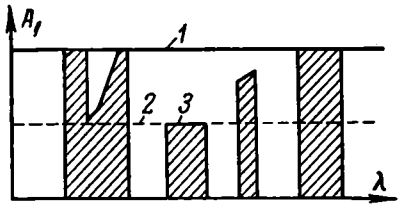

Fig. 16-2. Dependence of absorptivity on wavelength:
1—black body; *2*—grey body; *3*—body with selective absorption

The fraction of incident radiation passing through the body is called *transmitted radiation* E_{tr}, W/m²

$$E_{tr} = DE_{in} \qquad (16\text{-}16)$$

where D is the *total transmissivity of the body*. Bodies whose total transmissivity is unity are referred to as absolutely transparent or diathermanous bodies (thin layers of dry air, layers of monoatomic gases).

Bodies characterized by $0 \leqslant D \leqslant 1$ are called *semitransparent* (glass, quartz, sapphire). The total transmissivity of many solids and fluids is assumed to be zero, since they are practically opaque.

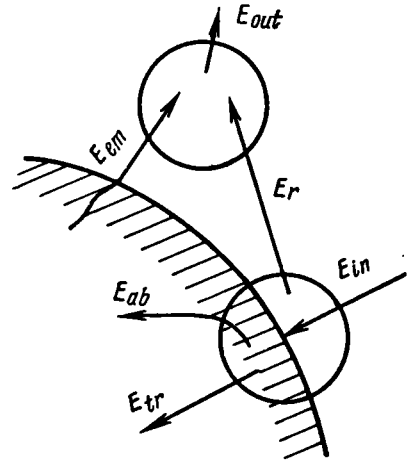

Fig. 16-3. Classification of radiation fluxes

Concurrently occurring processes of mutual emission, reflection and transmission of radiant energy in different systems of bodies are known as *radiant heat exchange*, and the component bodies of a given emitting system may be at the same temperature. For a body participating in radiant heat exchange with other bodies, based on the law of conservation of energy, the following equation of heat balance can be written (Fig. 16-3):

$$E_{in} = E_{ab} + E_r + E_{tr} \qquad (16\text{-}17)$$

or

$$A + R + D = 1$$

if relationship (16-17) is divided by E_{in} and the previously presented relationships are taken into account.

The sum of the radiant fluxes emitted and reflected from the surface of a given body is called the *outgoing radiation* or the *radiation leaving the surface of the body* (Fig. 16-3) and equal to

$$E_{out} = E + E_r = E + RE_{in} \qquad (16\text{-}18)$$

Outgoing radiation depends on the physical properties and temperature not only of the body concerned, but also of the surrounding bodies. Outgoing radiation also depends, in addition, on the shape, size and relative arrangement of

the bodies in space. Because of these factors, the physical properties of outgoing radiation and of emitted radiation are different. Their radiation spectra are also different. If the surface of a body is capable of ideally diffuse radiation and ideally diffuse reflection, the radiation leaving the surface will also be ideally diffuse. The concept of outgoing radiation was first introduced by O.E. Vlasov in 1929.

Radiant heat exchange between bodies is determined by *net radiation* which is equal to the difference between the heat flux that is incident up on a given body and the heat flux emitted by that body into the surroundings. The net radiant flux (or radiation) can be found by different methods depending on the arrangement of a fictitious calculation surface (Fig. 16-4).

By the first method (Nusselt's), it is determined from the heat balance composed in respect to surface $a - a$, located inside the body near its surface, allowing for $E_{tr} = 0$:

$$q_{net} = E - E_{ab} = E - AE_{in} \quad (16\text{-}19)$$

The second method of determining net radiation (suggested by O.E. Vlasov) consists in determining q_{net} from the balance composed in respect to the fictitious surface $b - b$, located outside the body, but close to its surface [16.2]:

$$q_{net} = E_{out} - E_{in} \quad (16\text{-}20)$$

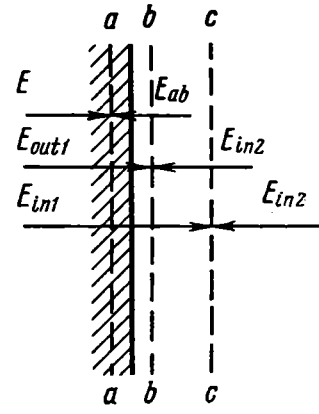

Fig. 16-4. Determining the net heat flux

Here, due to the small distance, the entire radiant energy emitted by the body reaches the fictitious surface $b - b$.

In the general case the net radiant flux is determined by the difference between two opposite radiant fluxes incident upon the fictitious surface $c - c$ (Yu. A. Surinov [16.3]):

$$q_{net} = E_{in1} - E_{in2} \quad (16\text{-}21)$$

The net radiant flux may be positive, negative or equal to zero (in the event of equilibrium radiation).

Net radiation and the radiation leaving a surface can be related mathematically by reducing Eq. (16-20) into

$$E_{out} = q_{net} + E_{in} \quad (16\text{-}22)$$

and rearranging Eq. (16-19) as follows

$$E_{in} = \frac{E - q_{net}}{A} \quad (16\text{-}23)$$

substituting Eq. (16-23) into Eq. (16-22), we obtain

$$E_{out} = q_{net}\left(1 - \frac{1}{A}\right) + \frac{E}{A}$$

or

$$E_{out} = \frac{E}{A} - \frac{R}{A} q_{net} \quad (16\text{-}24)$$

For a black body $A = 1$ and $E_{out} = E_0$, outgoing radiation degenerates into emitted radiation.

It follows then that all kinds of hemispherical radiation, except the emitted radiation, are linear functions of incident radiation. The emitted radiation of a body is linked to other kinds of radiation and is correlated with them through the outgoing radiation.

Volumetric radiation. Characteristic of a medium filling some volume (space) of a system and which is capable of emitting, absorbing and scattering radiation, are the radiant energy densities*. By analogy with what was said above we can speak here of the densities of emitted, absorbed, scattered and of other kinds of radiation.

By the total and spectral radiant energy densities of emitted radiation are meant the radiation fluxes emitted by unit volume of a medium per unit time in all directions within the limits of the solid angle $\omega = 4\pi$:

$$\eta = dQ/dV; \quad \eta_\lambda = dQ_\lambda/dV \tag{16-25}$$

η and η_λ are measured in W/m^3.

The radiation fluxes reduced to the entire volume will be expressed by relationships

$$Q = \int_V \eta \, dV; \quad Q_\lambda = \int_V \eta_\lambda \, dV \tag{16-26}$$

The spectral and total radiation fluxes are related by

$$Q = \int_0^\infty Q_\lambda \, d\lambda = \int_0^\infty \int_V \eta_\lambda \, d\lambda \, dV \tag{16-27}$$

The density of *absorbed volumetric radiation*

$$\eta_{ab} = \alpha \eta_{in} \tag{16-28}$$

and the density of *scattered volumetric radiation*

$$\eta_{scat} = \beta \eta_{in} \tag{16-29}$$

represent, just as in relationships (16-14) and (16-15), a certain fraction of the density of the incident volumetric radiation. The quantities α and β are called respectively the *volumetric absorption coefficient* and the *volumetric scattering coefficient*. The sum of the two coefficients is referred to as the volumetric extinction coefficient of an attenuating medium (k).

By analogy with relationships (16-18), by the density of *outgoing volumetric radiation* is meant the sum of the densities of the emitted and scattered radiations:

$$\eta_{out} = \eta + \eta_{scat} = \eta + \beta \eta_{in} \tag{16-30}$$

Hence, in the event of volumetric radiation reflected radiation is replaced by scattered radiation and the concepts of absorptivity and reflectivity are replaced by the volumetric absorption and scattering coefficients.

By analogy with Eq. (16-19) the net radiant energy density is expressed by the relationship

$$\eta_{net} = \eta - \eta_{ab} = \eta - \alpha \eta_{in} \tag{16-31}$$

The last two dependences can be used to derive equations relating the densities of the net and outgoing volumetric radiations, similar to Eqs. (16-24):

$$\eta_{out} = \eta_{net} (1 - 1/\alpha) + \eta/\alpha \tag{16-32}$$

* Volumetric radiation is also characterized by irradiance denoted by u, J/m^3.

or

$$\eta_{out} = \eta/\alpha - \beta/\alpha\eta_{net} \tag{16-33}$$

The considered kinds of emissive power and radiation densities are the main characteristics of radiant heat exchange on boundary surfaces and in the volume of a medium filling an emitting system.

The density fields of different kinds of radiation depend on the geometry of the emitting system, temperature distribution and also on the distribution of optical properties both over the volume and on system boundaries.

Radiant flux vector. It determines the direction of the most intensive transport of radiant energy at the considered point in the radiation field. The vector is equal numerically to the net radiation flux transported per unit time through unit area perpendicular to an arbitrary direction of radiation transport, i.e. equal to the difference between the radiation fluxes that are incident upon the indicated surface from the two sides. This is clear from the following. The elementary flux passing through area dF (Fig. 16-5) will be expressed by the scalar product of the radiant flux vector $\vec{q}_r$ by $\vec{dF}$:

$$\left.\begin{aligned} dQ &= \vec{q}_r \vec{dF} = q_r\, dF \cos\psi \\ \text{or} \\ dQ &= q_n\, dF = q_r \cos\psi\, dF \end{aligned}\right\} \tag{16-34}$$

where q_n is the projection of the radiant flux vector on the normal to the surface.

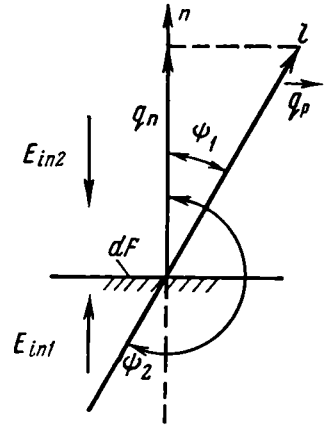

Fig. 16-5. Determining the radiation vector

For radiation fluxes that are incident upon one side of area dF the angle $\psi_1 < \pi$ and $dQ_{in1} = E_{in1}dF$ is positive; for the other side of the area the angle $\psi_2 > \pi$ and dQ_{in2} is negative. Hence, the radiation passing through area dF will be equal to the net radiation flux [see Eq. (16-21)]

$$dQ = dQ_{in1} - dQ_{in2} = dQ_{net}$$

whence we obtain

$$Q_{net} = \int_F \vec{q}_r \vec{dF} = \int_F q_l \cos\psi\, dF = Q_{in1} - Q_{in2} \tag{16-35}$$

It follows from Eq. (16-35) that the projection of the radiant flux vector on the normal to the surface expresses net radiation [16.4].

If one side of the surface considered is exposed to radiation, then the radiant flux vector expresses incident radiation.

Let us find the density of net radiation, using Eq. (16-12) for incident radiation by integrating over two hemispheres. Allowing for $\cos\psi_1 = -\cos\psi_2$, we obtain:

$$\left.\begin{aligned} q_{net} &= E_{in1} - E_{in2} = \int_{2\pi} I_1 \cos\psi\, d\omega + \int_{2\pi} I_2 \cos\psi\, d\omega \\ \text{or} \\ q_{net} &= \int_{4\pi} I \cos\psi\, d\omega \end{aligned}\right\} \tag{16-36}$$

The components of the net radiant energy density in the direction of the Ox-, Oy-, Oz-axes are components of the radiant flux vector

$$\vec{q}_r = \mathbf{i}q_{netx} + \mathbf{j}q_{nety} + \mathbf{k}q_{netz} = \int_{4\pi} \vec{I} d\omega \qquad (16\text{-}37)$$

It follows that the radiant flux vector is determined by the vector integral of radiation intensity over a spherical solid angle.

Equation (16-37) is the integral form of the radiant flux vector. In addition to the integral, the radiant flux vector may have a gradient form.

The radiant flux vector acquires the gradient form when radiation heat transfer is treated as a process of emission of discrete particles—photons. If the mean free path of photons is comparatively small, then, by analogy with heat conduction in gases, radiation heat transfer is effected by the diffusion of radiant energy in a photon gas. It is then appropriate to introduce a fictitious thermal conductivity at the expense of radiation λ_{rad}. The radiant flux vector then acquires the gradient form, similar to Fourier's law for the heat flux vector:

$$\vec{q}_r = -\lambda_{rad} \operatorname{grad} T, \quad \lambda_r = 16/3 \frac{\sigma}{\alpha} T^3 \text{ [16.5]} \qquad (16\text{-}38)$$

The diffusion picture of the displacement of radiant-energy carriers is true for conditions, close to thermodynamic equilibrium, for a grey medium having a great optical thickness (Sec. 18-2), for emitting systems of simple geometry and others.

The above considered classification of kinds of radiation was suggested by Yu.A. Surinov [16.3, 16.6].

16-3. Laws of Thermal Radiation

(*a*) *Planck's Law*

The laws of thermal radiation have been obtained for black bodies and conditions of thermodynamic equilibrium.

Black body (*equilibrium*) radiation is radiation in the presence of which all component bodies of a given emitting system acquire the same temperature. Thermal radiation is of a dynamic nature: with equal temperatures each of the component bodies both emits and absorbs radiant energy, but in equal quantities ($Q_{net} = 0$).

The emission of energy in respect to wavelength is not uniform and depends on temperature. Planck's law establishes the relation of the spectral emissive power, wavelength and temperature

$$E_{0\lambda} = \frac{2\pi c_1}{\lambda^5} \left(e^{\frac{c_2}{\lambda T}} - 1\right)^{-1} \qquad (16\text{-}39)$$

where λ = wavelength, m;
$c_1 = 5.944 \times 10^{-17}$ = Planck's first constant, W-m^2;
$c_2 = 1.4388 \times 10^{-2}$ = Planck's second constant [16.7], m-K;
e = base of natural (Naperian) logarithms;
T = absolute temperature of body, K.

The spectral emissive power $E_{0\lambda}$ is measured in W/m^3 [see Eqs. (16-13)].

Planck's law was obtained theoretically. According to this law for every wavelength there is an appropriate magnitude of the emissive power $E_{0\lambda}$ (Fig. 16-6). The emissive power, characterized by individual isotherms, passes through a maximum. With wavelengths $\lambda \to 0$ and $\lambda \to \infty$ the emissive power tends to zero. Radiant energy possesses the properties of thermal radiation at wavelengths between 0.4 and 800 micron (visible and infrared radiation). The intensity of radiation increases considerably with rising temperature.

Radiation characterized by a spectrum similar to that of black-body radiation (at $T = \text{const}$) is identified as non-selective radiation.

Inasmuch as Planck's law was formulated for a black body emitter, then as far as non-black bodies are concerned it expresses the maximum possible radiation flux. For non-black bodies the radiation spectrum depends, in addition to wavelength, on physical properties and is determined experimentally.

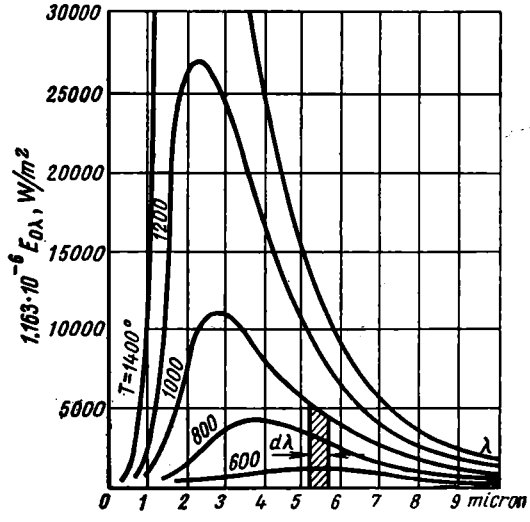

Fig. 16-6. Graphical presentation of Planck's law

(b) *Rayleigh-Jeans' Law*

Planck's law has two limiting cases one of which is that when the product λT is large compared with the constant c_2. With this provision we can confine ourselves to only two terms of the exponential function (16-39) expanded into a series with the $\frac{c_2}{\lambda T}$ exponent:

$$e^{\frac{c_2}{\lambda T}} = 1 + \frac{1}{1!}\left(\frac{c_2}{\lambda T}\right) + \frac{1}{2!}\left(\frac{c_2}{\lambda T}\right)^2 + \dots$$

Then Eq. (16-39) becomes:

$$E_{0\lambda} = \frac{2\pi c_1 T}{c_2 \lambda^4} \tag{16-40}$$

This relationship expresses Rayleigh-Jeans' law.

(c) *Wien's Law of Deviation*

The second extreme case corresponds to a small value of the product λT as compared to constant c_2. Then, the unity present in the denominator of Eq. (16-39) can be neglected, and the relationship becomes Wien's law (1893):

$$E_{0\lambda} = \frac{2\pi c_1}{\lambda^5} e^{-\frac{c_2}{\lambda T}} \tag{16-41}$$

Coordinates of the maximum values of the emissive power (Fig. 16-6) can be obtained from the extreme value of (16-39). For this purpose the derivative of the function is found for the wavelength. Equating the derivative to zero, we

obtain the following transcendental equation:

$$e^{-\frac{c_2}{\lambda_{max}T}} + \frac{c_2}{5\lambda_{max}T} - 1 = 0$$

whose solution is

$$\frac{c_2}{\lambda_{max}T} = 4.965$$

from which

$$\lambda_{max}T = 2.8978 \times 10^{-3} \qquad (16\text{-}42)$$

where λ_{max} is the wavelength corresponding to a maximum intensity of radiation; the product $\lambda_{max}T$ is measured in units of m·K. Relationship (16-42) expresses Wien's law of deviation, according to which maximum spectral intensity of radiation shifts towards the shorter waves with rising temperature.

The maximum emissive power of a black body can be found from Planck's law, if $\lambda = \lambda_{max}$ is replaced by the value from Eq. (16-42). We then find that

$$(E_{0\lambda})_{max} = c_3 T^5 \qquad (16\text{-}43)$$

where the constant $c_3 = 1.307$ W/m³·K⁵.

From Eq. (16-43) it follows that the magnitude of the maximum emissive power varies proportionally with the fifth power of the absolute temperature of the body.

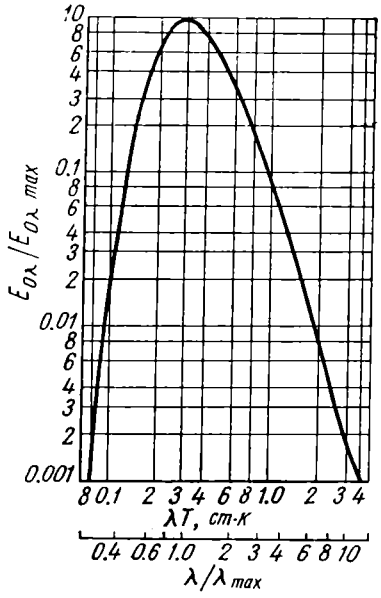

Fig. 16-7. Graphical presentation of Planck's law in dimensionless form

(d) *Planck's Law in Dimensionless Form*

Wien's law of deviation (16-43) permits Planck's law (16-39) to be expressed in dimensionless form:

$$\frac{E_{0\lambda}}{(E_{0\lambda})_{max}} = \frac{2\pi c_1}{c_3(\lambda T)^5}\left(\frac{1}{e^{\frac{c_2}{\lambda T}} - 1}\right) = \varphi(\lambda T) \qquad (16\text{-}44)$$

If we substitute the value of T from (16-42) in (16-44), then the latter acquires the form of:

$$\frac{E_{0\lambda}}{(E_{0\lambda})_{max}} = f\left(\frac{\lambda}{\lambda_{max}}\right) \qquad (16\text{-}45)$$

Planck's law is represented here graphically not by a family of isotherms, as illustrated in Fig. 16-6, but by a single curve that holds for any wavelength and temperature of the body (Fig. 16-7).

The maximum of this relationship corresponds to the values $\frac{E_{0\lambda}}{(E_{0\lambda})_{max}} = 1$ and $\frac{\lambda}{\lambda_{max}} = 1$.

(e) *Stefan-Boltzmann's Law*

The Stefan-Boltzmann law relates the hemispherical total emissive power and temperature. This relationship was first established by Stefan in 1879, measuring the radiation emitted from a model of a black body, well before the appea-

rance of Planck's quantum theory. Later, in 1884, the relationship was found theoretically (departing from the laws of thermodynamics) by Boltzmann and that is why the law is called Stefan-Boltzmann's law. Stefan-Boltzmann's law can also be obtained from Planck's law. For the emissive power E_0, W/m², Stefan-Boltzmann's law may be presented in the following form:

$$E_0 = \int_0^\infty E_{0\lambda}\, d\lambda = \sigma_0 T^4 \qquad (16\text{-}46)$$

where σ_0 is Stefan-Boltzmann's constant.

To facilitate practical calculations, Eq. (16-46) is usually presented in the following form

$$E_0 = c_0 \left(\frac{T}{100}\right)^4 \qquad (16\text{-}47)$$

where $c_0 = 5.6687 = 5.67$ is the radiation constant of a black body, measured in W/m²·K⁴.

To express the net radiant energy density in vacuum, u_0, J/m³, at an envelope temperature, T, Stefan-Boltzmann's law reduces into [16.8]:

$$u_0 = \frac{4\sigma_0}{c_{ph}} T^4 \qquad (16\text{-}48)$$

where c_{ph} is the velocity of photons, m/s.

It follows from Eqs. (16-47) and (16-48) that the total radiation density varies in proportion to the fourth power of absolute temperature. At $T=0$ total radiation is also zero.

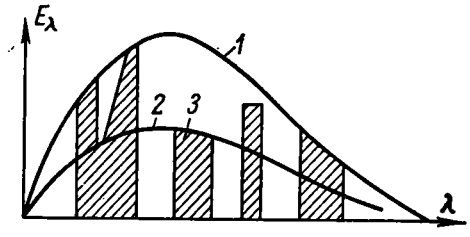

Fig. 16-8. Intensity of radiation depending on wavelength:
1—black body radiation; *2*—grey body radiation; *3*—selective radiation

The Stefan-Boltzmann law also applies to grey bodies. It is then assumed that, as with black bodies, the inherent radiation of a grey body is proportional to the fourth power of the absolute temperature, but the emissive power is smaller than that of black bodies at the same temperature, as illustrated in Fig. 16-8. The law thus acquires the following form for grey bodies:

$$E = \varepsilon E_0 = \varepsilon c_0 \left(\frac{T}{100}\right)^4 = c \left(\frac{T}{100}\right)^4 \qquad (16\text{-}49)$$

where $\varepsilon = E/E_0 = c/c_0$ = integral *emissivity* of a grey body;
c = its radiation constant (or factor), W/m²·K⁴.

Thus, by total emissivity is meant the ratio of the total radiation density of a surface to that of a black body at the same temperature.

Stefan-Boltzmann's law is strictly valid for a grey body to the same extent as the assumption that emissivity remains strictly constant, independent of temperature. Actually, however, the emissivity of a grey body depends on its nature, temperature, state of the emitting surface, and is usually determined by experiment. The radiation coefficient (constant) then characterizes the intensity of the radiation emitted by the body involved. The radiation coefficient is equal numerically to the emitted radiation reduced to unit area and unit time and also to unit fourth-power temperature difference between the body and surrounding bodies which are at absolute zero temperature.

(f) Kirchhoff's Law of Radiation

Kirchhoff's law (1882) establishes the quantitative relation between the emissivity and absorptivity of grey and black bodies. It can be derived from the heat balance of an emitting system consisting of a relatively large enclosure with heat-insulated walls, with two bodies inside it. Under conditions of thermodynamic equilibrium the radiant energy of each of the three bodies is equal to the absorbed energy:

$$E_1 = E_{ab1} = A_1 E_{in1} = A_1 E_0 \quad E_2 = E_{ab2} = A_2 E_{ab2} = A_2 E_0$$

whence we get

$$E_1/A_1 = E_2/A_2 = \ldots = E_0 = f(T) \tag{16-50}$$

Relationship (16-50) expresses Kirchhoff's law according to which the ratio of radiant energy to absorbed energy is independent of the nature of bodies and is equal to the radiation of a black body at the same temperature. From Eqs. (16-49) and (16-50) we get:

$$E = \varepsilon E_0 = A E_0$$

resulting in that the absorptivity and emissivity of a body are numerically equal, i.e.

$$\varepsilon = A \tag{16-51}$$

Relationships (16-50) and (16-51) are also true for spectral quantities, then spectral emissivity is expressed as follows

$$\varepsilon_\lambda = \frac{E_\lambda}{E_{0\lambda}} = f(\lambda T)$$

The emissivities corresponding to directional radiation, i.e. related to a definite direction of radiation, are represented by the following relationships:

$$\varepsilon_\psi = \frac{E_\psi}{E_{0\psi}}$$

and (16-52)

$$\varepsilon_{\psi\lambda} = \frac{E_{\psi\lambda}}{E_{0\psi\lambda}}$$

Kirchhoff's law holds for grey bodies only when spectral quantities are correlated.

In a thermodynamically equilibrium system with an intermediate absorbing medium the densities of emitted and absorbed radiation fluxes are numerically equal for each elementary volume of the medium [16.8]:

$$\eta = \eta_{ab} = \alpha \eta_{in} = \alpha \int_{4\pi} I_0 \, d\omega = 4\alpha E_0 = 4\pi\alpha I_0 \tag{16-53}$$

Relationship (16-53) expresses Kirchhoff's law for a medium.

(g) Lambert's Cosine Law

Stefan-Boltzmann's law determines the summary radiation from the surface of a body in all directions of a hemisphere.

Lambert's law (1760) determines the radiant energy emitted by a body in all directions. In accordance with this law, the directional radiation from a

black body is in proportion to the radiation emitted in the direction of the normal to the surface and the cosine of the angle between them.

For the directional radiation flux Lambert's law becomes:

$$I_\psi = I_n \cos \psi \tag{16-54}$$

where I_ψ and I_n are the total directional densities of radiation, respectively in the direction determined by angle ψ (Fig. 16-1) and in the direction of the normal to the surface.

An important corrollary of the Lambert's law for the brightness, or radiosity, of black body radiation is determined by Eq. (16-10). Substituting Eq. (16-54) into Eq. (16-10), we obtain:

$$I = \frac{I_\psi}{\cos \psi} = I_n = \text{idem} \tag{16-55}$$

Hence, if radiation obeys Lambert's law, radiosity is independent of direction, i.e. is constant. Then, Eq. (16-54) acquires the following appearance:

$$I_\psi = I \cos \psi \tag{16-54'}$$

Let us relate the radiosity and density of hemispherical radiation. The radiation density for an area element in a given direction is expressed by relationship (16-12)

$$dE_\psi = I d\omega \cos \psi$$

The solid angle $d\omega$ is that at which the elementary area of the surface is seen from any point of the elementary area of another body:

$$d\omega = \frac{dF}{r^2} \tag{16-56}$$

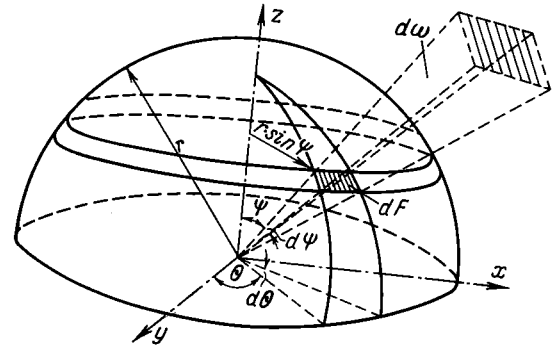

Fig. 16-9. Determining the solid angle in spherical coordinates

where dF is the elementary area subtended on the surface of a sphere of unit radius r (Fig. 16-9). The elementary area can be presented as the product of two elementary arcs: $rd\psi$ and $r \sin \psi \, d\omega$ where ψ is the angle complementary to the angle of latitude; θ is the angle of longitude.

Hence,

$$d\omega = d\psi \sin \psi d\theta \tag{16-57}$$

Substituting into Eq. (16-12) the value of $d\omega$ from Eq. (16-57), allowing for Eq. (16-55) and integrating, we obtain the relationship for hemispherical radiation density:

$$E = I \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin \psi \cos \psi \, d\psi = I\pi \tag{16-58}$$

whence

$$I = E/\pi \tag{16-59}$$

where E is the density of integral hemispherical radiation of a black body, determined by the Stefan-Boltzmann law; I is the angular intensity or brightness of integral radiation.

From equation (16-59) it follows that the intensity of radiation in the direction normal to the emitting surface is smaller than the total hemispherical ra-

diation in all directions by a factor of π. This relationship is also true for spectral radiation. Substituting the values of radiation brightness obtained from Eqs. (16-12) and (16-11), we get:

$$\left.\begin{aligned} dE_\psi &= E/\pi\, d\omega \cos\psi \\ dE_{\psi\lambda} &= E_\lambda/\pi\, d\omega \cos\psi \end{aligned}\right\} \tag{16-60}$$

These equations express the cosine law or Lambert's law.

Lambert's law is true for both black and diffuse-radiation bodies, but many real bodies do not obey it. Thus, for polished metals, for instance, the radiation

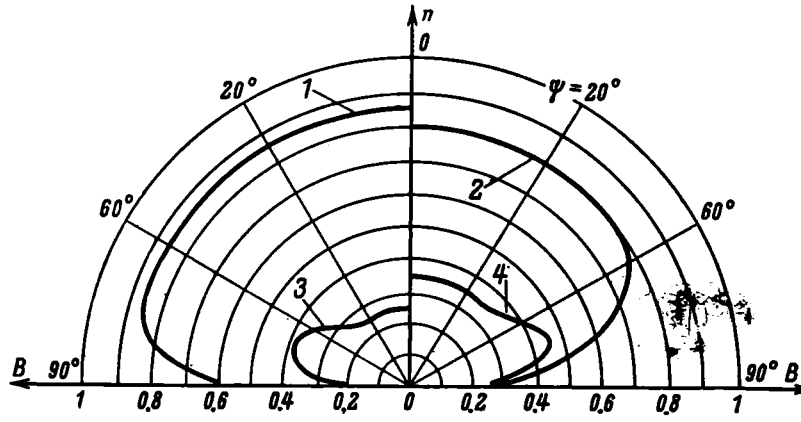

Fig. 16-10. Dependence of radiosity on the direction of radiation: 1—corundum; 2—oxidized copper; 3—polished aluminium bronze; 4—polished bismuth

brightness at $\psi = 60$ °C to 80 °C is higher than that in the direction normal to their surface. With a further increase in the angle ψ brightness drops abruptly to zero (Fig. 16-10). With materials like corundum and copper oxide, radiation brightness along the normal is greater than the brightness in other directions.

If we know the dependence on the angle ψ of the angular intensity of the integral radiation I_ψ of a real body or of its monochromatic hemispherical radiation $I_{\psi\lambda}$, the density of integral hemispherical radiation is determined by the following relationships:

$$E = 2\pi \int_0^{\pi/2} I(\psi) \sin\psi \cos\psi\, d\psi \tag{16-61}$$

$$E = 2\pi \int_0^{\infty} d\lambda \int_0^{\pi/2} I_\lambda(\psi) \sin\psi \cos\psi\, d\psi \tag{16-62}$$

n) Black Temperature

In addition to emissivity, black body radiation is characterized by the so-called *black temperature.*

By black temperatures is meant such fictitious temperature which a given body should have to emit black radiation. Each body may be characterized by a number of black temperatures, depending on the kind of radiation it emits. When comparing the total integral radiation fluxes of a body at a temperature T with those of a black body, the black temperature is called the *radiation* (T_r).

By definition

$$E(T) = E_0(T_r)$$

or according to Stefan-Boltzmann's law the actual and radiation temperatures of the body are expressed by the relationship

$$T = T_r \sqrt[4]{\frac{1}{\varepsilon}}$$

When the black temperature is determined by comparing the colour of a given body with that of a black body, it is referred to as the *colour temperature* (T_c), and monochromatic radiation is usually considered only for some two wavelengths (two colours):

$$\frac{E_{\lambda 1}(T)}{E_{\lambda 2}(T)} = \frac{E_{0\lambda 1}(T_c)}{E_{0\lambda 2}(T_c)}$$

Using (16-41), (16-52), and formula (16-59), we get the relation between T and T_c

$$\frac{1}{T_c} - \frac{1}{T} = \frac{\ln \varepsilon_{\lambda 1}/\varepsilon_{\lambda 2}}{c_2\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)}$$

When comparing the brightness, the black temperature is referred to as the *brightness temperature* (T_b). By definition

$$I_\lambda(T) = I_{0\lambda}(T_b)$$

Substituting $I_\lambda(T) = \varepsilon_\lambda I_{0\lambda}(T_b)$ and using formula (16-59), we get the relation between T and T_b:

$$\frac{1}{T} - \frac{1}{T_b} = \frac{\lambda}{c_2} \ln \varepsilon_\lambda$$

The given relationships between T and T_r, T_c and T_b are the basis for the optical methods of measuring high temperatures [16.9-16.10].

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CHAPTER

17

EXCHANGE OF THERMAL RADIATION BETWEEN SOLIDS SEPARATED BY A TRANSPARENT (DIATHERMANOUS) MEDIUM

17-1. Investigation Methods for Processes of Radiation Heat Transfer

Problems in radiation heat transfer may be concerned with: determination of various kinds of radiation by the given temperatures, optical properties of surfaces of solids, their geometry and size (direct problem); determination of surface temperatures of bodies by given radiation fluxes, optical and geometrif cal properties of bodies (inverse problem), solution of combined problems when for some bodies radiation fluxes are given, and for others—temperatures, and it is necessary to determine temperatures for some bodies and radiation fluxes for others. Here only direct problems shall be considered, in which the determination of net radiation is of practical interest.

The exchange of thermal radiation in various emitting systems is investigated by means of the following methods: the method of multiple reflections, that of outgoing radiation fluxes, the net-radiation method; and also the algebraic, integral and differential methods.

When the method of multiple reflections is applied track is kept of the change in radiant energy taking place during the individual stages of attenuation of absorptions and reflections taking place as a given body exchanges radiation with surrounding bodies. This method vividly reveals the mechanism of radiation heat transfer in concrete emitting systems. But being of a very detailed nature, however, the method of multiple reflections involves painstaking calculations. Its application to intricate geometrical system therefore is complicated.

The method of outgoing radiation fluxes and the net-radiation method involve investigation of radiation heat transfer (or exchange) with the aid of quantities characterizing the ultimate results of heat exchange between the bodies of a given emitting system, therefore, both methods are classified as total flux ones.

The total flux methods do not permit a vivid revelation of the physical picture of the process of radiation heat transfer, but they make it possible to obtain design data without painstaking calculations.

The integral method synthesizes the ideas obtained by means of the multiple reflections and total flux methods. It is based on the integral equations worked out for various kinds of radiation. The integral equations describe the processes of radiation heat transfer with random distribution of the physical properties of the emitting system of bodies and of the intermediate medium, continuously depending on coordinate points. They are of a general and strict nature, make it possible to get a full idea of the essence of the phenomena involved in radiation heat transfer and to study these phenomena in intricate geometrical systems. The solution of integral equations, however, involves considerable difficulties.

Integral equations can be simplified by approximating them with the aid of algebraic equations, then the integral method becomes an algebraic one.

The algebraic method permits determination of both surface-mean radiation for surfaces of individual bodies (individual sections—zones) and local radiation at some nodal points of an emitting system.

The analytical investigation methods for radiation heat transfer also include

the resolvent method (Sec. 17-11). When this method is applied the solutions of integral equations are expressed in terms of the so-called radiation resolvent, which gives the method its name. Then, instead of solving integral equations for various kinds of radiation, the problem is only to find the solution of the equation for the resolvent and this facilitates the problem considerably.

The resolvent method permits an algebraic approximation of the integral equation of the resolvent. The method is then referred to as the resolvent-zonic one (Sec. 17-12). A further development of this approach to the investigation of processes of radiation heat transfer is the iterative-zonic method [17.1]. The differential method is based on solving the differential equations describing radiation heat transfer in an attenuating and emitting medium (Chapter 18).

Alongside with the analytical investigation methods, experimental methods are used in studying radiation heat transfer, including the methods of light and electrical simulation (see [16-8]).

Thus, the intricacy of the processes of radiation heat transfer necessitated the application of a great number of various analytical and experimental methods of investigation. Below, problems shall be solved applying different methods, depending on the nature of the problem involved.

17-2. Radiation Heat Transfer in a System of Bodies with Infinite Parallel Flat Surfaces

(a) *Emitting System Without Screens*

Method of multiple reflections. Let us consider an emitting system of bodies *1* and *2* (Fig. 17-1) whose dimensions are large in comparison with the distance between them. The absorptivities A_1 and A_2 and emissivities ε_1 and ε_2 are independent of temperature and the coordinates of a point on the surface. The temperatures T_1 and T_2 and the densities of emitted radiation do not change along the surfaces of these bodies. There is no heat transfer by conduction and convection; the processes of radiation heat transfer are independent of time (they are steady-state). Let us find the amount of net radiation. For this purpose let us trace the motion of radiant energy emitted by the first body:

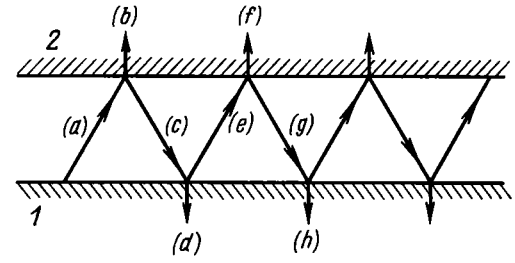

Fig. 17-1. System of parallel flat bodies

body *1* emits E_1; (a)
body *2* absorbs $E_1 A_2$; (b)
body *2* reflects back to body *1* $E_1 (1 - A_2)$; (c)
body *1* absorbs from the radiation reflected by body *2* $E_1 (1 - A_2) A_1$; (d)
body *1* reflects from the radiation reflected from body *2* $E_1 (1 - A_2) (1 - A_1)$; (e)
body *2* absorbs from the radiation reflected by body *1* $E_1 (1 - A_2) (1 - A_1) A_2$; (f)
body *2* absorbs from the radiation reflected from body *1* $E_1 (1 - A_2) (1 - A_1) (1 - A_2)$; (g)
body *1* again absorbs from the radiation reflected from body *2* $E_1 (1 - A_2) (1 - A_1) (1 - A_2) A_1$; etc. (h)

Similar relationships [(a)-(g)] are true for the second body, and only the subscripts 1 and 2 change in the formulae.

The intensity of net radiation can be found from formula (16-19).

The intensity of absorbed radiation is the sum of the absorbed fraction of emitted radiation and the fraction of absorbed radiation coming from body *2*.

The quantity of energy absorbed by body *1* from its own inherent radiation is determined by the sum of the energies (d), (h):

$$E_1(1+k+k^2+\ldots)(1-A_2)A_1 = E_1\left(\frac{1}{1-k}\right)(1-A_2)A_1 \qquad (17\text{-}1)$$

taking into consideration that

$$k = (1-A_1)(1-A_2)$$

As follows from the relationships (a), (b), body *1* absorbs the following quantity of energy from the radiation emitted from body *2*

$$E_2(1+k+k^2+\ldots)A_1 = \frac{E_2A_1}{1-k} \qquad (17\text{-}2)$$

Then

$$q_n = E_1 - E_2\left(\frac{1}{1-k}\right)(1-A_2)A_1 - E_2\left(\frac{A_1}{1-k}\right)$$

The above expression permits a final equation for the density of net radiation q_n, W/m^2, coming to body *1*:

$$q_n = q_{1,2} = \frac{E_1A_2 - E_2A_1}{A_1+A_2-A_1A_2} = \frac{\frac{E_1}{A_1} - \frac{E_2}{A_2}}{\frac{1}{A_1}+\frac{1}{A_2}-1} \qquad (17\text{-}3)$$

Method of outgoing radiation [17.2]. Here the net radiation is determined by relationship (16-20):

$$q_{1,2} = E_{leav1} - E_{leav2} \qquad (17\text{-}4)$$

Let us express outgoing radiation fluxes with the aid of Eq. (16-18), accounting for $R = 1 - A$ (since it is assumed that $D = 0$):

$$\left.\begin{aligned} E_{leav1} &= E_1 + (1-A_1)E_{leav2} \\ E_{leav2} &= E_2 + (1-A_2)E_{leav1} \end{aligned}\right\} \qquad (17\text{-}5)$$

The system of equations (17-5) yields

$$E_{leav1} = \frac{E_1+E_2-A_1E_2}{A_1+A_2-A_1A_2}$$

$$E_{leav2} = \frac{E_1+E_2-A_2E_1}{A_1+A_2-A_1A_2}$$

Substituting these values into Eq. (17-4) we obtain again relationship (17-3) for $q_{1,2}$.

Net-radiation method [17.3]. When this method is applied to determine $q_{1,2}$, the radiation leaving each of the considered bodies will, in accordance

with Eq. (16-24), be expressed by relationships

$$\left.\begin{aligned} E_{leav1} &= q_{1,2}\left(1-\frac{1}{A_1}\right)+E_1/A_1 \\ E_{leav2} &= q_{2,1}\left(1-\frac{1}{A_2}\right)+E_2/A_2 \end{aligned}\right\} \tag{17-6}$$

Under conditions of steady-state radiation $q_{1,2} = -q_{2,1}$. Substituting Eq. (17-6) into Eq. (17-4), we obtain a relationship identical with Eq. (17-3) but in a simpler way than when applying the method of multiple reflections. Let us find now the final calculation formula for $q_{1,2}$, replacing for this purpose in Eq. (17-3) the values of the emitted radiations with their expressions in terms of given temperatures in accordance with Stefan-Boltzmann's law:

$$E_1 = c_0\varepsilon_1\left(\frac{T_1}{100}\right)^4; \quad E_2 = c_0\varepsilon_2\left(\frac{T_2}{100}\right)^4 \tag{17-7}$$

Then we find that

$$q_{1,2} = \frac{c_0\frac{\varepsilon_1}{A_1}\left(\frac{T_1}{100}\right)^4 - c_0\frac{\varepsilon_2}{A_2}\left(\frac{T_2}{100}\right)^4}{\frac{1}{A_1}+\frac{1}{A_2}-1} \tag{17-8}$$

If, in accordance with Eq. (16-51), we assume that $\varepsilon = A$, then formula (17-8) simplifies as follows:

$$q_{1,2} = \frac{c_0\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right]}{\frac{1}{A_1}+\frac{1}{A_2}-1} = c_0A_{1,2}\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right] \tag{17-9}$$

where $A_{1,2}$ is the *reduced absorptivity*. For the geometric system considered it is expressed by the relationship

$$A_{1,2} = \frac{1}{\frac{1}{A_1}+\frac{1}{A_2}-1} \tag{17-10}$$

The quantity

$$c_{1,2} = \frac{c_0}{\frac{1}{A_1}+\frac{1}{A_2}-1} = \frac{1}{\frac{1}{c_1}+\frac{1}{c_2}-\frac{1}{c_0}} \tag{17-11}$$

is referred to as the *reduced radiation factor* and measured in units of $W/m^2 \cdot K^4$. The reduced radiation factor characterizes the intensity of net radiation for the considered system of two or an arbitrary number of bodies. It is equal numerically to the net radiation reduced to unit area of the considered body, to unit time and to unit difference between the fourth-power temperatures of that body and the surrounding bodies. The obtained radiation factor depends on the optical and geometrical properties of the given emitting system of bodies. With $c_2 = c_0$, $c_{1,2} = c_1$; at $c_1 = c_2 = c_0$ $c_{1,2} = c_0$.

The total net flux will be:

$$Q_{1,2} = q_{1,2}F = c_{1,2}F\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right] \tag{17-12}$$

Relationships (17-9) and (17-12) show the net radiation flux to be directly proportional to the reduced radiation factor, surface of the body and the fourth-power temperature difference, while in processes of heat transfer by conduction

and convection the rate of heat flow is in proportion to the first-power temperature difference. This explains the greater importance of radiation heat transfer at high temperatures, compared with heat transfer by conduction and convection.

Having found $q_{1,2}$, E_{out} is determined from Eq. (17-6), then other kinds of radiation are found by formulae in Sec. 16-2.

(*b*) *Radiation Heat Transfer with Screens*

One screen. Let us consider a system composed of parallel flat bodies *1* and *2* and ascreen placed between them (Fig. 17-2). Screens are arranged orthogonally with respect to the direction of radiation and are made of materials with a large reflectivity and thermal conductivity (polished thin sheets of aluminium, copper, etc.). As the screens reemit the radiation that is incident upon them in the direction opposite to that of the radiation emitted from the system of bodies, net radiation diminishes in accordance with the number of screens installed and their optical properties. Let us assume that the absorptivity of bodies *1* and *2* and the screen are the same. The thermal resistance offered to heat flow by the screen $(\delta/\lambda)_{sc}$ is negligibly small. The problem is to find the net radiation flux and screen temperature T_{sc}.

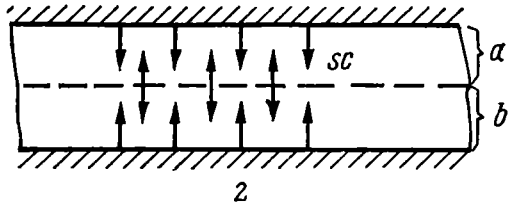

Fig. 17-2. System of flat parallel bodies with one screen

The considered emitting system consists of an aggregate of systems *a* and *b*, identical with the system shown in Fig. 17-1, for which we can use the previously derived relationship (17-9) expressing the net radiation flux:

$$\left.\begin{aligned} q_{1sc} &= A_{1sc}c_0\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_{sc}}{100}\right)^4\right] \\ q_{2sc} &= A_{2sc}c_0\left[\left(\frac{T_{sc}}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right] \end{aligned}\right\} \qquad (17\text{-}13)$$

here, as assumed in the problem, $A_{1sc} = A_{2sc} = A_{1,2}$. Besides, under steady-state conditions $q_{1sc} = q_{2sc} = q_{(1,2)sc}$. Then, from Eq. (17-13) we find T_{sc}:

$$\left(\frac{T_{sc}}{100}\right)^4 = -\frac{1}{2}\left[\left(\frac{T_1}{100}\right)^4+\left(\frac{T_2}{100}\right)^4\right]$$

Substituting this value into Eq. (17-13), we obtain the expression for net radiation $q_{(1,2)sc}$, W/m², equal to:

$$q_{(1,2)sc} = \frac{1}{2}A_{1,2}c_0\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right] \qquad (17\text{-}14)$$

where the reduced absorptivity $A_{1,2}$ of the system of bodies *1* and *2* is determined with the aid of relationship (17-10).

Comparing Eq. (17-14) with the relationship (17-9) derived for a system of bodies without a screen, we see that one screen reduces net radiation by half.

Arbitrary number of screens. Let us consider a more general case when an arbitrary number of screens *n* are installed one after another; the absorptivities of the screens differ and are not equal to these of bodies *1* and *2*, i.e.

$$A_{sc1} \neq A_{sc2} \neq \ldots \neq A_{scn} \neq A_1 \neq A_2 \qquad (17\text{-}15)$$

Let us find the net radiation and temperatures for the screens assuming, as above, that the resistance offered by them to heat flow is negligibly small.

Let us use again relationship (17-9) for the obtained aggregate of $n + 1$ parallel flat systems a, b, c . . . (Fig. 17-3):

$$\left.\begin{aligned} q_{1,\,sc1} &= A_{1,\,sc1}c_0\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_{sc1}}{100}\right)^4\right] \\ q_{sc1,\,sc2} &= A_{sc1,\,sc2}c_0\left[\left(\frac{T_{sc1}}{100}\right)^4-\left(\frac{T_{sc2}}{100}\right)^4\right] \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ q_{scn,\,2} &= A_{scn,\,2}c_0\left[\left(\frac{T_{scn}}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right] \end{aligned}\right\} \qquad (17\text{-}16)$$

The unknown quantities in this system of $n + 1$ equations are the temperatures of the screens and net radiation.

Let us find the net radiation flux, expressing the temperature differences from Eqs. (17-16) as follows:

$$\left.\begin{aligned} \frac{q_{1,\,sc1}}{c_0}\frac{1}{A_{1,\,sc1}} &= \left(\frac{T_1}{100}\right)^4-\left(\frac{T_{sc1}}{100}\right)^4 \\ \frac{q_{sc1,\,sc2}}{c_0}\frac{1}{A_{sc1,\,sc2}} &= \left(\frac{T_{sc1}}{100}\right)^4-\left(\frac{T_{sc2}}{100}\right)^4 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{q_{scn,\,2}}{c_0}\frac{1}{A_{scn,\,2}} &= \left(\frac{T_{scn}}{100}\right)^4-\left(\frac{T_2}{100}\right)^4 \end{aligned}\right\} \qquad (17\text{-}17)$$

Under steady-state conditions $q_{1,sc1} = q_{sc1,sc2} = \ldots = q_{scn,2} = q_{(1,2)sc}$.

Upon addition of Eqs. (17-17) we obtain the relationship

$$\frac{q_{(1,\,2)sc}}{c_0}\left(\frac{1}{A_{1,\,sc1}}+\frac{1}{A_{sc1,\,sc2}}+\ldots+\frac{1}{A_{scn,\,2}}\right)=\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4$$

when the net radiation flux is

$$q_{(1,\,2)sc} = c_0A_{(1,2)\,sc}\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right] \qquad (17\text{-}18)$$

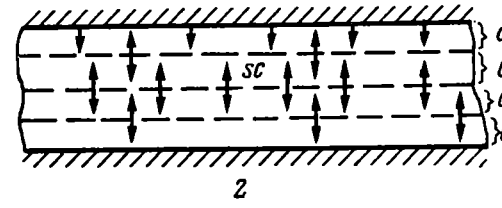

Fig. 17-3. System of flat parallel bodies with an arbitrary number of screens

where the reduced absorptivity for the system of bodies with screens considered is expressed by the relationship

$$A_{(1,\,2)sc} = \frac{1}{\frac{1}{A_{1,\,sc1}}+\frac{1}{A_{sc1,\,sc2}}+\ldots+\frac{1}{A_{scn,\,2}}}$$

Let us transform this dependence, applying Eq. (17-10) to each of the systems a, b, c . . . Then,

$$\frac{1}{A_{(1,\,2)sc}}=\left(\frac{1}{A_1}+\frac{1}{A_{sc1}}-1\right)+\left(\frac{1}{A_{sc1}}+\frac{1}{A_{sc2}}-1\right)+\ldots+\left(\frac{1}{A_{scn}}+\frac{1}{A_2}-1\right)$$

or

$$\frac{1}{A_{(1,\,2)sc}}=\left(\frac{1}{A_1}+\frac{1}{A_2}-1\right)+\left(\frac{2}{A_{sc1}}-1\right)+\left(\frac{2}{A_{sc2}}-1\right)+\ldots+\left(\frac{2}{A_{scn}}-1\right)$$

The final form of the above relationship is as follows:

$$A_{(1,2)sc}=\left[\frac{1}{A_{1,2}}+\sum_{i=1}^{n}\left(\frac{2}{A_{sci}}-1\right)\right]^{-1} \tag{17-19}$$

where $A_{1,2}$ is the absorptivity of the system of bodies *1* and *2* in the absence of screens. The sum from i equals 1 to i equals n contains the quantity characterizing the thermal resistance due to the presence of screens, and this resistance increases with the number of screens. In the private case, when the equality

$$A_{1,sc1}=A_{sc1,sc2}=\ldots=A_{scn,2}$$

holds the system of equations (17-17) reduces into

$$q_{(1,2)sc}=\frac{1}{n+1}A_{1,2}c_0\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]=\frac{1}{n+1}q_{1,2} \tag{17-20}$$

It follows from Eq. (17-20) that when n screens are used between bodies exchanging radiant energy net radiation diminishes $(n+1)$ times. The same result can be obtained directly from Eq. (17-18).

The reduced absorptivity can be replaced in relationship (17-18) with the reduced radiation factor:

$$c_{(1,2)sc}=A_{(1,2\,sc}c_0=\left[\frac{1}{c_{1,2}}+\sum_{i=1}^{n}\left(\frac{2}{c_{sci}}-\frac{1}{c_0}\right)\right]^{-1} \tag{17-21}$$

The full net radiation flux is determined by multiplying the radiation flux by the area of the body.

It follows from the obtained calculation formulae that the distance between screens and the heated surface of body *1* does not effect the net radiation flux.

The unknown screen temperatures are determined from Eqs. (17-17) in which the quantity $q_{(1,2)sc}$ is already known, i.e.

$$\left(\frac{T_{sc1}}{100}\right)^4=\left(\frac{T_1}{100}\right)^4-\frac{q_{(1,2)sc}}{c_0}\frac{1}{A_{1,sc1}}$$

$$\left(\frac{T_{sc2}}{100}\right)^4=\left(\frac{T_{sc1}}{100}\right)^4-\frac{q_{(1,2)sc}}{c_0}\frac{1}{A_{sc1,sc2}}=\left(\frac{T_1}{100}\right)^4-\frac{q_{(1,2)sc}}{c_0}\left(\frac{1}{A_{1,sc1}}+\frac{1}{A_{sc1,sc2}}\right)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left(\frac{T_{scn}}{100}\right)^4=\left(\frac{T_1}{100}\right)^4-\frac{q_{(1,2)sc}}{c_0}\left(\frac{T}{A_{1,sc1}}+\frac{1}{A_{sc1,sc2}}+\ldots+\frac{1}{A_{sc(n-1),n}}\right)$$

Calculating the net radiation flux, account should also be taken of the transport (transfer) of heat by conduction and convection through the medium filling the space between the screens.

If the distance between the bodies and the screens and the pressure of the medium are reduced, the rate of heat transfer decreases. If the gas medium is under a low pressure, a multiple screen system of this kind can be used as a reliable thermal protection (screen-type insulation).

17-3. Radiation Coefficients of Solids and Its Determination

To determine net heat fluxes, we have to know the radiation constants or factors. The radiation factor is a complex function depending on the nature of the emitting body, its temperature, the state of its surface, and with metals—on the

extent of surface oxidation. The radiation constant of pure polished metals is small. Thus, for instance, the magnitude of the radiation constant of metals at a temperature of 100 °C is less than 0.1 of that of a black body. Metals are characterized by a high reflectivity, since, due to the great electric conductivity, the beam penetrates but a small depth. The radiation constant of pure metals can be determined theoretically. The relative radiation constant (emissivity) of metals is related to their specific electrical resistance ρ_{sc} by the following equation:

$$\varepsilon_n = 3.49 \sqrt{\rho_{sc}} \frac{T}{100}$$

where ρ_{sc} is the resistivity at 0°C, Ohm-mm²/m.

Experimental data are available on the resistivity ε_n of metals [17.4-17.5].

Oxide films forming on the surface of metals sharply increase emissivity, so that it may exceed 0.5 [17.6-17.7]. The emissivity of alloys is larger than that of pure metals. The emissivity of semiconductors is greater than 0.8 at 100 °C, but it diminishes with rising temperature. Hard-melting compounds (carbides, borides, silicides) possess an emissivity of 0.5 and above. The radiation constant of dielectric materials is considerably higher than that of pure metals, and usually diminishes with rising temperature.

The absorptivity of solids depends on the same factors as emissivity. It also depends on the nature of the incident radiation, which explains why the directly measured emissivity of a real body may differ from its directly measured absorptivity under similar conditions. In most practical cases this difference is slight and absorptivity A can be determined using data on emissivities.

In tests it is usually the relative radiation constant that is measured, since direct measurement of absorptivity entails considerable difficulties.

Integral radiation constants are experimentally studied by means of the following methods: radiation, calorimetric, regular regime and the method of continuous constant-rate heating. In all of them the heat transferred by convection and conduction must be insignificant compared with that transferred by way of radiation.

(a) *Radiation Method*

The radiation method is relative one. It is based on comparing the quantity of heat radiated by the solid investigated with that of a black body or another solid (standard), whose radiation factor or emissivity is known. Radiant energy is absorbed by a special device containing a differential thermocouple. One of the thermocouple junctions is exposed to the radiation falling from the investigated body, and the other junction—is exposed to the radiation that is incident upon it from the reference solid. The net radiation flux is determined by thermoemf of a differential thermocouple, measured by the galvanometer.

The radiation factor of the flat surface investigated is calculated by relationship (17-9) which can be presented in the form

$$q_{1,2} = c\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right] = k\varphi$$

whence

$$c = \frac{k\varphi}{\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right]} \qquad (17\text{-}22)$$

where c is the radiation factor measured in units of W/m²·K⁴.

The instrument constant k is determined by preliminary tests in which the investigated solid is replaced by a solid of a known radiation constant:

$$k=\frac{c_{sc}}{\varphi_{sc}}\left[\left(\frac{T_{sc}}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]$$

where φ and φ_{sc} = deviations in galvanometer readings observed when conducting tests with the investigated body and the reference solid respectively;
T_1 and T_{sc} = their absolute temperatures;
T_2 = absolute temperature of the radiation detector, K [17.4, 17.7, 17.8].

(*b*) *Calorimetric Method*

The calorimetric method is based on direct measurement of net. It is, therefore, considered absolute. The radiation factor is also determined by the relationship (17-9):

$$c=\frac{q_{1,2}}{\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]} \tag{17-23}$$

The investigated body may be of different shape, provided the surface of the enclosure is considerably larger than the surface of the body, or that the radiation constant of the enclosure is close to that of a black body [17.7, 17.9].

(*c*) *Regular Regime Method*

The *regular regime method* is based on the following relationship, taken from (3-93) (at Bi $\leqslant 0.1$ and Fo $\geqslant 0.3$-0.5):

$$\alpha=\frac{mC}{F} \tag{17-24}$$

where C is the full heat capacity of the body investigated; and F—its surface.

If the investigated body participates only in radiation heat transfer, the heat-transfer coefficient will be equal to the radiation heat-transfer coefficient found from the relationship

$$\alpha=\alpha_r=\frac{q_{1,2}}{T_1-T_2}=cf \tag{17-25}$$

Then, substituting the value of α_r from Eq. (17-25) in Eq. (17-24), we obtain a calculation formula for the radiation constant:

$$c=m\frac{C}{Ff} \tag{17-26}$$

where the temperature factor f, K^3 is denoted by

$$f=\frac{\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4}{T_1-T_2} \tag{17-27}$$

with account taken of the fact that $q_{1,2}$ is found from Eq. (17-23).

From Eq. (17-26) it follows that the method of regular regime is restricted to finding the rate of cooling n in the sequence usual for the regular regime (Sec. 3-10).

(d) The Method of Constant Rate Heating

This method also pertains to the regular regime method. In distinction to the preceding method in which the investigated body is cooled at a constant temperature of the surrounding medium, when the considered method is applied the temperature of the surroundings varies in time at a constant rate. The tests are conducted with a Fourier number Fo $\geqslant 0.3$ to 0.5.

The test specimen, of simple geometric shape (for instance, in the shape of a hollow cylinder), is placed into a thick-walled jacket whose inside dimensions differ but little from the outside dimensions of the test specimen. The medium filling the small clearance between the jacket and specimen is kept under a low pressure, precluding, thereby, heat transfer in it by conduction and convection. The test specimen-and-jacket system is heated at a constant rate ($\partial t/\partial \tau = \text{const}$).

The radiation factor is determined from Eq. (17-12)

$$Q_{1,2} = c_{1,2} F_1 \left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4 \right] \tag{17-28}$$

where T_1 and T_2 are the temperatures of the specimen and jacket.

The net radiation flux is determined by the mass G, specific heat capacity c_p and the rate of heating of the specimen $\partial t/\partial \tau$

$$Q_{1,2} = G c_p \frac{\partial t}{\partial \tau} \tag{17-29}$$

Then, the reduced radiation coefficient for the system "test specimen-jacket" can be found in the relationship

$$c_{1,2} = \frac{Q_{1,2}}{\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4 F_1} = \frac{1}{\frac{1}{c_1} + \frac{1}{c_2} - \frac{1}{c_0}} \tag{17-30}$$

and after that—the unknown value of the radiation coefficient for the test specimen, c_1, with a given radiation coefficient for the block, c_2.

17-4. Radiation Heat Transfer Between a Body and Its Envelope

a) Emitting System Without Screens

Consider two bodies, one of which serves as an enclosure for the other, as illustrated in Fig. 17-4. Body *1* is convex, and body *2* concave. Their surfaces F_1 and F_2, and absorptivities A_1 and A_2 are known, and also their temperatures T_1 and T_2, with $T_1 > T_2$.

The unknown magnitude of net radiation is found with the aid of formula (16-21). In the presence of a diathermal intercepting (intermediate) medium we can write:

$$Q_{1,2} = Q_{ef1} - \bar{\varphi}_{2,1} Q_{ef2} \tag{17-31}$$

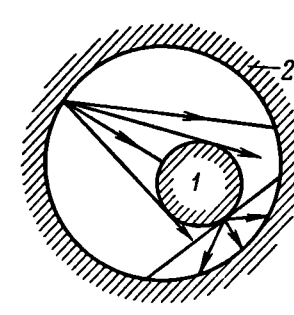

Fig. 17-4. System consisting of a convex body and an envelope

where $\bar{\varphi}_{2,1}$ is *the mean view (configuration) factor*. It characterizes the fraction of the total radiation getting from the second body on the first body in relation to the total radiation outgoing from the second body. The view factor $\bar{\varphi}_{1,2} = 1$,

since all the energy emitted from the first body gets on the second body. The view factor $\bar{\varphi}_{1,1} = 0$ which is in accordance with the assumption that body *1* is convex. The quantity $\bar{\varphi}_{2,2} \neq 0 = 1 - \bar{\varphi}_{2,1}$ characterizes the fraction of energy emitted by the second body on itself. The net radiation flux can be determined applying the net-radiation method [17.3].

Then in accordance with Eq. (16-24) we get:

$$\left.\begin{aligned} Q_{ef1} &= Q_{1,2}\left(1-\frac{1}{A_1}\right)+\frac{Q_1}{A_2} \\ Q_{ef2} &= Q_{2,1}\left(1-\frac{1}{A_2}\right)+\frac{Q_1}{A_2} \end{aligned}\right\} \tag{17-32}$$

Substituting (17-32) in Eq. (17-31), and taking into consideration that $Q_{1,2} = -Q_{2,1}$, we obtain:

$$Q_{1,2} = \frac{\frac{Q_1}{A_1} - \frac{Q_2}{A_2}\bar{\varphi}_{2,1}}{\frac{1}{A_1} + \left(\frac{1}{A_2} - 1\right)\bar{\varphi}_{2,1}} \tag{17-33}$$

The emitted radiation fluxes can be expressed by Stefan-Boltzmann's law in terms of the given temperatures

$$\begin{aligned} Q_1 &= F_1\left(\frac{T_1}{100}\right)^4 c_0\varepsilon_1 \\ Q_2 &= F_2\left(\frac{T_2}{100}\right)^4 c_0\varepsilon_2 \end{aligned} \tag{17-34}$$

Substituting formulae (17-34) into Eq. (17-33), we obtain:

$$Q_{1,2} = \frac{c_0\left[\left(\frac{T_1}{100}\right)^4 \frac{\varepsilon_1}{A_1} F_1 - \left(\frac{T_2}{100}\right)^4 \frac{\varepsilon_2}{A_2} F_2\bar{\varphi}_{2,1}\right]}{\frac{1}{A_1} + \bar{\varphi}_{2,1}\left(\frac{1}{A_2} - 1\right)} \tag{17-35}$$

To determine the unknown quantity $\bar{\varphi}_{2,1}$ let us assume tentatively that bodies *1* and *2* are at the same temperature ($T_1 = T_2$). Then, $Q_{1,2} = 0$. As a result, it follows from Eq. (17-35) that

$$\frac{\varepsilon_1}{A_1} F_1 - \frac{\varepsilon_2}{A_2} F_2\bar{\varphi}_{2,1} = 0$$

whence

$$\bar{\varphi}_{2,1} = \frac{\varepsilon_1}{A_1}\frac{A_2}{\varepsilon_2}\frac{F_1}{F_2} \tag{17-36}$$

Thus, in the general case the view factor depends on the geometry of the emitting system and on its optical properties. Consequently, Eq. (17-36) can be used to express finally the net radiation flux, dividing in this equation the numerator and denominator by ε_1/A_1:

$$Q_{1,2} = \frac{c_0 F_1\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right]}{\frac{1}{\varepsilon_1} + \frac{A_2}{\varepsilon_2}\left(\frac{1}{A_2} - 1\right)\frac{F_1}{F_2}} \tag{17-37}$$

The usual assumption is that $\varepsilon = A$, then Eq. (17-36) reduces into

$$\bar{\varphi}_{2,1} = \frac{F_1}{F_2} \tag{17-38}$$

and the mean view, or angle, factor turns into a purely geometrical characteristic.

The expression for the net radiation flux then acquires the following appearance

$$Q_{1,2}=\frac{\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]c_0F_1}{\frac{1}{A_1}+\left(\frac{1}{A_1}-1\right)\frac{F_1}{F_2}} \tag{17-39}$$

or its shorter form is

$$Q_{1,2}=\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]c_0F_1A_{1,2} \tag{17-40}$$

where the reduced absorptivity of the system of bodies

$$A_{1,2}=\frac{1}{\frac{1}{A_1}+\left(\frac{1}{A_1}-1\right)\frac{F_1}{F_2}} \tag{17-41}$$

If $A_1 = 1$ or $\frac{F_1}{F_2} \to 0$, then $A_{1,2} = 1$.

Let us introduce the concept of the reduced radiation coefficient of a system of bodies, $c_{1,2}$, $W/(m^2 \cdot K)^4$

$$c_{1,2}=\frac{1}{\frac{1}{c_1}+\left(\frac{1}{c_2}-\frac{1}{c_0}\right)\frac{F_1}{F_2}} \tag{17-42}$$

then, instead of Eq. (17-40), we obtain

$$Q_{1,2}=\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]F_1c_{1,2} \tag{17-43}$$

where $Q_{1,2}$ is measured in Watt.

In the private case when the surfaces $F_1 \approx F_2$ (Fig. 17-5) the view factor $\varphi_{2,1} = 1$, meaning that all energy emitted from body *1* is incident on body *2*, and we come to the solution obtained above for the system of flat parallel bodies.

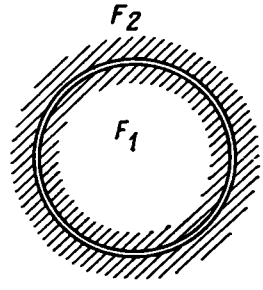

Fig. 17-5. System of body with an envelope, at $F_1 \approx F_2$;
F_1 and F_2—areas of the body and envelope respectively

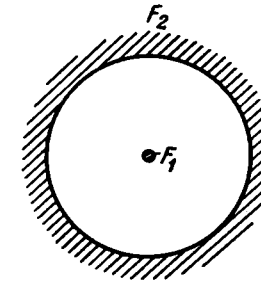

Fig. 17-6. System of body with an envelope, at $F_1 \ll F_2$

If one of the bodies is small compared with the other, $F_1 \ll F_2$ (Fig. 17-6), then $\varphi_{2,1} \to 0$, and $c_{1,2} \to c_1$. With $c_2 = c_0$ the same result follows from formula (17-42).

The given above relationships for $Q_{1,2}$ are valid for concentrically and eccentrically arranged spherical surfaces and also for arbitrary non-concave surfaces with an envelope.

(b) Heat Transfer in the Presence of Screens

One screen. Unlike flat systems, the reduction in the rate of heat transfer in the presence of screens depends on their arrangement with respect to the emitting body, since the magnitudes of the view factors vary in accordance with the arrangement of screens. Screening or shielding is most effective when a cylindrical or spherical screen is placed near the body that is at the higher temperature.

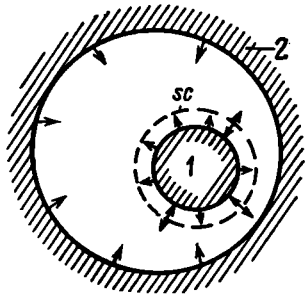

Fig. 17-7. System of cylindrical bodies with a screen:
1—body *1*; *2*—body *2*; *sc*—screen

When applied to systems "body *1*—screen" and "screen—body *2*" (Fig. 17-7), the following equations can be written for the net radiation flux, if we assume that in the general case $A_1 \neq A_{sc} \neq A_2$:

$$\left.\begin{aligned} Q_{1sc} &= A_{1sc}c_0F_1\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_{sc}}{100}\right)^4\right] \\ Q_{sc2} &= A_{sc2}c_0F_{sc}\left[\left(\frac{T_{sc}}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right] \end{aligned}\right\} \quad (17\text{-}44)$$

where, with account taken of Eq. (17-4), the reduced absorptivities are expressed by relationships

$$\left.\begin{aligned} A_{1sc} &= \frac{1}{\frac{1}{A_1}+\frac{F_1}{F_2}\left(\frac{1}{A_{sc}}-1\right)} \\ A_{sc2} &= \frac{1}{\frac{1}{A_{sc}}+\frac{F_{sc}}{F_2}\left(\frac{1}{A_2}-1\right)} \end{aligned}\right\} \quad (17\text{-}45)$$

For a steady-state thermal regime $Q_{1sc} = Q_{sc2} = Q_{(1,\,2)sc}$ and

$$A_{1sc}F_1\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_{sc}}{100}\right)^4\right]=A_{sc2}F_{sc}\left[\left(\frac{T_{sc}}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]$$

from which the temperature of the screen is

$$\left(\frac{T_{sc}}{100}\right)^4=\frac{A_{1sc}F_1\left(\frac{T_1}{100}\right)^4+A_{sc2}F_2\left(\frac{T_2}{100}\right)^4}{A_{1sc}F_1+A_{sc2}F_{sc}} \quad (17\text{-}46)$$

The unknown net radiation flux can be found from Eq. (17-44), dividing the numerator and denominator by $A_{1sc}A_{2sc}F_{sc}$ and taking account of Eq. (17-46) for the screen temperature:

$$\left.\begin{aligned} Q_{(1,\,2)sc} &= A_{(1,\,2)sc}c_0\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]F_1 \\ \text{or} \\ Q_{(1,\,2)sc} &= c_{(1,\,2)sc}\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]F_1 \end{aligned}\right\} \quad (17\text{-}47)$$

where the reduced absorptivity and radiation coefficient of the system are respectively expressed by equations

$$A_{(1,\,2)sc}=\frac{1}{\frac{1}{A_{1sc}}+\frac{F_1}{F_{sc}}\frac{1}{A_{sc2}}}\;;\quad c_{(1,\,2)sc}=\frac{1}{\frac{1}{c_{1sc}}+\frac{F_1}{F_2}\frac{1}{c_{sc2}}}$$

Taking Eqs. (17-41) and (17-42) into account, we have:

$$A_{(1,2)sc}=\frac{1}{\frac{1}{A_{1,2}}+\frac{F_1}{F_{sc}}\left(\frac{2}{A_{sc}}-1\right)};\quad c_{(1,2)sc}=\frac{1}{\frac{1}{c_{1,2}}+\frac{F_1}{F_{sc}}\left(\frac{2}{c_{sc}}-\frac{1}{c_0}\right)} \tag{17-48}$$

where $A_{1,2}$ and $c_{1,2}$ are reduced absorptivity of the system in the absence of a screen, determined from Eqs. (17-41), (17-42). The quantity $A_{(1,2)sc}$ will be smaller when $F_{sc}\to F_1$ and when quantity A_{sc} is small.

The second item in the denominator $\frac{F_1}{F_{sc}}\left(\frac{2}{A_{sc}}-1\right)$ characterizes the resistance offered to radiation heat transfer attributed to the screen.

An arbitrary number of screens, installed successively between bodies *1* and *2*, permit us to consider the produced system as a totality of systems "body *1*—screen *1*", "screen *1*—screen *2*", etc. (Fig. 17-8) and apply to them the relations derived for body with envelope.

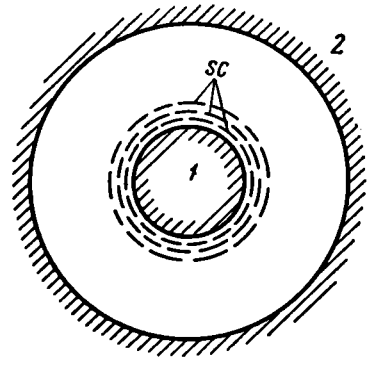

Fig. 17-8. System of cylindrical bodies with an arbitrary number of screens

If again we assume $A_1\neq A_{sc1}\neq\ldots\neq A_{scn}\neq A_2$ and $c_1\neq c_{sc1}\neq\ldots\neq c_{scn}\neq c_2$, the following system of equations can be written:

$$\left.\begin{aligned}
Q_{1,sc1}&=c_0A_{1,sc1}\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_{sc1}}{100}\right)^4\right]F_1\\
Q_{sc1,sc2}&=c_0A_{sc1,sc2}\left[\left(\frac{T_{sc1}}{100}\right)^4-\left(\frac{T_{sc2}}{100}\right)^4\right]F_{sc1}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
Q_{scn,2}&=c_0A_{scn,2}\left[\left(\frac{T_{scn}}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]F_{scn}
\end{aligned}\right\} \tag{17-49}$$

Expelling from Eq. (17-49) the unknown screen temperatures and allowing that under steady-state conditions $Q_{1\,sc1}=Q_{sc1,\,sc2}=\ldots=Q_{(1,\,2)sc}$, we can find the expression for the net radiation flux

$$Q_{(1,2)sc}=\frac{c_0\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]}{\frac{1}{A_{1,sc1}F_1}+\frac{1}{A_{sc1,sc2}F_{sc1}}+\ldots+\frac{1}{A_{scn,2}F_{scn}}}=$$
$$=\frac{F_1c_0\left[\left(\frac{T_1}{100}\right)^4-\left(\frac{T_2}{100}\right)^4\right]}{\frac{1}{A_{1,sc1}}+\frac{1}{A_{sc1,sc2}}\frac{F_1}{F_{sc1}}+\ldots+\frac{1}{A_{scn,2}}\frac{F_1}{F_{scn}}} \tag{17-50}$$

Using the previously derived relationship (17-45) for the reduced absorptivity, we can write:

$$A_{1,sc1}=\frac{1}{\frac{1}{A_1}+\frac{F_1}{F_{sc1}}\left(\frac{1}{A_{sc1}}-1\right)};\quad A_{sc1,sc2}=\frac{1}{\frac{1}{A_{sc1}}+\frac{F_{sc1}}{F_{sc2}}\left(\frac{1}{A_{sc2}}-1\right)} \tag{17-51}$$

Substituting the relationships (17-51) into expression (17-50) and making some transformations, we obtain the final expression for the net radiation flux

$Q_{(1,\,2)sc}$, W:

$$Q_{(1,2)sc} = c_0 A_{(1,2)sc} F_1 \left[\left(\frac{T_1}{100} \right)^4 - \left(\frac{T_2}{100} \right)^4 \right] \tag{17-52}$$

where the reduced absorptivity of the emitting system considered will be expressed by the relationship

$$A_{(1,2)sc} = \frac{1}{\frac{1}{A_{1,2}} + \sum_{i=1}^{n} \frac{F_1}{F_{sci}} \left(\frac{2}{A_{sci}} - 1 \right)} \tag{17-53}$$

The derived relationships (17-52) and (17-53) are generalized relationships (17-47) and (17-48) applicable to a radiating system with an arbitrary number of screens.

Comparing these relationships, it is evident that the resistance offered to radiant heat exchange in such a system increases in proportion to the number of screens fitted.

With $F_1 = F_{sc1} = \ldots = F_2$ relationships (17-50) and (17-51) reduce to Eqs. (17-19) and (17-21) previously derived for systems of bodies with parallel flat surfaces.

17-5. Exchange of Thermal Radiation Between Bodies Arbitrarily Located in Space. View (Angle, Configuration) Factors

Let us consider two black bodies with isothermal surfaces at temperatures T_1 and T_2. There is no self-irradiation of these bodies ($\bar{\varphi}_{1,1} = \bar{\varphi}_{2,2} = 0$), nor do the bodies exchange heat with other bodies. The bodies are homogeneous and isotropic; radiosity does not depend on direction. The problem is to find the net radiation flux. For this purpose consider on each of the bodies involved elementary areas dF_1 and dF_2 (Fig. 17-9), infinitely small compared with the distance r between their centres at points M and N.

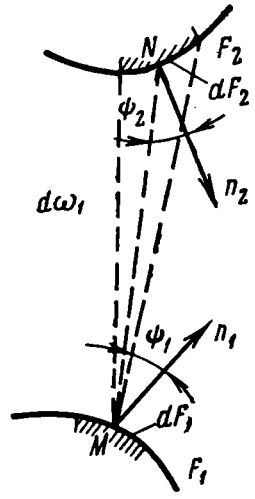

Fig. 17-9. System of two bodies arbitrarily arranged in space:
n_1 and n_2 — normals to respective areas; ψ — the angle formed by normals and the direction of radiation

Let us find the elementary radiation flux that is incident upon area dF_1 from elementary area dF_2, using Eq. (16-12):

$$d^2Q_{in1} = I_{\psi_2} d\omega_2 dF_2 = I_2 \cos \psi_2 d\omega_2 dF_2 \tag{17-54}$$

The elementary radiation flux emitted by dF_2 that is incident upon dF_1 is

$$d^2Q_{in2} = I_{\psi_1} d\omega_1 dF_1 = I_1 \cos \psi_1 d\omega_1 dF_1 \tag{17-55}$$

The elementary solid angles can be presented by relationships

$$d\omega_1 = \frac{dF_2 \cos \psi_2}{r^2}; \qquad d\omega_2 = \frac{dF_1 \cos \psi_1}{r^2}$$

The brightness of radiation (radiosity) emitted from the two elementary areas is expressed through the density of hemispherical radiation in accordance with

formula (16-59). Equations (17-54) and (17-55) then acquire the following form:

$$d^2Q_{in1} = E_2 \cos\psi_2 \frac{dF_1 \cos\psi_1}{\pi r^2} dF_2 \tag{17-56}$$

$$d^2Q_{in2} = E_1 \cos\psi_1 \frac{dF_2 \cos\psi_2}{\pi r^2} dF_1 \tag{17-57}$$

Let us introduce the following notations:

$$d\varphi_{1,2} = \frac{\cos\psi_1 \cos\psi_2}{\pi r^2} dF_2; \quad d\varphi_{2,1} = \frac{\cos\psi_2 \cos\psi_1}{\pi r^2} dF_1 \tag{17-58}$$

where the quantities $d\varphi_{1,2}$ and $d\varphi_{2,1}$ are referred to as the *elementary view factors of radiation*. Then,

$$\left.\begin{aligned} d^2Q_{in1} &= E_2\, d\varphi_{2,1}\, dF_2 = dQ_2\, d\varphi_{2,1} \\ d^2Q_{in2} &= E_1\, d\varphi_{1,2}\, dF_1 = dQ_1\, d\varphi_{1,2} \end{aligned}\right\} \tag{17-59}$$

In formula (17-59) dQ_1 and dQ_2 are the total hemispherical elementary heat fluxes emitted by the areas dF_1 and dF_2.

It follows from the last relationships that

$$d\varphi_{1,2} = \frac{d^2Q_{in2}}{dQ_1}; \quad d\varphi_{2,1} = \frac{d^2Q_{in1}}{dQ_2} \tag{17-60}$$

Thus, the elementary view (angle, configuration) factor characterizes the fraction of the total radiant energy leaving an elementary area of one body that is incident upon the elementary area of the other body. The product of the elementary view factor by the corresponding elementary area is called the *elementary surface-surface direct exchange area* and is denoted by

$$d^2H_{1,2} = d\varphi_{1,2}\, dF_1; \quad d^2H_{2,1} = d\varphi_{2,1}\, dF_2 \tag{17-61}$$

Accordingly, the relationships (17-59) reduce to

$$d^2Q_{in1} = E_2\, d^2H_{2,1}; \quad d^2Q_{in2} = E_1\, d^2H_{1,2} \tag{17-62}$$

Let us find the local radiation fluxes from the elementary areas dF_1 and dF_2 that are incident upon the finite surfaces F_1 and F_2. For this purpose let us integrate relationships (17-59) in respect to F_2 and F_1, allowing for the fact that at $T = \text{const}$ the radiation fluxes emitted from black bodies are constant over the surface of each body.

With account taken of Eq. (17-58) let us denote the integrals of the elementary view factors as follows:

$$\left.\begin{aligned} \varphi_{1,2} &= \int_{F_2} d\varphi_{1,2} = \int_{F_2} \frac{\cos\psi_1 \cos\psi_2}{\pi r^2} dF_2 \\ \varphi_{2,1} &= \int_{F_1} d\varphi_{2,1} = \int_{F_1} \frac{\cos\psi_2 \cos\psi_1}{\pi r^2} dF_1 \end{aligned}\right\} \tag{17-63}$$

The local incident radiation fluxes are expressed by relationships

$$\left.\begin{aligned} dQ_{in1} &= E_2 \varphi_{2,1}\, dF_2 \\ dQ_{in2} &= E_1 \varphi_{1,2}\, dF_1 \end{aligned}\right\} \tag{17-64}$$

From these relationships we have:

$$\left.\begin{aligned} \varphi_{1,2} &= \frac{dQ_{in2}}{dQ_1} \\ \varphi_{2,1} &= \frac{dQ_{in1}}{dQ_2} \end{aligned}\right\} \tag{17-65}$$

$\varphi_{1,2}$ and $\varphi_{2,1}$ are known as *local view factors*. According to Eq. (17-65) the local view factor characterizes the fraction of radiant energy emitted by one elementary area dF_1 (dF_2) falling upon a body of area F_2 (F_1), with respect to the total energy dQ_1 (dQ_2) emitted by dF_1 (dF_2) over the entire hemisphere.

Let us find the net radiation flux, subtracting for this purpose the second relationship from Eq. (17-64):

$$dQ_{1,2} = dQ_{in2} - dQ_{in1} = E_2\varphi_{1,2}\,dF_1 - E_1\varphi_{1,2}\,dF_2 \tag{17-66}$$

To determine $Q_{1,2}$, W, Eq. (17-66) should be integrated, introducing preliminarily the concept of *mean view factors*:

$$\bar{\varphi}_{1,2} = \frac{1}{F_1}\int_{F_1}\varphi_{1,2}\,dF_1; \qquad \bar{\varphi}_{2,1} = \frac{1}{F_2}\int_{F_2}\varphi_{2,1}\,dF_2 \tag{17-67}$$

The mean view factors make it possible to find:

$$Q_{in1} = E_2\bar{\varphi}_{2,1}F_2 \text{ and } Q_{in2} = E_1\bar{\varphi}_{1,2}F_1 \tag{17-68}$$

The net radiation flux is determined by the difference

$$Q_{1,2} = E_1\bar{\varphi}_{1,2}F_1 - E_2\varphi_{2,1}F_2 \tag{17-69}$$

Relationships (17-68) make it possible to express the mean view factors in terms of the corresponding heat fluxes

$$\bar{\varphi}_{1,2} = \frac{Q_{in2}}{Q_1}; \quad \bar{\varphi}_{2,1} = \frac{Q_{in1}}{Q_2} \tag{17-70}$$

Consequently, the mean view factor characterizes the fraction of the energy emitted by finite surface F_1 that is incident upon the other body with a finite surface F_2.

The mean surface-surface direct exchange areas are represented by

$$\bar{H}_{1,2} = F_1\bar{\varphi}_{1,2}; \qquad \bar{H}_{2,1} = F_2\bar{\varphi}_{2,1} \tag{17-71}$$

The mean surface-surface direct exchange area of the first body in respect to the second body is the fraction of the surface of the first body whose total radiation flux is equivalent to the radiation flux from the first body that is incident upon the second body.

Using Eq. (17-71) and Stefan-Boltzmann's law, Eq. (17-69) is transformed into equation

$$Q_{1,2} = c_0\left[\left(\frac{T_1}{100}\right)^4\bar{\varphi}_{1,2}\;F_1 - \left(\frac{T_2}{100}\right)^4\bar{\varphi}_{2,1}F_2\right] \tag{17-72}$$

where $Q_{1,2}$ is measured in Watt.

Allowing for the fact that in accordance with relationships (17-63) and (17-67) the mean surface-surface direct exchange areas $\bar{H}_{1,2} = \bar{H}_{2,1}$, relationship (17-72) can be presented in the form

$$Q_{1,2} = c_0\bar{H}_{1,2}\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right] \tag{17-73}$$

When determining the net radiation flux it is assumed that the quantities $\bar{\varphi}_{1,2}$, $\bar{\varphi}_{2,1}$, $\bar{H}_{1,2}$, $\bar{H}_{2,1}$ are given.

It follows from the foregoing that the view factors $d\varphi$, φ and $\bar{\varphi}$ are purely geometrical characteristics of an emitting system, since they are determined by the geometry of the bodies and their arrangement in space. In general, they

also depend on the optical properties of the emitting system [relationship (17-36)].

The net radiation flux for a system consisting of two grey bodies can be found from relationship (17-69) provided the radiation fluxes emitted by the surfaces are replaced in the relationship with the radiation fluxes leaving the surfaces, i.e. with the outgoing radiation fluxes:

$$Q_{1,2} = E_{out1} F_1 \bar{\varphi}_{1,2} - E_{out2} F_2 \bar{\varphi}_{2,1} \tag{17-74}$$

17-6. Geometrical Properties of Radiant Fluxes

From Eqs. (17-69), (17-73) and (17-74) it follows that to calculate net radiant fluxes we must have data on the size of the mutual radiation surfaces or the view factors.

The view factors and surface-surface direct exchange areas characterize certain geometrical properties of emitting systems with a transparent intercepting (intermediate) medium.

They include the following properties: reciprocity (mutual symmetry), the enclosure, coincidence, shading and non-concavity properties.

The property of reciprocity consists in that the surface-surface direct exchange areas of two bodies between which radiant exchange takes place are equal to each other, no matter which of the areas of the two bodies is the emitting one. So, in accordance with relationships (17-58) and (17-61) we find that the elementary surface-surface direct exchange areas are equal to

$$\left.\begin{aligned} d^2H_{1,2} &= d^2H_{2,1} \\ \text{or} \\ dF_1\, d\varphi_{1,2} &= dF_2\, d\varphi_{2,1} \end{aligned}\right\} \tag{17-75}$$

The mean surface-surface direct exchange areas are also numerically the same:

$$\left.\begin{aligned} \overline{H}_{1,2} &= \overline{H}_{2,1} \\ \text{or} \\ F_1\bar{\varphi}_{1,2} &= F_2\bar{\varphi}_{2,1} \end{aligned}\right\} \tag{17-76}$$

This equality also follows from relationship (17-71), if relationships (17-67) and (17-63) are substituted into it successively. In addition, the relationship can be obtained from the conditions of thermodynamic equilibrium, using Eq. (17-72), when $T_1 = T_2$ and $Q_{1,2} = 0$. If one of the view factors is known, the other can be determined from the relationship

$$\bar{\varphi}_{1,2} = \bar{\varphi}_{2,1} \frac{F_2}{F_1}$$

In the general case the considered body can exchange heat with all bodies surrounding it. This condition makes it possible to derive the relationships describing the *enclosure property.*

According to the law of conservation of energy, the radiation fluxes from body *1* that are incident upon bodies *i* shall amount to the radiation emitted from body *1*:

$$\sum_{i=1}^{n} Q_{in1,i} = Q_1 \text{ or} \tag{17-77}$$

$$\sum_{i=1}^{n} \overline{\varphi}_{1,i} Q_1 = Q_1$$

whence

$$\sum_{i=1}^{n} \overline{\varphi}_{1,i} = 1 \qquad (17\text{-}78)$$

Thus the sum of the view factors of a given body exchanging heat with other bodies is equal to unity, and the sum of the surface-surface direct exchange areas is equal to the surface of the body considered, since, in accordance with Eqs. (17-71) and (17-78)

$$\sum_{i=1}^{n} \overline{H}_{1,i} = F_1 \quad \sum_{i=1}^{n} \overline{\varphi}_{1,i} = F_1 \qquad (17\text{-}79)$$

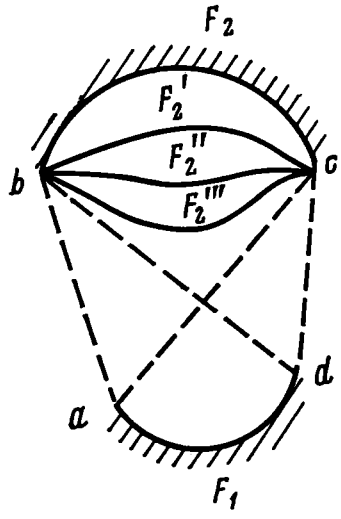

Fig. 17-10. Determining the coincidence property

The view factor between surface F_2 and surface F_1 does not depend on the configuration (or geometry) of surface F_2, if the surface F_2 (F_2', F_2'', F_2''') is subtended within the system of external (*ab* and *cd*) and internal (*ac* and *bd*) intercepting lines (Fig. 17-10). This property is in full accordance with the property of reciprocity and the enclosure property and is referred to as the *property of coincidence* of radiation fluxes.

The shading property consists in that the net radiation flux travelling from one body to another is equal to zero, if a non-transparent body is placed between the two bodies

$$Q_{1,2} = 0; \quad \overline{\varphi}_{1,2} = 0; \quad \overline{\varphi}_{2,1} = 0$$

With a plain and convex body there is no self-irradiation, and consequently

$$\overline{\varphi}_{1,1} = \overline{\varphi}_{2,2} = 0 \qquad (17\text{-}80)$$

For concave bodies it differs from zero:

$$\overline{\varphi}_{1,1} \geqslant 0; \qquad \overline{\varphi}_{2,2} \geqslant 0 \qquad (17\text{-}81)$$

Relationship (17-80) expresses the *property of non-concavity*, and Eq. (17-81)—the property of concavity.

From the foregoing it follows that each of the geometric properties of radiation fluxes makes it possible to derive relationships concerning the mutual surfaces and view factors which can be used for calculating the latter.

17-7. Investigating Radiation Exchange in an Arbitrary Enclosure With the Aid of an Algebraic Method

Black bodies. Let us consider an enclosure (Fig. 17-11) of given geometry and size built up of a finite number n of isothermal bodies with surfaces $F_1, F_2, \ldots$ $\ldots, F_n$, respectively at temperatures $T_1, T_2, \ldots, T_n$ [17.10-17.11]. This open system is converted into an enclosure by means of fictitious surfaces possessing black body properties ($R = 0$) at $T = 0$ ($E_0 = 0$).

The problem is to find the net radiation flux for the surfaces of all bodies making up the enclosure.

For the ith body the net radiation flux is determined from relationship (16-19):

$$q_{neti} = E_i - E_{ini};\ i = 1, \ldots, n \qquad (17\text{-}82)$$

E_{ini} is the radiation incident upon body i from each of the $k = 1, \ldots, n$ surfaces of the system. The incident radiation flux only from the kth surface amounts to

$$E'_{ini} = \frac{E_k \bar{\varphi}_{i,k} F_k}{F_i} \qquad (17\text{-}83)$$

Let us make use of the property of the reciprocity rule expressed by Eq. (17-76). Then

$$E'_{ini} = E_k \bar{\varphi}_{k,i} \qquad (17\text{-}84)$$

The radiation from all component bodies of the system that is incident upon body i is expressed by the algebraic sum

$$E_{ini} = \sum_{k=1}^{n} E_k \bar{\varphi}_{k,i} \qquad (17\text{-}85)$$

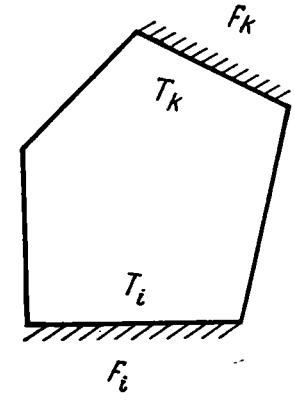

Fig. 17-11. Enclosure consisting of n isothermal surfaces

On the other hand, the radiation emitted from the body onto all bodies of the system will, in accordance with the law of conservation of energy, be equal to

$$E_i = \sum_{k=1}^{n} E_i \bar{\varphi}_{k,i} \qquad (17\text{-}86)$$

Substituting the obtained relationships (17-85) and (17-86) into the initial Eq. (17-82), we obtain

$$q_{neti} = \sum_{k=1}^{n} (E_i - E_k) \bar{\varphi}_{k,i} \qquad (17\text{-}87)$$

where $\bar{\varphi}_{k,i}$ is the mean view factor between surface i and surface $k = 1, \ldots, n$; q_{neti} is measured in W/m^2.

Expressing the radiation emitted from a surface in terms of temperatures, following Stefan-Boltzmann's law, we obtain:

$$q_{neti} = \sum_{k=1}^{n} c_0 \bar{\varphi}_{k,i} \left[\left(\frac{T_i}{100} \right)^4 - \left(\frac{T_k}{100} \right)^4 \right] \qquad (17\text{-}88)$$

The summation includes $k = i$, if body i, is concave ($\bar{\varphi}_{i,i} \neq 0$).

Relationships (17-87) and (17-88) are applied successively to each surface $i = 1, \ldots, n$, with the mean view factors being considered as given quantities.

Grey bodies. In the event of a grey-body enclosure the initial conditions should also include the optical properties of all the bodies making up the enclosure, A_i, R_i, which are assumed to be constant for each body. The net radiation flux is determined with the aid of relationship (16-20), according to which

$$q_{neti} = E_{outi} - E_{ini}$$

Relationships (17-85) and (17-86) reduce into:

$$E_{ini} = \sum_{k=1}^{n} E_{outk}\bar{\varphi}_{k,\,i} \tag{17-89}$$

$$E_{outi} = \sum_{k=1}^{n} E_{outi}\bar{\varphi}_{k,\,i} \tag{17-90}$$

Substituting Eqs. (17-89) and (17-90) into Eq. (16-20), we obtain the following system of algebraic equations for the net radiation flux:

$$Q_{neti} = \sum_{k=1}^{n} (E_{outi} - E_{outk})\,\bar{\varphi}_{k,\,i} \tag{17-91}$$

It follows from Eq. (17-91) that determining q_{neti} it is first necessary to find the radiation fluxes leaving the surfaces (outgoing radiation) with the aid of relationship (16-18):

$$E_{outi} = E_i + R_i E_{ini} \tag{17-92}$$

Substituting into it the value of incident radiation from Eq. (17-89), we obtain

$$E_{outi} = E_i + R_i \sum_{k=1}^{n} E_{outk}\bar{\varphi}_{k,\,i} \tag{17-93}$$

or

$$E_{outi} - R_i \sum_{k=1}^{n} E_{outk}\bar{\varphi}_{k,\,i} = E_i; \quad i = 1, \ldots, n \tag{17-94}$$

The total radiation flux leaving a surface is determined by multiplying both sides of Eq. (17-94) by the surface F_i. Then, making use of the reciprocity property or rule, expressed by Eq. (17-76), the system of equations (17-94) can be reduced to

$$Q_{outi} - R_i \sum_{k=1}^{n} Q_{outk}\bar{\varphi}_{i,\,k} = Q_i \tag{17-95}$$

Thus, the obtained system of n algebraic equations can be used to determine $E_{out1}, \ldots, E_{outn}$ unknown quantities, which are then used to find q_{neti} from the system of equations (17-91), with account taken of F_i.

Other radiation fluxes are determined from relationships (16-14) and (16-18), used to classify radiation fluxes in Sec. 16-2, in terms of the determined outgoing radiation:

$$\left.\begin{aligned} F_{abi} &= A_i E_{ini} = \frac{A_i}{R_i}(E_{outi} - A_i E_{0i}) \\ E_{ri} &= E_{outi} - E_i; \qquad E_{ini} = \frac{E_{outi} - A_i E_{0i}}{R_i} \end{aligned}\right\} \tag{A}$$

By analogy with the calculation method treated use can be made of systems of algebraic equations worked out for other kinds of radiation.

Let us work out a system of equations for incident radiation, using for this purpose relationship (17-89) and substituting into the latter the value of E_{outk} from Eq. (16-18):

$$E_{ini} = \sum_{k=1}^{n} (E_k + R_k E_{ink})\,\bar{\varphi}_{k,\,i}$$

or

$$E_{ini} - \sum_{k=1}^{n} R_k E_{ink} \bar{\varphi}_{k,\,i} = \sum_{k=1}^{n} E_k \bar{\varphi}_{k,\,i}; \quad i = 1, \ldots, n \tag{17-96}$$

When applied to find total radiation fluxes Eq. (17-96) is multiplied by F_i, then yielding

$$Q_{ini} - \sum_{k=1}^{n} R_k Q_{ink} \bar{\varphi}_{k,\,i} = \sum_{k=1}^{n} Q_k \bar{\varphi}_{i,\,k} \tag{17-97}$$

where incident radiation Q_{in} is measured in Watt.

Radiation fluxes and the total fluxes of other kinds of radiation are determined by the formula derived for E_{ini} (Sec. 16-2):

$$\left.\begin{aligned} &E_{outi} = E_i + R_i E_{ini}; \quad q_{neti} = E_{outi} - E_{ini} \\ &E_{abi} = A_i E_{ini}; \quad E_{ri} = R_i E_{ini} \end{aligned}\right\} \tag{B}$$

Let us work out a system of algebraic equations for net radiation, substituting for this purpose into relationship (17-89) the value of E_{ini} and E_{outk}, in accordance with Eqs. (16-23) and (16-24), in terms of net radiation q_{net}. Then the system of equations becomes:

$$\frac{q_{neti}}{A_i} - \sum_{k=1}^{n} \frac{R_k}{A_k} q_{netk} \bar{\varphi}_{k,\,i} = \sum_{k=1}^{n} (E_{0i} - E_{0k}) \bar{\varphi}_{k,\,i} \tag{17-98}$$

since relationship (17-86) is true for E_{0i}. Here denoted by E_{0i} and E_{0k} are the radiation fluxes from a black body at temperatures T_i and T_k.

The total net radiation flux is represented by the following system of algebraic equations:

$$\frac{Q_{neti}}{A_i} - \sum_{k=1}^{n} \frac{R_k}{A_k} Q_{netk} \bar{\varphi}_{i,\,k} = \sum_{k=1}^{n} (E_{0i} - E_{0k}) \bar{H}_{k,\,i} \tag{17-99}$$

Let us determine now the other radiation fluxes in terms of net radiation, using relationships (16-23), (16-24); (16-15) and (16-19):

$$\left.\begin{aligned} &E_{ini} = E_{0i} - \frac{q_{neti}}{A_i}; \quad E_{outi} = E_{0i} - \frac{R_i}{A_i} q_{neti} \\ &E_{ri} = R_i \left(E_{0i} - \frac{q_{neti}}{A_i}\right); \quad E_{abi} = q_{neti} - E_i \end{aligned}\right\} \tag{C}$$

Let us work out a system of algebraic equations for reflected and absorbed radiation.

Using relationships (16-15), (17-89) and (16-18), we find the unknown system of relationships permitting to determine the emissive power and the density of reflected radiation:

$$E_{ri} - R_i \sum_{k=1}^{n} E_{rk} \bar{\varphi}_{k,\,i} = R_i \sum_{k=1}^{n} E_k \bar{\varphi}_{k,\,i} \tag{17-100}$$

$$Q_{ri} - R_i \sum_{k=1}^{n} Q_{rk} \bar{\varphi}_{i,\,k} = R_i \sum_{k=1}^{n} Q_k \bar{\varphi}_{i,\,k} \tag{17-101}$$

Relationships (16-14), (16-15), (16-24) and (16-18) can be used to work out the following system of equations for other kinds of radiation expressed in terms

of the reflected radiation determined with the aid of formula (17-100):

$$\left.\begin{aligned} E_{ini} &= \frac{E_{ri}}{R_i}; \quad E_{abi} = \frac{A_i}{R_i} E_{ri} \\ q_{neti} &= A_i \left(E_{0i} - \frac{E_{ri}}{R_i}\right); \quad E_{outi} = A_i E_{0i} + E_{ri} \end{aligned}\right\} \tag{D}$$

By analogy with the foregoing it is possible to find a system of algebraic equations for absorbed radiation:

$$E_{abi} - A_i \sum_{k=1}^{n} E_{obk} \overline{\varphi}_{k,\,i} = A_i \sum_{k=1}^{n} E_k \overline{\varphi}_{k,\,i} \tag{17-102}$$

$$Q_{abi} - A_i \sum_{k=1}^{n} Q_{abk} \varphi_{i,\,k} = A_i \sum_{k=1}^{n} Q_k \overline{\varphi}_{i,\,k} \tag{17-103}$$

and a system of elementary equations for other kinds of radiation:

$$\left.\begin{aligned} E_{ini} &= \frac{E_{abi}}{A_i}; \quad E_{ri} = R_i \frac{E_{abi}}{A_i}; \\ E_{outi} &= A_i E_{0i} - \frac{E_{abi} R_i}{A_i}; \quad q_{neti} = E_i - E_{abi} \end{aligned}\right\} \tag{E}$$

The systems of algebraic equations derived and their corresponding elementary relationships for various kinds of radiation indicate that all radiation fluxes depend on temperature, geometry and optical properties of the bodies making up an emitting system.

For the private case when the ith body exchanges heat only with one body and $k = 1$, each of the systems of algebraic equations reduces into one equation. So, for instance, Eq. (17-98) reduces into:

$$\frac{q_{neti}}{A_i} - \frac{R_i}{A_i} q_{net1} \varphi_{1i} = (E_{0i} - E_{01}) \overline{\varphi}_{1i} \tag{17-104}$$

Under steady-state conditions $q_{neti} = -q_{net1}$. With $\overline{\varphi}_{1i} = 1$ and $R_1 = 1 - A_1$ we find the unknown value:

$$q_{net} = \frac{E_{0i} - E_{0,\,1}}{\frac{1}{A_i} + \frac{1}{A_1} - 1}$$

which coincides with formula (17-3), previously derived by other methods.

17-8. Zoning Investigation Method for Radiation Heat Transfer

Section 17-7 was concerned with an emitting system of isothermal bodies. If some body is devoid of an isothermal surface, it is subdivided into smaller finite sections (zones) each of which can be considered as isothermal.

Then, the calculation formulas for the mean radiation fluxes shall be valid only for these sections.

If the temperatures of individual bodies vary substantially, their surfaces are subdivided into elementary (infinitesimal) areas and local rates of radiation heat transfer are determined at individual points, instead of the mean radiation fluxes.

Let us consider an enclosure consisting of nonisothermal grey bodies of known geometry and size (Fig. 17-12) with given distributions of temperature and optical properties. It is necessary to determine the various radiation fluxes.

Each of the surfaces of bodies i and k is subdivided into infinitesimal areas dF_i and dF_k within the limits of which temperature and optical properties may be assumed constant [17.10, 17.13]. Let the fixed area dF_i have the point M, and the current infinitesimal area dF_k—point N.

The elementary area dF_k emits radiation that is incident upon area dF_i at a rate $E_{out\,N_k}\, d\varphi_{M_i N_k}$ and the radiation flux from the entire surface F_k is equal to

$$E_{in\,M_i} = \int_{F_k} E_{out\,N_k}\, d\varphi_{M_i,\,N_k}$$

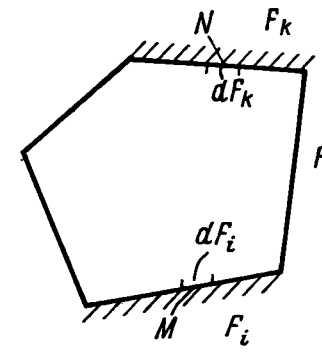

Fig. 17-12. Enclosure built-up of isothermal bodies

The radiation incident at point M_i on the area DF_i of body i from all surfaces of the system $k = 1, \ldots, n$ is represented by the sum of integrals

$$E_{in\,M_i} = \sum_{k=1}^{n} \int_{F_k} E_{out\,N_k}\, d\varphi_{M_i,\,N_k} \qquad (17\text{-}89')$$

where $d\varphi_{M_i,\,N_k}$ is the elementary view factor between area dF_k with point M and area dF_i with point M.

Thus, the final system of algebraic equations (17-89) is replaced here with a final system of integral equations the number of which corresponds to the number of elementary areas singled out on each system surface, and this system of integral equations makes it possible to find the distribution of local incident radiation fluxes.

Just as Eq. (17-89), relationship (17-89′) is one of the most important in the theory of radiation heat transfer.

Systems of integral equations for other kinds of radiation fluxes are derived in the same manner as the system of algebraic equations (Sec. 17-7). So, the system of integral equations for the local value of the density of outgoing radiation is obtained from (16-18) by substituting into it the value of E_{in} from Eq. (17-89′):

$$E_{out\,M_i} - R_i \sum_{k=1}^{n} \int_{F_k} E_{out\,N_k}\, d\varphi_{M_i,\,N_k} = E_{M_i} \qquad (17\text{-}94')$$

The local incident and net radiation fluxes can be calculated by means of the following systems of integral equations:

$$E_{in\,M_i} - \sum_{k=1}^{n} \int_{F_k} R_{N_k} E_{in\,N_k}\, d\varphi_{M_i,\,N_k} = \sum_{k=1}^{n} \int_{F_k} E_{0,\,N_k} A_{N_k}\, d\varphi_{M_i,\,N_k} \qquad (17\text{-}96')$$

$$\frac{E_{net\,M_i}}{A_{M_i}} - \sum_{k=1}^{n} \int_{F_k} \frac{R_{N_k}}{A_{N_k}} E_{net\,N_k}\, d\varphi_{M_i,\,N_k} = \sum_{k=1}^{n} \int_{F_k} (E_{0,\,M_i} - E_{0,\,N_k})\, d\varphi_{M_i,\,N_k} \qquad (17\text{-}98')$$

where $E_{0,\ M_i}$ and $E_{0\,N_k}$ are black body radiation fluxes respectively at the temperatures at points M_i and N_k.

The obtained systems of integral equations (17-94′), (17-96′) and (17-98′) conform with the systems of elementary equations (A), (B) and (C) derived for other kinds of radiation, with the only difference consisting in that in the latter equations the position of points M and N must refer to elementary (infinitesimal) zones dF_i and dF_k.

The investigation method for radiation heat transfer considered above is referred to as the *zoning* method of investigation.

17-9. Conditions for Mutual Conversion of Integral and Algebraic Equations of Radiation Heat Transfer

Under certain conditions integral equations may reduce into algebraic, i.e. they shall yield the same result. Let us find the conditions under which the integral and algebraic equations shall be in strict agreement with each other, which means that the corresponding local and mean radiation fluxes must be equal. For incident radiation, for instance, the above said is expressed by relationship

$$E_{ini} = E_{in\,M_i} \tag{17-105}$$

According to Eq. (17-89), the mean intensity of incident radiation

$$E_{ini} = \sum_{k=1}^{n} E_{outk}\bar{\varphi}_{k,\,i}$$

On the other hand, the mean E_{ini} can be found as the integral mean with the aid of Eq. (17-89′) from Sec. (17-8):

$$E_{ini} = \frac{1}{F_i}\int_{F_i} E_{in\,M_i}\,dF_i = \sum_{k=1}^{n}\int_{F_k} E_{out\,N_k}\,d\varphi_{i,\,N_k} \tag{17-106}$$

where $\varphi_{i,\ N_k}$ is the mean elementary view (configuration) factor determined by the relationship

$$d\varphi_{i,\,N_k} = \frac{1}{F_i}\int_{F_i} \varphi_{M_i,\,N_k}\,dF_i$$

Comparing expressions (17-89′) and (17-106) it is clear that the equality (17-105) exists when condition

$$d\varphi_{M_i,\,N_k} = d\varphi_{i,\,N_k} \quad (i,\ k = 1,\ \ldots,\ n) \tag{17-107}$$

is satisfied, i.e. when the elementary view factor is equal to its mean value.

The systems of integral equations describing other kinds of radiation also reduce into corresponding algebraic equations. If condition (17-107) is not observed strictly, the systems of algebraic equations shall but approximately describe processes of thermal radiation heat transfer.

Let us examine the conditions under which algebraic equations reduce into integral ones. As applied to incident radiation the mean heat flux is determined by the system of algebraic equations (17-89) derived in Sec. 17-7:

$$E_{ini} = \sum_{k=1}^{n} E_{outk}\bar{\varphi}_{k,\,i} \quad k = 1,\ \ldots,\ n$$

where the subscript k indicates individual bodies or zones of the emitting system which are at constant temperatures and have invariable optical properties. In the limiting case it is assumed that the number of zones $n \to \infty$, and the surfaces of zones contract into points and $F_k \to 0$.

The mean radiation fluxes then acquire real values at individual points; the mean view factors between zones turn into elementary view factors; summation over individual zones is replaced with integration over the entire surface F of the emitting system. The final system of algebraic equations (17-89′) reduces into an integral equation describing the continuous change of the intensity of incident radiation, depending on the position of point M on the surface:

$$E_{in\,M} = \int_F E_{out\,N}\, d\varphi_{M,\,N} \qquad (17\text{-}89'')$$

In the limiting case the system of algebraic equations (17-94), (17-98) and others, describing different kinds of radiation, reduces into corresponding integral equations that are strict and exact.

17-10. Integral Investigation Method for Radiation Heat Transfer

The integral method is applied to investigate intricate problems in thermal radiation heat transfer, when the initial system is of complex geometrical shape and the distribution of temperature and optical properties is arbitrary over the surface of the system.

As applied to the conditions indicated above, an exact solution of problems of this kind is based on integral equations, whence the name of the investigation method. In Sec. 17-9 integral equations were derived from algebraic.

Integral equations can also be derived independently making use of the fundamental Eq. (16-12) of radiation heat transfer. As applied to incident radiation, expressed in terms of radiosity, this equation acquires the appearance:

$$d^2Q_{in\,N} = I_N\, d\omega_N\, dF_N \cos\psi_N$$

Using Lambert's law and also Eqs. (16-56) and (16-58), we get

$$d^2Q_{in\,M} = E_N\, dF_N\, d\varphi_{N,\,M}$$

where, as before, M and N are respectively the fixed and current points of the surface F of the system considered (Fig. 17-12).

The radiation flux leaving area dF_N that is incident upon area dF_M is equal to

$$dE_{in\,M} = \frac{d^2Q_{in\,M}}{dF_M} = E_N\, d\varphi_{N,\,M}\, \frac{dF_N}{dF_M}$$

Let us use the property of reciprocity expressed by Eq. (17-75). Then, upon integration, we obtain:

$$E_{in\,M} = \int_F E_N\, d\varphi_{M,\,N} \qquad (17\text{-}108)$$

For a grey body the outgoing radiation will replace the emitted radiation and relationship (17-108) reduces into Eq. (17-89″).

Relationship (17-89″) makes it possible to derive integral equations for other radiation fluxes.

The method of deriving integral equations is similar to that used to derive algebraic equations (Sec. 17-7). So, for instance, to derive the integral equation describing the intensity of outgoing radiation, Eq. (16-18) is used as above, but

instead of Eq. (17-89), describing incident radiation, Eq. (17-89″) is taken. The integral equation describing the distribution of E_{out} over the surface of an emitting system $F = \sum_{i=1}^{n} F_i$ has the following appearance:

$$E_{out\,M} - R_M \int_F E_{out\,N}\, d\varphi_{M,N} = E_M \qquad (17\text{-}94'')$$

The integral equation for the density of incident radiation is obtained by substituting in Eq. (17-89″) the found value of $E_{out\,M}$

$$E_{in\,M} - \int_F R_N E_{in\,N}\, d\varphi_{M,N} = \int_F A_N E_{0N}\, d\varphi_{M,N} \qquad (17\text{-}96'')$$

where E_{0N} is the radiation density of a black body at the temperature at point N on the emitting surface.

Thus, instead of the systems of algebraic equations we obtained integral equations describing the continuous distribution of all kinds of radiation fluxes over an emitting surface.

Integral equations for net, reflected and absorbed radiation can be obtained in the same way as the system of equations (17-98), (17-100), (17-102), and, therefore they are presented below in the final form, omitting intermediate computation:

$$\frac{q_{net\,M}}{A_M} - \int_F \frac{R_N}{A_N} q_{net\,N}\, d\varphi_{M,N} = \int_F (E_{0M} - E_{0N})\, d\varphi_{M,N} \qquad (17\text{-}98'')$$

$$E_{rM} - R_M \int_F E_{rN}\, d\varphi_{M,N} = R_M \int_F E_N\, d\varphi_{M,N} \qquad (17\text{-}100'')$$

$$E_{abM} - A_M \int_F E_{abN}\, d\varphi_{M,N} = A_M \int_F E_N\, d\varphi_{M,N} \qquad (17\text{-}102'')$$

In the case considered, the elementary equations (A), (B), (C), (D), (E) corresponding to the integral equations derived above (Sec. 17-7) are applied to individual points M on the surface of the emitting system.

Each of the systems, consisting of any one of the integral equations (17-94″), (17-96″), (17-98″), (17-100″) or (17-102″) and a corresponding set of elementary equations (A) through (E), worked out for other kinds of radiation, is equivalent to the others and is characterized by its method of computation.

17-11. Resolvent Investigation Method for Radiation Heat Transfer

Exact analytical solutions of integral equations (Sec. 17-10) were obtained only in private problems [17.10]. In general various approximate solutions are practised [17.10, 17.12, 17.13] including the solution obtained by the iterative method. Let us consider this method, applying it to an arbitrary geometric enclosure composed of grey bodies with a given distribution of temperature and of optical properties over the boundary surface of the enclosure. The problem is to find various radiation fluxes.

Under these conditions the radiation leaving the surface (also referred to as the outgoing radiation) is expressed by the relationship

$$E_{out\,M} = E_M + R_M \int_F E_{out\,N}\, d\varphi_{M,N} \qquad (17\text{-}94'')$$

When the iterative method is applied the result of each successive approximation is used as the initial value for the next approximation. An arbitrary number of approximations is made and in the limit they lead to a solution of the accuracy required. As an initial zero approximation let us assume for point M (Fig. 17-13) that $E^{(0)}_{out} = 0$.

Similarly for current points N $E^{(0)}_{out\,N} = 0$ is true. Then, to solve the integral equation (17-94″), $E_{out\,N}$ must be replaced with its value, equal to zero, resulting in that

$$E^{(1)}_{out\,M} = E_M \qquad (17\text{-}109)$$

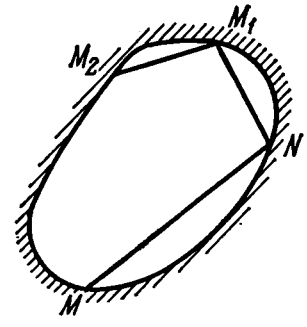

Fig. 17-13. Iterative solution

Identical relationships are valid for all current points N.

Thus, only emitted radiation is accounted for a first approximation, and no account is taken of the intermediate multiple reflections. Hence, this is a rough approximation. Let us substitute into the integral equation the result of the first approximation and find that

$$E^{(2)}_{out\,M} = E_M + R_M \int_F E_N\, d\varphi_{M,N} = E_M + E^{(1)}_{rM} \qquad (17\text{-}110)$$

where, in addition to the emitted, a single reflection is also accounted for.

Substituting the solution (17-110), obtained as a second approximation, into integral equation (17-94″), we find the solution for the third approximation, then for the fourth and successive approximations. With account taken of n intermediate reflections the solution for the $(n + 1)$ approximation will be represented by the power series:

$$E^{(n+1)}_{out\,M} = E_M + R_M \int_F E_N\, d\varphi_{M,N} + R_M \int_F \int_F R_{M1} E_{M1}\, d\varphi_{M,M1}\, d\varphi_{M1,N} + \ldots$$

$$\ldots + R_M \underbrace{\int_F \ldots \int_F}_{n} R_{M1} R_{M2} \ldots R_{M(n-1)}\, d\varphi_{M\,M1}\, d\varphi_{M1,M2} \ldots d\varphi_{M(n-1),N} \qquad (17\text{-}111)$$

The first term in Eq. (17-111) expresses emitted radiation, the other items—the radiation reflected from unit area at point M. Hence, the relationship can be presented in the following abbreviated form

$$E_{out\,M} = E_M + \sum_{i=1}^{n} E^{(i)}_{rM} \qquad (17\text{-}112)$$

In the general case an infinite number of reflections ($i \to \infty$) may take place. Solution (17-111) is presented in the form:

$$E_{out\,M} = E_M + R_M \int_F E_N\, d\Phi_{M,N} \qquad (17\text{-}113)$$

where the term $d\Phi_{M,\,N}$ is known as the *exchange factor* expressed by the infinite series

$$d\Phi_{M,\,N} = d\varphi_{M,\,N} + \sum_{i=1}^{n} d\varphi_{(M,\,N)i} \tag{17-114}$$

The quantities $d\varphi_{(M,\,N)i}$ characterize the view factor between infinitesimal elements with account taken of one, two, . . ., n intermediate reflections. Relationship (17-111) can be used to express the view factors between infinitesimal elements with account taken of one, two, . . ., n intermediate reflections with the aid of relationships

$$\begin{aligned}
d\varphi_{(M,\,N)1} &= \int_F R_{M1}\, d\varphi_{M,\,M1}\, d\varphi_{M1,\,N} \\
d\varphi_{(M,\,N)2} &= \int_F \int_F R_{M1} R_{M2}\, d\varphi_{M,\,M1}\, d\varphi_{M1,\,M2}\, d\varphi_{M2,\,N} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
d\varphi_{(M,\,N)n} &= \underbrace{\int_F \int_F}_{n} R_{M1} R_{M2} \cdots R_{M(n-1)}\, d\varphi_{M,\,M1}\, d\varphi_{M1,\,M2} \cdots d\varphi_{M(n-1),\,N}
\end{aligned} \tag{17-115}$$

where $R_{M1}, \ldots, R_{M(n-1)}$ are the reflectivities at points $M_1, \ldots, M_{(n-1)}$.

Thus, in distinction to $d\varphi_{M,\,N}$ the exchange factor $d\Phi_{M,\,N}$ accounts for multiple reflections and is an optical and geometrical characteristic since, in addition to the geometry of the system, it accounts for its reflectivity properties.

The exchange factor can be expressed in terms of the radiation resolvent:

$$d\Phi_{M,\,N} = \Gamma_{M,\,N}\, dF_N \tag{17-116}$$

and the view factor through the kernel:

$$d\varphi_{M,\,N} = K_{M,\,N}\, dF_N \tag{17-117}$$

where $\Gamma_{M,N}$ and $K_{M,N}$ are the resolvent and kernel of the integral equation.

The radiation resolvent and the kernel have a certain physical meaning. The resolvent $\Gamma_{M,N}$ is the ratio of the elementary radiation flux from area dF_N that is incident upon unit area at point M, with account taken of the multiple reflections from the system boundary, to the elementary hemispherical radiation emitted from area dF_N. In other words, $\Gamma_{M,\,N}$ is the ratio of the elementary exchange factor between area dF_N and area dF_M to area dF_N [Eq. (17-116)]. By analogy, and in accordance with Eq. (17-117), the kernel of the integral equation, $K_{M,N}$, is the ratio of the elementary view factor between areas dF_N and dF_M to the area dF_N.

Let us present now solution (17-113) in the resolvent form:

$$E_{out\,M} = E_M + R_M \int_F E_N \Gamma_{M,\,N}\, dF_N \tag{17-118}$$

Comparing the initial integral equation for the radiation leaving the surface, Eq. (17-94″), with its solutions presented by Eqs. (17-113) and (17-118), it can be seen that the latter equation contains under the sign of the integral the function E_N, characterizing emitted radiation, instead of the unknown function $E_{out\,N}$ expressing outgoing radiation, and account of the multiple reflections

is taken now by the exchange factor and radiation resolvent. Consequently, all the intricacies involved in the problem and its solution are concentrated in determining the radiation resolvent.

Solutions of integral equations in terms of the radiation resolvent can also be obtained for other kinds of radiation. So, for incident radiation the solution of Eq. (17-89″) has the appearance:

$$E_{in\,M} = \int_F E_N \Gamma_{M,\,N}\, dF_N \tag{17-119}$$

Then, Eqs. (17-119) and (16-19) are used to find a solution for net radiation in the following form:

$$E_{net\,M} = A_M \int_F E_N \Gamma_{M,\,N}\, dF_N - E_M \tag{17-120}$$

The radiation resolvent can be presented in the form of the following series, using for this purpose Eq. (17-114):

$$\Gamma_{M,\,N} = K_{M,\,N} + \sum_{i=1}^{n} K_{(M,\,N)i} \tag{17-121}$$

Relationship (17-121) in its turn can be reduced into the integral equations for the radiation resolvent [17.12]:

$$\Gamma_{M,\,N} = K_{M,\,N} + \int_F R_P \Gamma_{P,\,N} K_{M,\,P}\, dF_P \tag{17-122}$$

$$\Gamma_{M,\,N} = K_{M,\,N} + \int_F R_P K_{P,\,N} \Gamma_{M,\,P}\, dF_P \tag{17-123}$$

which are identical and can be equally applied.

Consequently, the solution of integral equations (17-94″), (17-89″), (17-96″) and others derived for various kinds of thermal radiation resolves itself in solving one of the integral equations (17-122) or (17-123) for the radiation resolvent, and this simplifies the task materially.

The considered method has been elaborated by Yu.A. Surinov [17.13] and it was given the name of the resolvent method.

17-12. Resolvent Zoning Method

This method is a variety of the zoning method considered in Sec. 17-8. Let us consider the essence of this method, applying it to the same problem [17.12-17.13]. The local values of outgoing radiation were described by the following system of integral equations:

$$E_{out\,M_i} = E_{M_i} + R_i \sum_{k=1}^{n} \int_{F_k} E_{out\,N_k}\, d\varphi_{M_i,\,N_k} \tag{17-124}$$

The solution of this system of equations is expressed in terms of emitted radiation and the resolvent or the exchange factor in the following form

$$E_{out\,M_i} = E_{M_i} + R_i \sum_{k=1}^{n} \int_{F_k} E_{N_k}\, d\Phi_{M_i,\,N_k} \tag{17-125}$$

The local value of the exchange factor between point M_i and the zone F_k ($k = 1, \ldots, n$) is expressed by the integral:

$$\Phi_{M_i, F_k} = \int_{F_k} d\Phi_{M_i, N_k} = \int_{F_k} \Gamma_{M_i, N_k} dF_{N_k} \tag{17-126}$$

The system of integral equations (17-125) then reduces into a finite system of algebraic equations:

$$E_{out\, M_i} = E_{M_i} + R_i \sum_{k=1}^{n} E_k \Phi_{M_i, F_k} \tag{17-127}$$

where $E_k = E_{N_k}$. Let us find the mean value of outgoing radiation for individual zones, integrating for this purpose Eq. (17-127) over the area of the zones F_i, then dividing the obtained result by the value of this area:

$$E_{out\, i} = E_i + R_i \sum_{k=1}^{n} E_k \Phi_{i,k}, \quad i = 1, \ldots, n, \tag{17-128}$$

$$E_{out\, i} = \frac{1}{F_i} \int_{F_i} E_{out\, M_i} dF_{M_i}; \quad \Phi_{i,k} = \frac{1}{F_i} \int_{F_i} \Phi_{M_i F_k} dF_{M_i} \tag{17-129}$$

Thus, to calculate $E_{out\, i}$, it is first necessary to calculate all the mean exchange factors $\Phi_{i,k}$.

Let us find a system of algebraic equations for these exchange factors, making use of the integral equation (17-123) derived for the radiation resolvent. As applied to zonal points M, N and P, corresponding to i, k and j (Fig. 17-12) the integral equation reduces in the finite system of equations for the resolvent:

$$\Gamma_{M_i, N_k} = K_{M_i, N_k} + \sum_{j=1}^{n} R_j \int_{F_i} \Gamma_{M_i, P_j} K_{P_j, N_k} dF_{P_j} \tag{17-130}$$

Let us integrate Eq. (17-130) over the surface F_k of zone k. Then, with account taken of relationship (17-126), we obtain:

$$\Phi_{M_i, F_k} = \varphi_{M_i, F_k} + \sum_{j=1}^{n} R_j \int_{F_j} \Gamma_{M_i, P_j} \varphi_{P_j, F_k} dF_{P_j} \tag{17-131}$$

where the local exchange factor

$$\Phi_{M_i, F_k} = \int_{F_k} \Gamma_{M_i, N_k} dF_{N_k} \tag{17-132}$$

the local view factors between points M and P and surface of zone k are respectively equal to:

$$\varphi_{M_i, F_k} = \int_{F_k} K_{M_i, N_k} dF_{N_k}; \quad \varphi_{P_j, F_k} = \int_{F_k} K_{P_j, N_k} dF_{N_k} \tag{17-133}$$

Now let us assume that the local and mean view factors are equal (the condition under which the integral equations reduce into algebraic equations stipulated in Sec. 17-9):

$$\varphi_{P_j, F_k} = \varphi_{j,k} = \frac{1}{F_j} \int_{F_j} \varphi_{P_j, F_k} dF_{P_j} \tag{17-134}$$

With account taken of Eq. (17-134) relationship (17-131) reduces into

$$\Phi_{M_i, F_k} = \varphi_{M_i, F_k} + \sum_{j=1}^{n} R_j \Phi_{M_i, F_j} \varphi_{j, k} \tag{17-135}$$

since

$$\int_{F_j} \Gamma_{M_i, P_j} d\Gamma_{P_j} = \Phi_{M_i, F_j} \tag{17-136}$$

Let us find the integral mean exchange factors:

$$\Phi_{i, k} = \frac{1}{F_i} \int_{F_i} \Phi_{M_i, F_k} dF_{M_i} = \varphi_{i, k} + \sum_{j=1}^{n} R_j \varphi_{j, k} \Phi_{i, j} \tag{17-137}$$

If we depart from relationship (17-122) for the radiation resolvent, and not from Eq. (17-123), then, instead of Eq. (17-137), we obtain:

$$\Phi_{i, k} = \varphi_{i, k} + \sum_{j=1}^{n} R_j \Phi_{j, k} \Phi_{i, j} \tag{17-138}$$

Thus, we obtained a system of algebraic equations (17-137) and (17-138) permitting calculation of the mean exchange factors, if the reflectivities of zones are known, and if the mean view, or configuration, factors were determined beforehand. Just as the latter, the exchange factors satisfy the enclosure and reciprocity properties described in Sec. 17-6.

Having found $\Phi_{i, k}$ we proceed to determine the outgoing radiation flux $E_{out\ i}$ for each zone $i = i, \ldots, n$ from Eq. (17-128). If necessary, the local radiation fluxes are calculated with the aid of the systems of equations (17-127) and (17-131) for some points of interest on the surface of zones. Other kinds of radiation fluxes are determined with the aid of relationships (A), Sec. 17-7, following from the classification of thermal radiation.

A similar calculation method is applied, if we depart from any other kind of radiation, incident, for instance.

The system of integral equations describing the incident radiation flux is expressed by the relationship

$$E_{in\ M_i} = \sum_{k=1}^{n} \int_{F_k} E_{out\ N_k} d\varphi_{M_i, N_k}$$

The solution of this system is presented by algebraic equations:

$$E_{in\ M_i} = \sum_{k=1}^{n} E_k \Phi_{M_i, F_k}, \quad i = 1, \ldots, n \tag{17-139}$$

$$E_{in\ i} = \frac{1}{F_i} \int_{F_i} E_{in\ M_i} dF_{M_i} = \sum_{k=1}^{n} E_k \Phi_{i, k} \tag{17-140}$$

The other radiation fluxes are found from system (C) described in Sec. 17-7 of this book.

Let us consider the system of algebraic equations derived for the density of net radiation:

$$E_{net\ i} = A_i \sum_{k=1}^{n} A_k \Phi_{i, k} (E_{0k} - E_{0i}) \tag{17-141}$$

where, as above, E_{0k} and E_{0i} represent black body radiation at the temperatures of the zones k and i. Knowing the zone area, we determine the total radiation fluxes.

In illustration, let us consider a flat parallel system built-up of grey bodies (Fig. 17-1). The exchange factors are determined with the aid of the system of equations (17-138), yielding

$$\Phi_{1k} = \varphi_{1k} + R_2\Phi_{2k}\varphi_{1,2}; \quad \Phi_{2k} = \varphi_{2k} + R_1\Phi_{1k}\varphi_{2,1}$$

Solving these equations together, we can find the local exchange factors:

$$\Phi_{1k} = \frac{\varphi_{1k} + R_2\varphi_{2,1}\varphi_{2k}}{1 - R_1R_2\varphi_{1,2}\varphi_{2,1}}; \quad \Phi_{2k} = \frac{\varphi_{2k} + R_1\varphi_{2,1}\varphi_{1k}}{1 - R_1R_2\varphi_{1,2}\varphi_{2,1}} \qquad (17\text{-}142)$$

For the case considered $\varphi_{1,1} = \varphi_{2,2} = 0$; $\varphi_{1,2} = \varphi_{2,1} = 1$. Hence,

$$\Phi_{1,2} = \Phi_{2,1} = \frac{1}{1 - R_1R_2}; \quad R = 1 - A$$

According to Eq. (17-141) the net radiation flux will be:

$$q_{net\,1} = A_1A_2(E_{0,1} - E_{0,2})\Phi_{1,2} = \frac{A_1A_2}{1 - R_1R_2}\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right]c_0$$

This result coincides with relationship (17-9).

17-13. Radiation Heat Transfer in the Presence of Reflecting Surfaces

In the above considered problems in radiation heat transfer we departed from the condition of ideal diffuse reflection with the directional radiosity being equally independent of the direction of incident radiation. In addition to ideally diffuse reflection, diffuse reflection is distinguished without a strongly pronounced direction and also directional (specular) reflection.

With specular reflection the radiant energy reflected from the surface of the body in all directions is equal to zero, except in one direction corresponding to the specular angle, equal to the incident angle (Fig. 17-14).

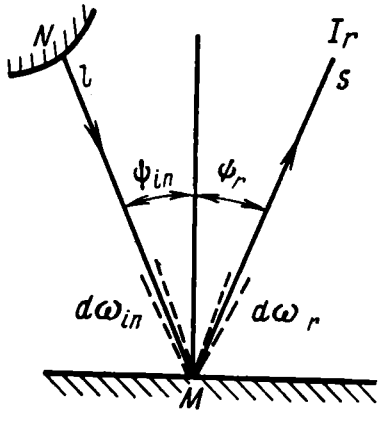

Fig. 17-14. Reflection of a specular surface: ψ_{in} and ψ_r – incident and specular angles

The radiosity of a reflected elementary beam depends on that of the incident elementary beam and on the reflectivity properties of the surface. The surface of a body may be closer to a diffuse or specular one, depending on its roughness (surface irregularities).

Roughness manifests itself in a different manner, depending on wavelength. A surface may happen to be rough for some wavelengths and smooth for others. With increasing wavelength the surface loses its roughness to a greater extent. A surface also loses its roughness with an increasing incident angle. That is why the state of a surface is characterized by optical roughness

$$\delta_{opt} = f(\delta_r, \lambda, \cos\psi_{in})$$

With a roughness height δ_r, corresponding to the condition

$$\delta_r \cos \psi_{in} \leqslant \lambda$$

the surface is considered to be optically smooth and has specular reflection, otherwise the surface is said to be optically rough and its reflection is diffuse. A private case of diffuse reflection is the ideally diffuse (isotropic) radiation characterized by equal directional radiosity.

The space distribution of the radiosity of reflected radiation is determined by dimensionless quantities including the *space function* and the *radiosity factor*.

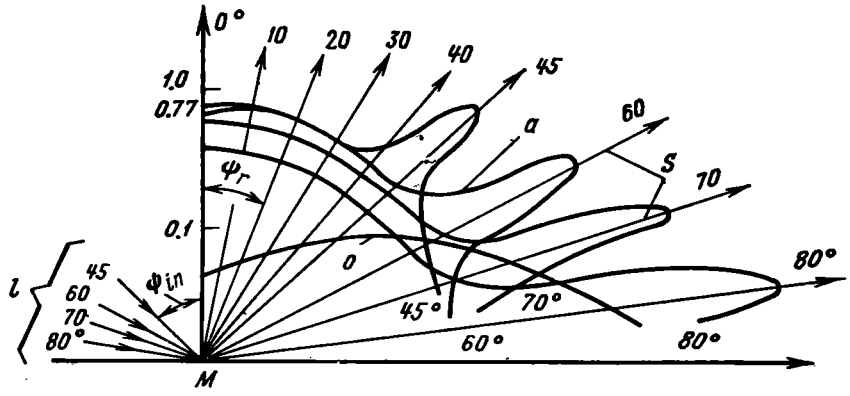

Fig. 17-15. Space factor of paints and varnishes versus incident and reflected angles: *a*— white brilliant surface; *b*— black dull surface

By the space function is meant the ratio of the directional radiosity of the investigated surface to the radiosity of a surface with ideally diffuse reflection (directional mean radiosity)

$$B_M(s, l) = \frac{dI_r}{\frac{1}{\pi} dE_r}; \quad E_r = E_{in} R_M(s, l) \tag{17-143}$$

By the radiosity factor is meant the ratio of the directional radiosity of the investigated surface to the radiosity of a surface characterized by ideally diffuse reflection and a reflectivity, equal to unity:

$$r_M(s, l) = \frac{dI_r}{\frac{1}{\pi} dE_{in}}; \quad E_r = E_{in} R_M(s, l) = E_{in} \tag{17-144}$$

These definitions pertain to individual points on the surface and to individual directions of incidence (l) and reflection (s) of radiant energy. They do not account for the effect of scattering by wavelength, since this kind of scattering is small compared with the effect of directional scattering. Here $R_M(s, l)$ is the *directional reflectivity*. The radiosity factor (space function) gives a visual representation of the nature of reflection. The space functions for paints and varnishes are given in Fig. 17-15 [17.14]. It follows from the illustration that brilliant surfaces are characterized by a narrow and elongated peak for the maximum specular component (curves *a*), dull surfaces have no peak (curve *b*).

Exact solutions of problems in radiation heat transfer with arbitrary reflection are based on the integral equations of radiation. However the integral equations derived in Sec. 17-10 are not valid in this case, since they were derived assuming that reflectivity is independent of direction.

The emissive power at point M is represented by the following integral equation derived for a surface with arbitrary reflection characteristics (Fig. 17-14) [17.15]:

$$E_{out\,M} = E_M + \int_{2\pi} R_M(l)\, I_{out\,N}(l) \cos\psi_{in}\, d\omega_{in} \qquad (17\text{-}145)$$

where, with a fixed direction of incident radiation l, the reflectivity at point M on the surface of the body is

$$R_M(l) = \frac{E_{r\,M}(s)}{E_{in\,M}(l)} = \frac{1}{\pi} \int_{2\pi} r_M(l, s) \cos\psi_r\, d\omega_r$$

For ideally diffuse reflection integral equation (17-145) reduces into Eq. (17-94) derived in Sec. 17-8, with account taken of relationships (16-56) and (17-58).

The integral equations for other kinds of thermal radiation also acquire a generalized appearance.

17-14. Methods of Determination of View Factors

(a) *Analytical Method*

The view (angle, configuration) factor is the main calculation quantity. It can be found by analytical, graphical-analytical methods and also by the method of angle factor algebra. Experimental methods include methods of electric circuit analogy and other methods of analogy [17.6, 17.10, 17.16-17.18].

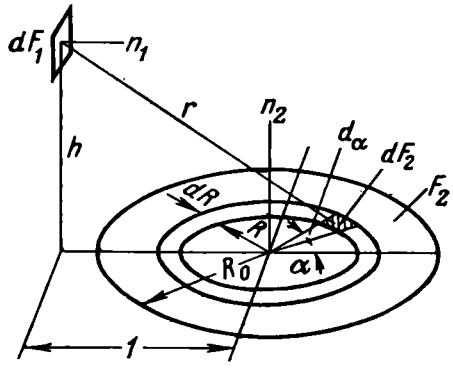

Fig. 17-16. System consisting of a plane (index *1*) and a circle (index *2*)

The analytical method is based on direct integration of the mathematical expression for an elementary view factor represented by Eq. (17-58). As an illustration, let us consider the emitting system given in Fig. 17-16, assuming that the bodies have diffuse reflection. Inasmuch as the view factor is determined by the magnitude of the angles between the emitting surface and normals, the scale of the system configuration can be changed so that one of the corresponding distances is equal to unity. Let us find the values of the quantities in relationship (17-58):

$$d\varphi_{2,1} = \frac{\cos\psi_1 \cos\psi_2}{\pi r^2}\, dF_2$$

The quantities in this expression are equal to:

$$r^2 = h^2 + 1 + R^2 + 2R\cos\alpha; \quad \cos\psi_1 = \frac{1 + R\cos\alpha}{r}$$

$$\cos\psi_2 = \frac{h}{r}; \quad dF_2 = R\, dR\, d\alpha$$

Then,

The local view factor between elementary area dF_1 and area of a circle F_2 is found by integrating relationship (17-146):

$$\varphi_{2,1}=\frac{h}{\pi}\int_0^{R_0} R\,dR\int_0^{\pi}\frac{1+R\cos\alpha}{(h^2+1+R^2+2R\cos\alpha)^2}\,d\alpha$$

Upon integration, we obtain:

$$\varphi_{2,1}=\frac{h}{2}\left[\frac{h^2+R_0^2+1}{\sqrt{(h^2+R_0^2+1)^2-4R_0^2}}-1\right]$$

According to this relationship, the local view factor can be found, if the radius R_0 of the circle and the distance h to the elementary area dF_1 are known.

Thus, the analytical method of solving integral equations involves direct integration of relationship (17-58) which is the mathematical description of the view factor.

(*b*) *Grapho-analytical* (*Unit-sphere*) *Method*

The grapho-analytical method of determining the view factor differs from the analytical one in that numerical integration is replaced by graphical construction. Let us consider the essence of this method. For that purpose we take an

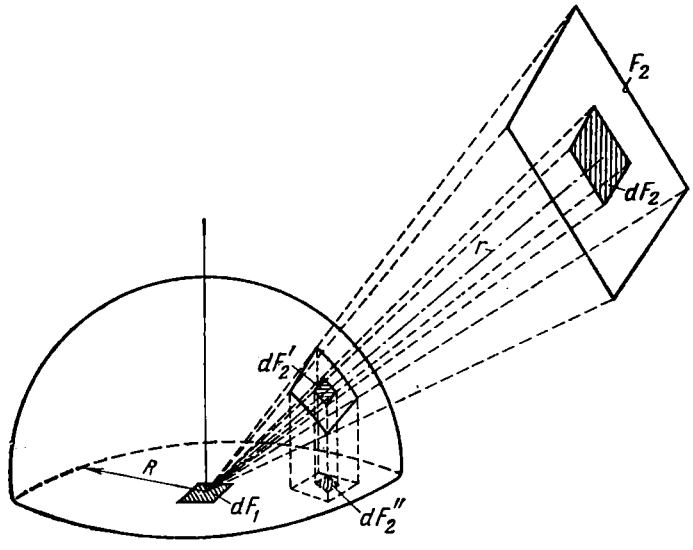

Fig. 17-17. Determination of the view factor by the grapho-analytical (unit-sphere) method

elementary area dF_1 on the surface of the emitting body *1* (Fig. 17-17), draw a contour intercepting the beams from body *1* that are incident upon body *2*, then we draw a hemisphere of radius R from its centre. The hemisphere leaves a circular trace of the same radius R on the plane of element dF_1. In addition, the radiation from the area dF_1 that is incident upon the surface of the second body will intercept a corresponding area dF_2', the projection of which will be expressed by the area dF_2''. Then, the local view factor $\varphi_{2,1}$ between area dF_1 and surface F_2 will be equal to the ratio of area dF_2'' to the area of a circle with a radius R. To prove this assumption, let us represent the elementary view factor

not by Eq. (17-58), but by the relationship for the solid angle

$$d\varphi_{2,1}=\frac{d\omega_1}{\pi}\cos\psi_1 \tag{17-147}$$

The value $d\omega_1\cos\psi_1$ is the projection of the elementary solid angle $d\omega_1$ on the plane of the emitting area dF_1. The local magnitude of the view factor is expressed by integral (17-147). The integral here is the sum of the projections of all elementary solid angles $d\omega_1$ corresponding to the entire surface F_2 on the plane of area dF_1 which was to be proved.

A number of mechanical and optical integrators have been developed, based on the grapho-analytical method [17.18].

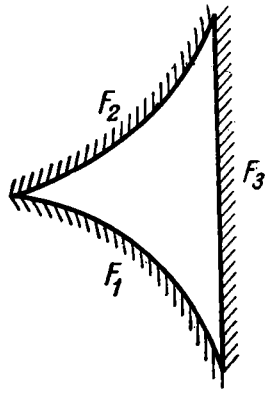

Fig. 17-18. Enclosure consisting of three convex (flat) solids with surfaces F_1, F_2 and F_3

(c) View Factor Algebra Method

When the method of view-factor algebra is applied integration is replaced by simple algebraic operations. The method is based on the geometrical properties of radiation fluxes (see Sec. 17-6).

Consider an enclosure composed of bodies, forming a cross section as shown in Fig. 17-18. The longitudinal dimensions of the bodies are assumed to be large, compared with the cross dimensions. The dimensions of the bodies are given. There is no self-irradiation ($\bar{\varphi}_{1,1}=\bar{\varphi}_{2,2}=\bar{\varphi}_{3,3}=0$). It is necessary to find the mean surface-surface direct exchange areas and the radiation coefficients.

Let us make use of the enclosure property described by Eq. (17-79). Then, for bodies with surfaces F_1, F_2 and F_3 the following relationships shall be valid:

$$\left.\begin{aligned}\bar{H}_{1,2}+\bar{H}_{1,3}&=F_1\\ \bar{H}_{2,1}+\bar{H}_{2,3}&=F_2\\ \bar{H}_{3,1}+\bar{H}_{3,2}&=F_3\end{aligned}\right\} \tag{17-148}$$

The mutuality property of radiant fluxes (17-76) makes it possible to reduce the number of unknowns in the last relationships from six to three, since

$$\bar{H}_{1,2}=\bar{H}_{2,1};\quad \bar{H}_{2,3}=\bar{H}_{3,2};\quad \bar{H}_{3,1}=\bar{H}_{1,3} \tag{17-149}$$

Then, upon summation of the left and right sides of Eq. (17-148)

$$\bar{H}_{1,2}+\bar{H}_{1,3}+\bar{H}_{2,3}=\frac{1}{2}(F_1+F_2+F_3) \tag{17-150}$$

Subtracting the equalities of (17-148) in turn from (17-150), we find the relationships determining the unknowns of the mutual emitting surfaces:

$$\left.\begin{aligned}\bar{H}_{2,3}&=\frac{F_2+F_3-F_1}{2}\\ \bar{H}_{1,3}&=\frac{F_1+F_3-F_2}{2}\\ \bar{H}_{1,2}&=\frac{F_1+F_2-F_3}{2}\end{aligned}\right\} \tag{17-151}$$

Making use of Eqs. (17-71) and (17-149), we obtain expressions for the corresponding mean geometrical view factors:

$$\left.\begin{aligned} \bar{\varphi}_{2,3} &= \frac{F_2+F_3-F_1}{2F_2}; \quad \bar{\varphi}_{1,3} = \frac{F_1+F_3-F_2}{2F_1}; \\ \bar{\varphi}_{1,2} &= \frac{F_1+F_2-F_3}{2F_1}; \quad \bar{\varphi}_{3,2} = \frac{F_3+F_2-F_1}{2F_3}; \\ \bar{\varphi}_{3,1} &= \frac{F_3+F_1-F_2}{2F_3}; \quad \bar{\varphi}_{2,1} = \frac{F_2+F_1-F_3}{2F_2} \end{aligned}\right\} \qquad (17\text{-}152)$$

When applied to systems of intricate geometry, various curvilinear outlines of cross-sections are replaced with simpler outlines made up of minimum-length lines [17.13, 17.19] (stretched tight strings) shown by the dotted lines in Fig. 17-19. This is in accordance with the coincidence properties of radiation fluxes (Sec. 17-6).

Let us find the mean mutual radiating surface and the mean view factor between two bodies of surface F_1 and F_2. Body BB' "sees" body $BCDS$, body SS' and body $NMLB'$. It does not exchange heat with body $S'PN$. Then, in accordance with Eq. (17-79) we car write:

$$\bar{H}_{1,\,BCDS} + \bar{H}_{1,2} + \bar{H}_{1,\,B'LMN} = F_1'$$

from which the unknown value of the mutual radiation surface will be expressed by the relationship

$$\bar{H}_{1,2} = F_1' - \bar{H}_{1,\,BCDS} - \bar{H}_{1,\,B'LMN} \qquad (17\text{-}153)$$

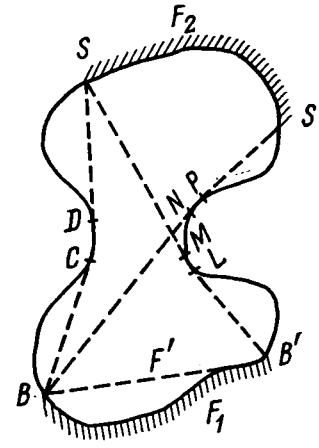

Fig. 17-19. Algebraic solution of radiant heat exchange in complex enclosures

where F_1' is the surface of radiating body *1*, determined by minimum-length lines.

The mutual radiation surface $\bar{H}_{1\,BCDS}$ can be found from the enclosure consisting of lines BB', $BCDS$ and $SMLB'$:

$$\bar{H}_{1,\,BCDS} = \frac{F_1' + BCDS - B'LMS}{2}$$

The enclosure system formed by lines BB', BN, and $B'LMN$ is found from

$$\bar{H}_{1,\,B'LMN} = \frac{F_1' + B'LMN - BN}{2}$$

Substituting the last two expressions in Eq. (17-153), we obtain

$$\bar{H}_{1,2} = F_1' - \frac{F_1' + BCDS - B'LMS}{2} - \frac{F_1' - B'LMN + BN}{2}$$

or

$$\bar{H}_{1,2} = \frac{B'LMS + BNPS'}{2} - \frac{BCDS + B'LMNPS'}{2} \qquad (17\text{-}154)$$

The line PS' is introduced in both half-sums of Eq. (17-154).

From the last relationship it follows that the mutual radiating surface $\bar{H}_{1,2}$ of the two surfaces is equal to half the length of the crossed strings stretched

between the ends of the lines representing them, less half the sum of the lengths of uncrossed strings similarly stretched between the ends of the surfaces.

To determine the mean view factor, we introduce the value of surface F_1' in the denominator of Eq. (17-154).

Let us use the above presented determination method for the view factor, applying it to a system comprising large flat parallel plates of equal width and comparatively great length. The width a (Fig. 17-20) and the distance between

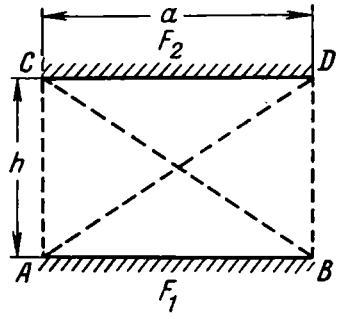

Fig 17-20. System of two flat parallel strips

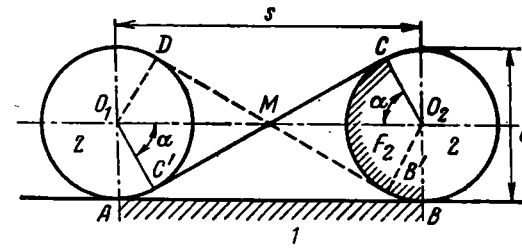

Fig. 17-21. System composed of radiating surface *1* and a bank of tubes *2*

the plates h are given. It is necessary to determine $\bar{H}_{1,2}$ and $\bar{\varphi}_{1,2}$. Let us introduce fictitious surfaces AC and BD. Then, we obtain an enclosure consisting of four bodies. The enclosure property will be expressed by the relationship

$$\bar{\varphi}_{1,2}+\bar{\varphi}_{1AC}+\bar{\varphi}_{1BD}=1; \quad \bar{\varphi}_{1,1}=0$$

from which

$$\bar{\varphi}_{1,2}=1-\bar{\varphi}_{1AC}-\bar{\varphi}_{1BD} \tag{17-155}$$

In accordance with Eq. (17-152),

$$\bar{\varphi}_{1AC}=\frac{F_1+AC-BC}{2F_1}; \quad \bar{\varphi}_{1BD}=\frac{F_1+BD-AD}{2F_1}$$

In these relationships: $AB = a$; $AC = BD = h$; $AD = BC = \sqrt{a^2+h^2}$.

Substitution of the found values in Eq. (17-155) gives the equation for the mean view factor and surface-surface direct exchange area:

$$\bar{\varphi}_{1,2}=\frac{2\sqrt{a^2+h^2}}{2a}-\frac{2h}{2a}=\sqrt{1+\left(\frac{h}{a}\right)^2}-\frac{h}{a}; \quad \bar{H}_{1,2}=\sqrt{a^2+h^2}-h \tag{17-156}$$

Using the principle of Eq. (17-154), we can obtain the magnitude of the surface-surface direct exchange from Eq. (17-156):

$$\bar{H}_{1,2}=\frac{AD+BC}{2}-\frac{AC+BD}{2}=\sqrt{a^2+h^2}-h$$

Let us consider an emitting system comprising plane AB and a single-row bank of tubes of infinite length. This condition makes it possible to place the radiating surface F_1 on the plane that is tangential to the surface of the tube bank, as shown in Fig. 17-21. The considered system has two enclosures $AC'CBB'A$ and $ABB'DC'A$. Then, in accordance with relationships (17-152), the mean view factor between plane F_1 and surface of tubes F_2 can be expressed as follows:

$$\bar{\varphi}_{2,1}=2\bar{\varphi}_{AB,\,BB'C}=\frac{AB+BB'C-ACC'}{AB}; \quad AB=S; \quad BB'C=\left(\frac{\pi}{2}+\alpha\right)\frac{d}{2}$$

from the right-angled triangle O_1MC' we have

$$\tan\alpha = \sqrt{\left(\frac{s}{d}\right)^2 - 1}; \quad \text{and} \quad \alpha = \arctan\sqrt{\left(\frac{s}{d}\right)^2 - 1}$$

$$AC'C = \left(\frac{\pi}{2} - \alpha\right)\frac{d}{2} + \tan\alpha$$

After substitution of all these values, we finally obtain:

$$\bar{\varphi}_{2,1} = 1 - \sqrt{1 - \left(\frac{d}{s}\right)^2} + \frac{d}{s}\arctan\sqrt{\left(\frac{s}{d}\right)^2 - 1} \qquad (17\text{-}157)$$

The view factor $\bar{\varphi}_{2,1}$ diminishes with an increase in the pitch of the tubes. Similarly, we can approximately determine the view factor for the second row of tubes of a bank. It is then assumed that, having passed through the tubes of the first row, radiant energy is distributed uniformly over the plane tangential to the tubes of the second row. The radiation falling on the surfaces of tubes of the second row will be $(1 - \bar{\varphi}_{2,1})$ times the radiant energy impinging on the tubes of the first row. Then, the common view factor for both rows of tubes will be

$$(\bar{\varphi}_{2,1})_{com} = \bar{\varphi}_{2,1} + (1 - \bar{\varphi}_{2,1})\bar{\varphi}_{2,1} = \bar{\varphi}_{2,1}(2 - \bar{\varphi}_{2,1})$$

if the ratio s/d is the same for the two rows of tubes. The view factors between tubes in a bank can be determined with the aid of special nomographic charts [17.20].

(d) The Method of Light Simulation (Modelling)

When determining view factors by experiment radiation fluxes are replaced with light (luminous) fluxes, since the two cases pertain to electromagnetic radiation. However, light simulation has a number of merits. The procedure is very convenient and simple, and makes it possible to eliminate the difficulties connected with measuring heat fluxes at high temperatures; side phenomena like heat transfer by convection and conduction are also eliminated; and experiments are carried out at low temperatures.

The general laws of simulation must, of course, be observed when constructing a light model. These laws include geometrical similarity of the model and identity of the optical properties making up the initial emitting system. The latter are characterized by certain absorptivities and reflectivities of the surfaces and also by their emissivities.

View factors are determined with the aid of the calculation formulae (17-60), (17-65) and (17-70) in which corresponding measured heat fluxes are considered, instead of radiation fluxes [17.7].

(e) The Method of Electrical Simulation (Electric-circuit Analogy)

The method of electrical simulation was employed in Sec. 3-12 to study conduction. There is also an analogy between the transfer of energy by radiation and the transport of a charge in an electric circuit. The similar mathematical descriptions of the indicated phenomena permit practical application of the analogy for solving different problems of radiation heat transfer.

The elementary radiation flux is represented by the relationship

$$dQ_{1,2} = c_0\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right]d\overline{H}_{1,2} = \frac{\Delta T}{R_T} \qquad (17\text{-}158)$$

where the thermal resistance is

$$R_T = \frac{\Delta T}{\Delta \overline{H}_{1,2}\left[\left(\frac{T_1}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4\right] c_0}$$

and

$$\Delta T = T_1 - T_2$$

On the other hand,

$$I = \frac{\Delta u}{R_{el}} \tag{17-159}$$

The formal analogy between Eqs. (17-158) and (17-159) makes it possible to use electrical models for an experimental investigation of view factors.

The principles for constructing electrical models are those for investigating conduction [17.10, 17.12, 17.21].

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CHAPTER

18

HEAT TRANSFER IN ABSORBING AND EMITTING MEDIA

18-1. Radiant Energy Transfer Equation

(*a*) *Energy Transfer Equations in an Absorbing Medium*

We have been considering above bodies with zero transmittance ($D = 0$). They only absorbed and reflected incident radiation from an external source of radiation and were, therefore, non-transparent or opaque, bodies.

Alongside with bodies of this kind there exist semi-transparent media capable of transmission of radiant energy (semiconductors, ceramics, glass, gases, vapours, etc.). When radiation passes through such a medium it is generally absorbed and scattered. In addition, the medium may be capable of emitting radiation, resulting in that the intensity of directional radiation (l) will change. The equation determining the variation of beam intensity, due to absorption, radiation and scattering, is referred to as the *radiant energy transfer equation.*

Let us first consider an absorbing medium in which one-dimensional transfer of radiation emitted from an external source takes place; the emitted or outgoing radiation of the medium is negligibly small compared with that of the external medium. As the radiation from the external source passes through the media, from its boundary to a given point, it diminishes gradually at the expense of absorption.

The continuous spectrum radiation from the external source that is incident upon the boundary surface of the medium ($I_{\lambda,\, l=0}$) is given. It is necessary to determine the law governing the variation of radiation density across the layer of the absorbing medium. It is assumed that as spectral radiation passes through the layer of the absorbing medium dl thick in the direction l its intensity diminishes in proportion to this intensity and an infinitely small path of beam dl:

$$dI_{\lambda,\, l} = -\alpha_\lambda I_{\lambda,\, l}\, dl \tag{18-1}$$

where α_λ is the spectral absorption coefficient of the medium. According to Eq. (18-1) it characterizes the relative variation of radiation intensity per unit beam length.

This expression is the basic law of transport of radiant energy in an absorbing medium. It can be presented in the form:

$$\frac{dI_{\lambda,\, l}}{I_{\lambda,\, l}} = \alpha_\lambda\, dl \tag{18-2}$$

Assuming that $l = 0$, $I_{\lambda,\, l} = I_{\lambda,\, l=0}$ (given value), integration of this equation gives:

$$I_{\lambda,\, l} = I_{\lambda,\, l=0} e^{-\int_0^l \alpha_\lambda\, dl} \tag{18-3}$$

Equation (18-3) permits determination of the spectral radiosity at each point in the direction l, for individual wavelengths.

The integral radiosity for individual radiation bands or for the entire spectrum is determined by integration within the limits of corresponding wavelengths.

The equation of radiant-energy transfer in an absorbing medium makes it possible to find its optical properties. For a given wavelength, the absorptivity of the medium is determined by the ratio of the radiant energy absorbed in a layer of thickness l to the energy that is incident upon the boundary surface of this layer:

$$A_\lambda = \frac{I_{\lambda,\, l=0} - I_{\lambda,\, l}}{I_{\lambda,\, l=0}} = 1 - e^{-\int_0^l \alpha_\lambda\, dl} \tag{18-4}$$

Let us introduce the concept of the optical thickness of a medium

$$L_\lambda = \int_0^l \alpha_\lambda\, dl$$

If the absorption coefficient of the beam is constant over the length of a beam, then its optical length will be:

$$L_\lambda = \alpha_\lambda l \tag{18-5}$$

Here l is the total thickness of the gas layer.

Then Eq. (18-3), expressing the attenuation of radiation intensity in the absorbing medium, acquires the following form:

$$I_{\lambda,\, l} = I_{\lambda,\, l=0} e^{-L_\lambda} \tag{18-6}$$

known as *Bouguer's law.*

The absorptivity of the medium in this case is then represented by relationship

$$A_\lambda = 1 - e^{-L_\lambda} \tag{18-7}$$

instead of by (18-4).

Under conditions of thermodynamic equilibrium, on the basis of Kirchhoff's law, the spectral absorptivity of a substance is equal to the spectral emissivity and, consequently:

$$\varepsilon_\lambda = A_\lambda = 1 - e^{-L_\lambda} \tag{18-8}$$

Thus, to determine the absorptivity and emissivity of a medium there should be available data on the absorption and radiation spectra, and also on the absorption coefficients for radiation of different wavelengths. Generally, the absorptivity coefficient of a medium depends on its physical nature, wavelength, temperature and pressure (for gases). In consequence, absorption coefficients happen to differ not only for individual spectral bands, but they also vary materially within the limits of a band. V. A. Fabricant applied Bouguer's law to media which intensify radiation, and which are used in lasers.

(*b*) *Energy Transport Equation for an Absorbing and Emitting Medium*

When thermal radiation (beams) passes through an absorbing medium the absorbed energy converts into heat and is again emitted by the medium. It was assumed above that a medium absorbing radiant energy does not reemit it noticeably. But in the more general case the intensity of the medium will diminish along the beam due to absorption, but it will also increase at the expense of emitted radiation. Then, instead of relationship (18-1), the radiation transfer equation acquires the following appearance [18.1]:

$$dI_l = (I_0 - I_l)\, \alpha_\lambda\, dl \tag{18-9}$$

This relationship can be derived from the heat balance. The fraction of incident radiant energy absorbed by the medium in a layer dl thick is determined as before by $I_l \alpha_\lambda \, dl$.

The intensity of the inherent radiation of the medium can be expressed by the intensity of radiation of a black body and absorptivity α_λ equal to $I_0 \alpha_\lambda \, dl$. The change in the intensity of radiation will then be expressed by the difference between the energy absorbed, and that emitted by the layer dl (for an equilibrium system), which leads to the differential equation (18-9), where, as before, I_l is the spectral intensity of radiation in the direction l and I_0 is the spectral intensity of black body radiation at medium temperature. The subscript "λ" is omitted here for simplicity. Relationship (18-9) can be reduced to another form, taking into account the fact that according to Kirchhoff's law (16-53) for an absorbing media $I_0 = \eta/4\pi\alpha_\lambda$:

$$dIl/dl = -\alpha_\lambda I_l + \eta/4\pi \tag{18-10}$$

Integration of Eq. (18-9) yields the relationship

$$I_l = I_{l=0} \exp\left(-\int_0^l \alpha_\lambda \, dl\right) + \int_0^l \alpha_\lambda I_0 \exp\left(-\int_{l'}^{-l} \alpha_\lambda \, dl''\right) dl' \tag{18-11}$$

The first term of Eq. (18-11) determines the intensity of the fraction of incident radiation $I_{l=0}$ passing from 0 to l; the second term is the intensity of the inherent radiation of all elements in layer dl', transmitted from l' to l, where $0 \leqslant l' \leqslant l$ and al'' lies on $l - l'$.

On the whole relationship (18-11) expresses the intensity of radiation in an absorbing and emitting medium as a function of point coordinates, direction l and wavelength.

In the private case of constant temperatures, constant optical properties of the medium and constant pressure (for gases) Eq. (18-11) acquires the following appearance:

$$I_l = I_{l=0} e^{-L} + I_0 (1 - e^{-L}) \tag{18-12}$$

According to (18-7) we have:

$$I_l = I_{l=0} (1 - A_\lambda) + I_0 A_\lambda \tag{18-13}$$

The intensity of radiation entering the medium on the boundary $l = 0$ is determined by the properties of surface (wall) confining the absorbing medium. For a diffuse wall (at $D = 0$)

$$I_{l=0} = \varepsilon_{\lambda w} \frac{E_{0w}}{\pi} + R_{\lambda w} \frac{E_{in}}{\pi} \tag{18-14}$$

where $\varepsilon_{\lambda w}$ and $R_{\lambda w}$ = spectral emissivity and reflectivity of the wall;
$I_{0w} = E_{0w}/\pi$ = spectral intensity of black body radiation at wall temperature;
$I_{in} = E_{in}/\pi$ = the same for radiation that is incident upon the wall.

For a grey wall $\varepsilon_{\lambda w}$ and $R_{\lambda w}$ are independent of wavelength.

Substituting Eq. (18-14) into relationship (18-13) and integrating in respect to the spectrum, we get:

$$I_l = \frac{1}{\pi} \int_0^\infty [(\varepsilon_{\lambda w} E_{0w} + R_{\lambda w} E_{in})(1 - A_\lambda) + A_\lambda E_0 A_\lambda] \, d\lambda \tag{18-15}$$

or

$$I_l = \frac{1}{\pi}\left[\varepsilon_w\left(\sigma T_w^4 - \int_0^\infty E_{0w}\alpha_\lambda\, d\lambda\right) + R_w\left(E_{in} - \int_0^\infty E_{in}\alpha_\lambda\, d\lambda\right) + \int_0^\infty E_{0\lambda}\alpha_\lambda\, d\lambda\right] \tag{18-16}$$

where $E_{0\lambda}$, E_0 are the spectral and integral radiation fluxes of a black body at medium temperature T.

Equation (18-16) permits determination of the integral mean absorptivity and emissivity of the medium:

$$A = \frac{1}{\sigma T_w^4}\int_0^\infty E_{0w}\alpha_\lambda\, d\lambda; \quad \varepsilon = \frac{1}{\sigma T^4}\int_0^\infty E_{0\lambda}\alpha_\lambda\, d\lambda \tag{18-17}$$

In the integral $\int_0^\infty E_{in}\alpha_\lambda\, d\lambda$ the quantity E_{in} consists of the energy emitted by the medium or other walls and that passed through the medium.

The knowledge of the integral properties of a medium is sufficient for solving analytically the problem involving heat transfer in a medium arranged inside a grey envelope.

The above relationships can be applied to a medium with a non-grey envelope provided its optical properties vary slightly with wavelength. If the medium is characterized, in addition, by radiation scattering (attenuating medium), then in the relationships (18-9) and (18-10) α_λ must be replaced by k_λ and η—by η_{out}. The coefficient k_λ is known as the *volumetric extinction coefficient* [18.2].

The analytical solutions based on the above cited equations of radiation heat transfer were derived for systems of simple geometry and some of them shall be considered below, in Sec. 18-4.

18-2. Optical Medium Thickness and Radiation Modes

One of the most important dimensionless characteristics of radiation is the optical thickness of a medium. In accordance with Eq. (18-5), it can be presented in the form

$$L_\lambda = \frac{l}{1/\alpha_\lambda}$$

The quantity $1/\alpha_\lambda$ is interpreted as the penetration distance or as the mean free path of photons. Indeed, if the absorption coefficient is small, the beam shall penetrate deeper into the medium without appreciable attenuation, i.e. the penetration distance shall be great. If the coefficient α_λ is large—the penetration distance is small.

It follows from the foregoing that the optical thickness is the ratio of the reference linear dimension to the penetration distance of radiation and that the role of the quantity $1/\alpha_\lambda$ is similar to that of mean free path of molecules and that the quantity L_λ^{-1} can be considered as Knudsen's photon number.

With $L_\lambda \ll 1$ the optical thickness of a medium is small and the medium is optically thin; with $L_\lambda \gg 1$ the medium is thick and is referred to as an optically thick medium. Just as with molecular transport of heat, depending on the magnitude of L_λ, different modes of radiant-energy transfer can be distinguished. The condition $L_\lambda \gg 1$ signifies that the mean free path of photons is much inferior to the reference dimension of a system.

A medium can be considered as some continuum of photons. Just as with molecular conductivity the transfer of radiant energy in a medium can be compared with diffusion transfer. Here the effect of interphoton collision is predominant. With $L_\lambda \geqslant 1$ the solution for the transfer equation coincides with relationship (16-38) [18.3-18.6].

When $L_\lambda \leqslant 1$, the free path of photons exceeds considerably the reference linear dimension of a system. The photons emitted from a medium get directly on the boundary surface without intermediate collisions and radiant interaction. This mode of radiant-energy transfer is called the mode of negligible self-absorption

In the limit, when $L_\lambda \to 0$, the medium does not participate in radiation heat transfer and the photons travel from surface to surface without intermediate emission and absorption. The values of optical thickness falling between $0 < L_\lambda < 1$ correspond to a transition mode of radiation heat transfer.

18-3. Integral Equations of Radiation Heat Transfer in Systems of Bodies with an Absorbing Participating Medium

Just as in the case of a diathermal participating medium considered in Sec. 17-10, exact solutions are based on integral equations of radiation heat transfer. For this purpose use is made of Eq. (18-10) expressing the variation of radiosity along a beam [18.5].

When applied to an arbitrary fixed point M in the radiation field the relationship for radiosity acquires the form

$$I_M = I_N e^{-L_\lambda} + \frac{1}{4\pi} \int_0^l \eta_P e^{-\Delta L_\lambda}\, dl' \tag{18-18}$$

where the optical thicknesses (distances) are:

$$L_\lambda = \int_0^l \alpha_{\lambda P}\, dl; \quad \Delta L_\lambda = \int_{l'}^l \alpha_{\lambda P}\, dl'' \tag{18-19}$$

where P = current point along the beam l;
N = point corresponding to $l = 0$.

Let us find the emissive power from Eq. (16-11) in terms of radiosity

$$E_{in\,M} = \int_{2\pi} I_n \cos\psi\, d\omega$$

Let us substitute the value of radiosity from Eq. (18-18), allowing for the fact that for a grey boundary surface $I_N = E_{out\,N}/\pi$, and the radiation leaving the surface, or outgoing radiation *out*, is determined by relationship (16-18).

We receive then the integral equation for the unknown emissive power at arbitrary point M:

$$E_{in\,M} - \int_{2\pi} R_N E_{in\,N}\, d\varphi_{(M,\,N)ab} - \frac{1}{4} \int_{2\pi} \int_0^l \eta_P\, dl'\, d\varphi_{(M,\,P)ab} = \int_{2\pi} E_N\, d\varphi_{(M,\,N)ab} \tag{18-20}$$

In the integral equation the quantities

$$\left.\begin{aligned} d\varphi_{(M,\,N)ab} &= e^{-L_\lambda} \frac{\cos\psi\, d\omega}{\pi} = e^{-L_\lambda}\, d\varphi_{M,\,N} \\ d\varphi_{(M,\,P)ab} &= e^{-l_\lambda} \frac{\cos\psi\, d\omega}{\pi} = e^{-L_\lambda}\, d\varphi_{M,\,P} \end{aligned}\right\} \tag{18-21}$$

express the elementary view factors with account taken of attenuation of radiation in a participating absorbing medium (subscript "ab"), and they are called the *general elementary view factors.* The calculation point M may lie both on the boundary surface and inside the space filled with the medium. When applied to a diathermic medium the view factors $d\varphi_{(M,N)ab}$ and $d\varphi_{(M,P)ab}$ reduce into $d\varphi_{M,N}$ and $d\varphi_{M,P}$, and the integral equation (18-20) becomes the integral equation (17-96), Sec. 17-10. The integral equation for the energy density of incident radiation is derived in a similar way:

$$\eta_{inM} = \int_{4\pi} I_M \, d\omega$$

Let us substitute into the above formula the value of radiosity expressed by equation (18-18):

$$\eta_{inM} = \frac{1}{\pi} \int_{4\pi} E_{out\,N} e^{-L_\lambda} \, d\omega + \frac{1}{4\pi} \int_{4\pi} \int_0^l \eta_P e^{-\Delta L_\lambda} \, dl' \, d\omega$$

The elementary solid angle can be expressed as follows:

$$d\omega(M, l) = \frac{dF_M \cos(n, l)}{r^2_{M,N}} = \frac{dF_P}{r^2_{M,P}} = \frac{dV_P}{r^2_{M,P}\, dl}$$

Then, the integral radiation for the energy density of incident radiation acquires the appearance:

$$\eta_{inM} = \frac{1}{\pi} \int_F E_{out\,N} e^{-L_\lambda} \frac{\cos\psi_M}{r^2_{M,N}} \, dF_N = \frac{1}{4} \int_V \eta_P \frac{e^{-\Delta L_\lambda}}{r^2_{M,N}} \, dV_P \qquad (18\text{-}22)$$

For a diathermic medium $\alpha_\lambda = 0$ and $e^{-L_\lambda} = e^{-\Delta L_\lambda} = 1$.

The integral Eqs. (18-20) and (18-21) permit determination of E_{inM} and η_{inM} for a definite wavelength.

Total radiation intensities are found by integration over the spectrum.

The method of obtaining the radiant energy densities for other kinds of radiation is similar to that treated in Sec. 17-10 for a diathermic medium. The differential relations (A) through (D) make it possible to find the densities of radiant energy on the basis of the integral equations derived for various kinds of radiation.

The integral equations for radiation with an absorbing participating medium can be solved by applying algebraic, zoning and resolvent approximations, just as for the diathermic medium.

18-4. Investigating Radiation Heat Transfer in a Flat Parallel Absorbing Medium by the Differential Method

Let us consider an absorbing non-isothermal medium bounded by two flat parallel walls with a finite distance between them. Let us assume that the grey non-transparent (opaque) walls reflect and emit radiant energy isotropically, that they are at constant but different temperatures ($T_1 > T_2$) and have different absorptivities A_1 and A_2 (Fig. 18-1). Let us also assume that the main mode of heat transfer is that of radiation and that a steady-state process takes place. It is necessary to find the distribution of the net radiation flux and temperature across the layer of the medium (one-dimensional problem).

The problem can be solved with the aid of the equation of transfer (differential method) or of the integral equation for radiation heat transfer (integral method).

Let us consider the differential method as applied to a layer with a large optical thickness (optically dense medium), $L \geqslant 1$ [18.4-18.8]. The radiant flux vector then is expressed by formula (16-38), derived on the basis of the diffusion concept of radiation heat transfer [18.6]:

$$q_n = -\frac{4}{3}\frac{\sigma}{\bar{\alpha}}\frac{d}{dx}(T^4) =$$

$$= -\frac{4}{3\bar{\alpha}}d\vartheta/dx, \quad \vartheta = \sigma T^4 \qquad (18\text{-}23)$$

where $\bar{\alpha}$ is the spectral mean absorption coefficient of the medium.

For the case considered the variation of the radiation flux

$$\frac{dq_n}{dx} = 0 \qquad (18\text{-}24)$$

Then, the differential equation of radiant energy transfer takes the form

$$\frac{d^2\vartheta}{dx^2} = 0 \qquad (18\text{-}25)$$

Integrating Eq. (18-25), we obtain relationships

$$d\vartheta/dx = c_1 \quad \text{and} \quad \vartheta = c_2 + c_1 x$$

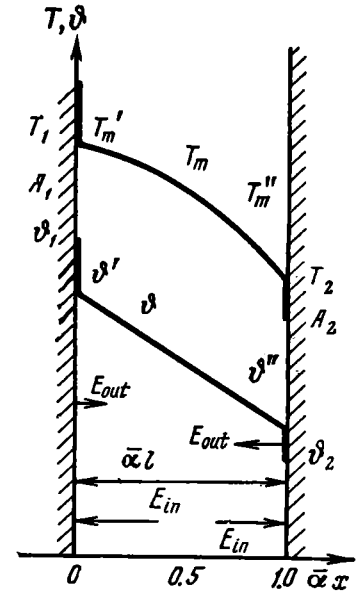

Fig. 18-1. Temperature and density distribution of net radiation

The integration constants are found from the boundary conditions: at $x = 0$ $\vartheta = \vartheta'$ and, consequently, $c_2 = \vartheta'$.

The constant c_1 is found from relationship (18-23)

$$q_n = -\frac{3}{4\bar{\alpha}}\frac{d\vartheta}{dx} = -\frac{3c_1}{4\bar{\alpha}}$$

Then the solution of Eq. (18-25) takes the form:

$$\vartheta = \vartheta' - \frac{3\bar{\alpha}}{4}q_n x \qquad (18\text{-}26)$$

The unknown quantities in Eq. (18-26) are ϑ and q_n.

The quantity q_n is determined with the aid of the system of equations characterizing radiation heat transfer at the medium-wall boundaries. According to Eqs. (16-20) and (16-24) we have:

$$q_n = E_{out} - E_{in} \qquad (18\text{-}27)$$

$$E_{out} = \frac{E_w}{A} - \left(\frac{1}{A} - 1\right)q_n \qquad (18\text{-}28)$$

In addition, the boundary conditions include the radiant energy density in the medium at the medium-wall boundary.

In conditions of thermodynamic equilibrium the radiant energy density can be expressed by Stefan-Boltzmann's law (Eq. 16-48):

$$u_0 c_{ph} = 4\sigma T_f$$

where T_f = absolute temperature of the medium;
c_{ph} = photon velocity.

Let us find the expression for the radiant energy density in the medium and on its boundary in terms of E_{in} and E_{out}, introducing for this purpose the mean radiosities:

$$\bar{I}_{out} = \frac{E_{out}}{\pi}; \quad \bar{I}_{in} = \frac{E_{in}}{\pi}$$

Then

$$u_0 c_{ph} = \int_{2\pi} \bar{I}_{out}\, d\omega + \int_{2\pi} \bar{I}_{in}\, d\omega = 2E_{out} + 2E_{in} \tag{18-29}$$

Solving Eqs. (18-27) through (18-29) together, we find

$$q_n = \frac{\sigma (T_w^4 - T_f^4)}{\frac{1}{A} - \frac{1}{2}} = \frac{\vartheta_w - \vartheta_f}{\frac{1}{A} - \frac{1}{2}} \tag{18-30}$$

Relationship (18-30) expresses the distribution of net radiation across the flat layer of the medium.

As applied to two boundary surfaces (at $x = 0$ and $x = l$) Eq. (18-30) reduces into:

$$q_n = \frac{\vartheta_1 - \vartheta'}{\frac{1}{A_1} - \frac{1}{2}}; \quad q_n = \frac{\vartheta'' - \vartheta_2}{\frac{1}{A_2} - \frac{1}{2}} \tag{18-31}$$

where ϑ_1 and ϑ_2 correspond to surface temperatures.

Let us express the temperature differences from Eq. (18-31)

$$\vartheta_1 - \vartheta' = q_n \left(\frac{1}{A_1} - \frac{1}{2}\right); \quad \vartheta'' - \vartheta_2 = q_n \left(\frac{1}{A_2} - \frac{1}{2}\right) \tag{18-32}$$

In addition, relationship (18-26) can be used to express the temperature difference across the layer of the medium:

$$\vartheta' - \vartheta'' = q_n \frac{3\bar{\alpha} x}{4} \tag{18-33}$$

Adding up the left- and right-hand sides of the last equations, we find the equation for the net radiation q_n, W/m^2, in a flat parallel system in the presence of a participating absorbing medium

$$q_n = \frac{\vartheta_1 - \vartheta_2}{\frac{1}{A_1} + \frac{1}{A_2} - 1 + \frac{3\bar{\alpha} l}{4}} = \frac{\sigma (T_1^4 - T_2^4)}{\frac{1}{A_1} + \frac{1}{A_2} - 1 + \frac{3\bar{\alpha} l}{4}} \tag{18-34}$$

Equations (18-26) and (18-34) are used to determine the distribution of temperature across the absorbing layer:

$$T^4 = T'^4 - q_n \frac{3}{4} \frac{\bar{\alpha}}{\sigma} x \tag{18-35}$$

Then, temperature differences are determined from Eq. (18-32):

$$T_1^4 - T_f^4 = \frac{q_n}{\sigma} \left(\frac{1}{A_1} - \frac{1}{2}\right); \quad T_f^4 - T_2^4 = \frac{q_n}{\sigma} \left(\frac{1}{A_2} - \frac{1}{2}\right) \tag{18-36}$$

and the temperature difference across the medium—from Eq. (18-33):

$$T'^4 - T''^4 = \frac{q_n}{\sigma} \frac{3}{4} \bar{\alpha} l \tag{18-37}$$

The distribution of net radiation in the participating medium shown in Fig. 18-1 is determined by Eq. (18-26) and the distribution of temperature, by Eq. (18-35).

The radiation flux changes linearly with optical thickness $\bar{\alpha}l$ with the variation being material. If the participating medium is diathermic or its optical thickness $L = \bar{\alpha}l \to 0$, then relationship (18-34) reduces into Eq. (17-9).

The temperature jumps occurring at the medium-wall boundaries (18-36) are proportional to the thermal resistance $(1/A - 1/2)$ offered to radiation heat transfer. Consequently, if the medium is optically thick a thin radiation layer, similar to the Knudsen one, exists at each wall surface, which explains the presence of temperature jumps and velocity jumps in a rarefied medium. The indicated regime is referred to as the *radiation slip regime.*

The considered differential method for a thin layer agrees sufficiently well with the results obtained by means of the integral method of investigation [18.9].

The exact differential methods, based on an immediate solution of equations of transfer, are given in [18.4, 18.7, 18.9].

18-5. Radiation from Gases and Vapours

Non-luminous and luminous gas media are distinguished. The luminescence of a gas medium is due to the presence of luminous particles of soot, coal and ash in it. Such a luminous gas medium is referred to as a *flame.*

The radiation from a flame is mainly determined by the radiation from the solid particles contained in it. The presence of a considerable number of fine suspended particles in a gas medium makes the latter turbid.

In addition to luminous flames containing suspended soot particles, classified as *turbid media* are dust-laden flows, for instance, dust clouds, mists. Turbid media are characterized by a considerable scattering of radiant energy.

Considered below are the peculiarities of radiation from or through a non-luminous gas medium, such as pure gases and vapours.

Monoatomic and biatomic gases (helium, hydrogen, oxygen, nitrogen, etc.) are practically transparent (diathermanous) to thermal radiation. Triatomic gases possess a high emissive power and absorptivity. Of the triatomic gases water vapour (H_2O) and carbon dioxide (CO_2) have found wide application in thermal energetics; as a rule, they are simultaneously present in furnace gases.

In contrast to solids and liquids radiation from gases is volumetric in nature, because thermal radiation involves all the microparticles of the gas. The absorptivity of gas, therefore, varies according to the density and thickness of the gas layer.

The larger the density and the thicker the layer, the greater its absorptivity. Radiation from gases is selective. They absorb and emit thermal radiation only in a definite band of wavelengths, and remain transparent in the greater part of the spectrum. Thus, three main absorption bands for CO_2 and H_2O can be selected. The main absorption bands can be signed out for each of them (Figs. 18-2, 18-3 and Table 18-1; the letters and figures correspond to various thickness of gas layer). From the table it follows that the absorption bands of the spectrum of CO_2 coincide partly with those of H_2O. Carbon dioxide possesses relatively narrow absorption bands, those of water vapour are wider. In consequence, the absorptivity and emissivity of water vapour are considerably higher than those of carbon dioxide.

TABLE 18-1

Interval of selective absorption for CO_2 and H_2O

Absorption bands	CO_2			H_2O		
	λ_1, μ	λ_2, μ	$\Delta\lambda$, μ	λ_1, μ	λ_2, μ	$\Delta\lambda$, μ
1	2.4	3.0	0.6	1.7	2.0	0.3
2	4.0	4.8	0.8	2.2	3.0	0.8
3	12.5	16.5	4.0	4.8	8.5	3.7
4	—	—	—	12	30	18

The width of individual radiation bands varies with gas temperature. The width of the bands increases with rising temperature, and absorptivity diminishes, because an increase in temperature results in a decrease in density.

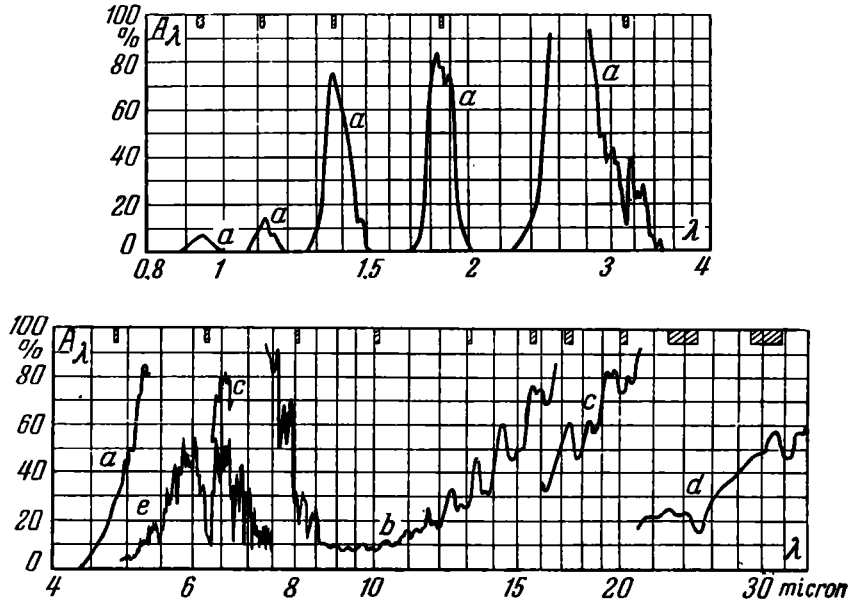

Fig. 18-2. Absorption bands for H_2O

The effect of band widening prevails over decrease in absorptivity, so that on the whole the emissive power of the gas increases with rising temperature.

The emissivity of carbon dioxide and water vapour based on test results can be presented by the following relationships:

$$\left.\begin{aligned} E_{CO_2} &= 3.5\,(pl)^{0.33}\left(\frac{T}{100}\right)^{3.5} \\ E_{H_2O} &= 3.5(pl)^{0.8}l^{0.6}\left(\frac{T}{100}\right)^{3} \end{aligned}\right\} \qquad (18\text{-}38)$$

According to these relationships, the emissivity of CO_2 increases proportionally to $T^{3.5}$, while the exponent of partial pressure p and of the layer thickness l is only 0.33. Hence CO_2 can possess a noticeable inherent radiation in a

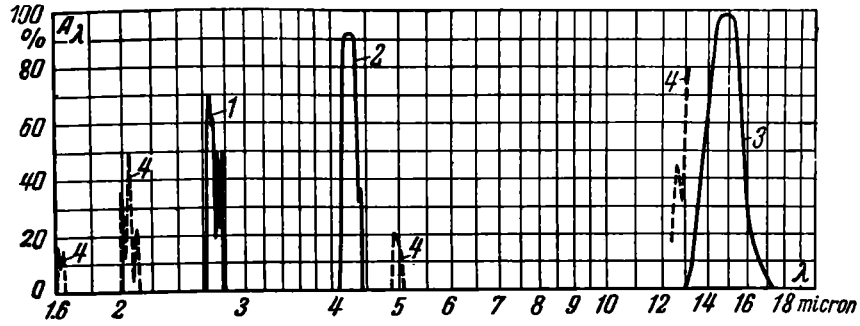

Fig. 18-3. Absorption bands for CO_2

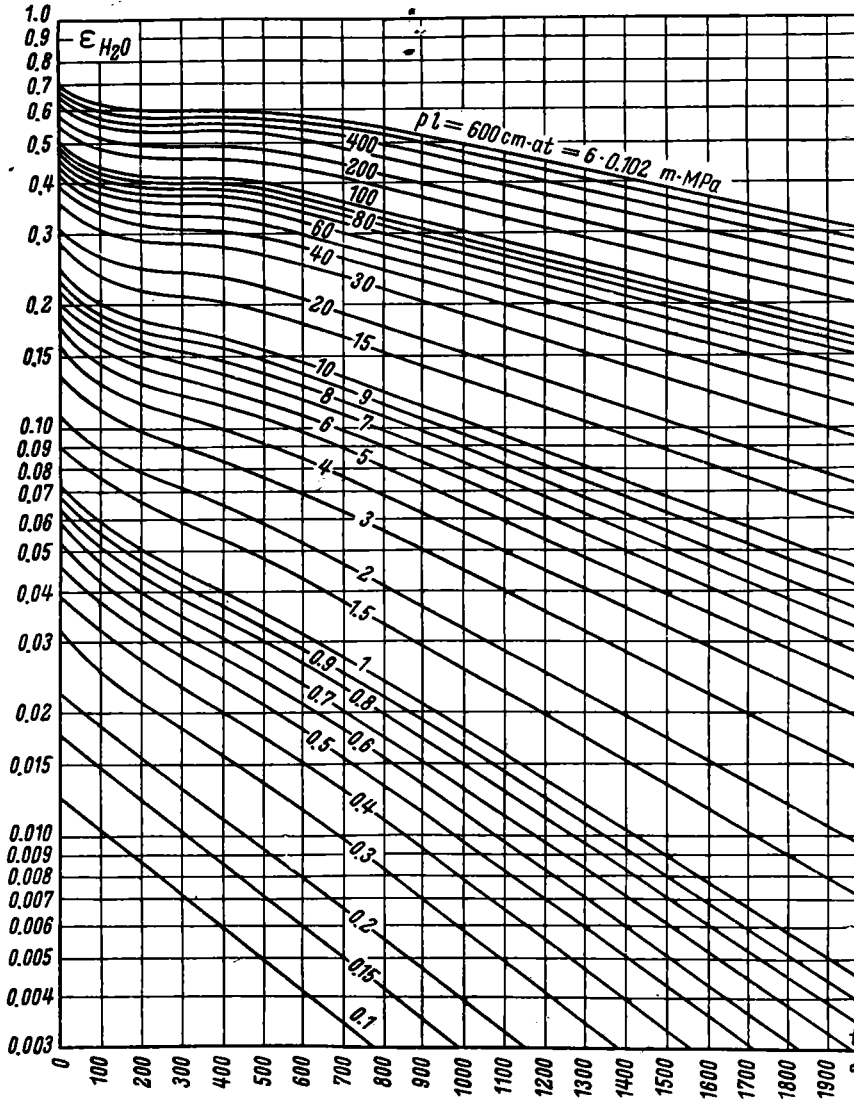

Fig. 18-4. Emissivity of water vapour (H_2O) in relation to temperature

relatively thin layer. This radiation, however, increases slowly with increasing thickness, and much more quickly with temperature.

Partial pressure (p) and length (l) have a much greater effect on the emissivity of H_2O than on CO_2, which is why the effect of CO_2 radiation prevails in thin layers, and that of H_2O in thick ones.

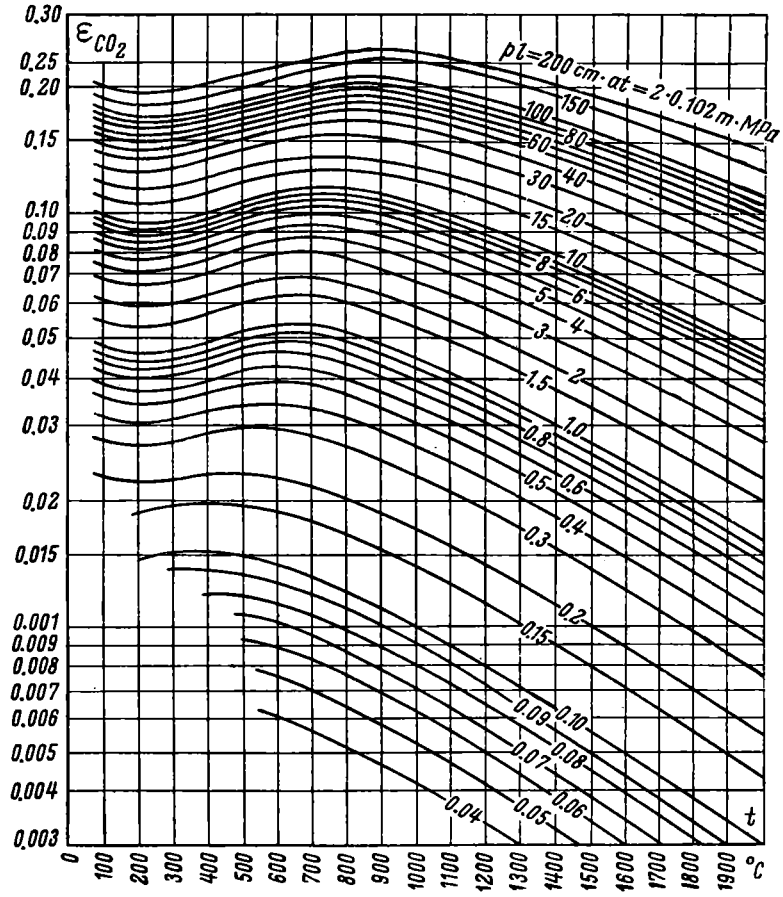

Fig. 18-5. Emissivity of carbon dioxide (CO_2) in relation to temperature

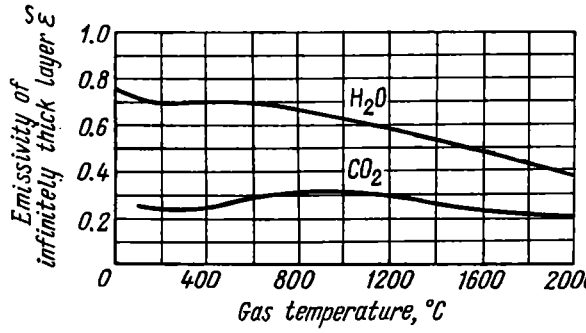

Fig. 18-6. Maximum emissivity of H_2O and CO_2 in relation to temperature

The relationships (18-38) show the radiation of gases to deviate considerably from the Stefan-Boltzmann law. Experimental data for emissivity of carbon dioxide (CO_2) and water vapour (H_2O) in relation to temperature and parameter (pl) are given in Figs. 18-4 and 18-5. The data on their maximum emissivity

with $pl \to \infty$ are plotted in Fig. 18-6, which shows them to be much less than unity, even in extreme cases. The maximum emissivity of water vapour is several times greater than that of carbon dioxide.

The emissivity of gas mixtures is rather less than the sum of the separately calculated emissivities of the components, because of the partial coincidence of radiation spectra (Fig. 18-7).

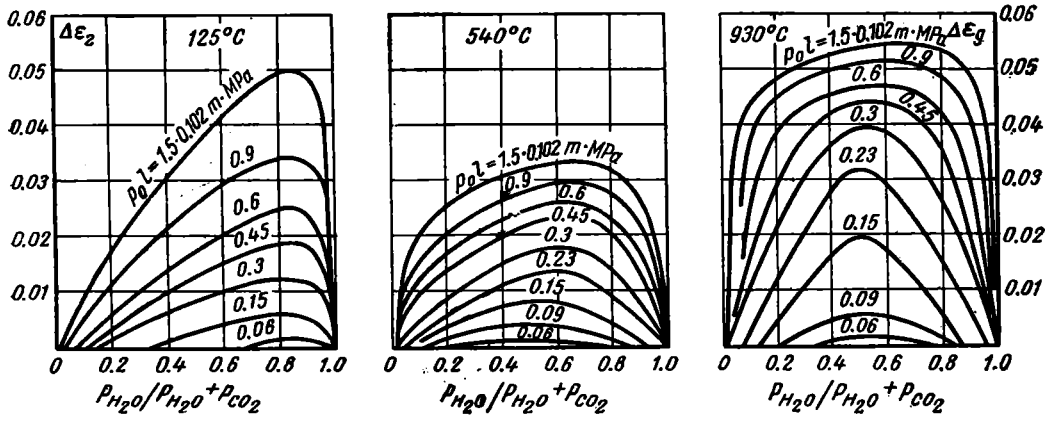

Fig. 18-7. Correction $\Delta\varepsilon_g$ accounting for mutual absorption:
p_0 = total gas pressure

In the combustion chambers of rockets and other systems the combustion products are at very high temperatures and pressures. Some data on the emissivity of water vapour at these conditions are given in [18.10-18.11].

In addition, the emissivity of gases can be determined by means of theoretical methods, such as the methods of statistical physics, quantum mechanics, molecular spectroscopy [18.1, 18.12].

18-6. Radiant-heat Exchange Between a Gas and an Enclosure

Let us assume that the gas and wall have constant temperatures T_g and T_w respectively. Both the gas and wall are assumed to be grey bodies, the radiation of the wall (enclosure) has a continuous spectrum. The gas medium emits selective grey radiation in individual bands e_1g_1, e_2g_2 (Fig. 18-8). In general the number of such bands may vary for different gases.

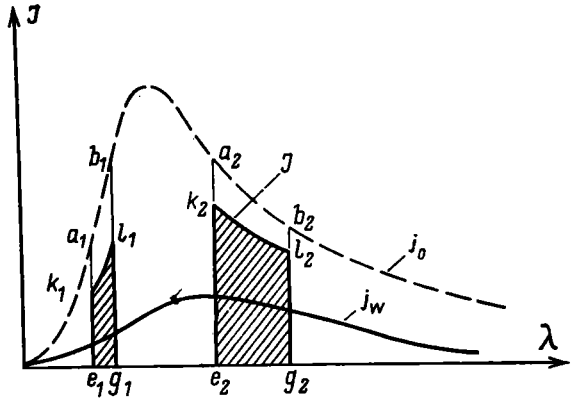

Fig. 18-8. Data on radiant-heat exchange between a gas and a wall

The gas medium exchanges heat with the enclosure only within the given spectrum bands. Beyond the spectrum bands there is radiant-heat interchange only between individual elements of the enclosure.

For an isothermal enclosure the net radiation will therefore be zero. The radiant flux emitted from the gas to the enclosure can then be expressed by the follo-

wing relationship

$$Q_{g,w} = (E_{ef,g} - E_{ef,w}) F_w \tag{18-39}$$

where F_w is the surface of the enclosure.

The radiation fluxes outgoing from, or leaving, a gaseous medium and a wall shall be determined with the aid of the net-radiation method.

As with the previously presented relationships for grey bodies (17-6), the fluxes may be represented as relations to separate bands of radiation [18.22]:

$$E_{ef,g} = (E_{0g})_{\Delta\lambda} + q_{g,w}\left(1 - \frac{1}{\varepsilon_{g\Delta\lambda}}\right)$$

$$E_{ef,w} = (E_{0w})_{\Delta\lambda} + q_{w,g}\left(1 - \frac{1}{\varepsilon_{w\Delta\lambda}}\right)$$

The black body radiation of the gas and enclosure corresponding to the maximum degree of emissivity can be expressed by the equations

$$(E_{0g})_{\Delta\lambda} = c_0 \left(\frac{T_g}{100}\right)^4 \overset{\infty}{\varepsilon}_g$$

$$(E_{0w})_{\Delta\lambda} = c_0 \left(\frac{T_w}{100}\right)^4 \overset{\infty}{\varepsilon}_w$$

and $\overset{\infty}{\varepsilon}_g$, $\overset{\infty}{\varepsilon}_w$ are respectively the maximum emissivities of the gas at gas and wall temperature (Fig. 18-6). The emissivity of gas $\varepsilon_{g\Delta\lambda}$ is determined by the equation:

$$\varepsilon_{g\Delta\lambda} = \frac{k_1 l_1 e_1 g_1 + k_2 l_2 e_2 g_2}{a_1 b_1 e_1 g_1 + a_2 b_2 e_2 g_2} = \frac{E_{g\Delta\lambda}}{(E_{0g})_{\Delta\lambda}} = \frac{\varepsilon_g E_{0g}}{\overset{\infty}{\varepsilon}_g E_{0g}} = \frac{\varepsilon_g}{\overset{\infty}{\varepsilon}_g}$$

Within the individual spectrum bands $\Delta\lambda$ the emissivity of the enclosure ε_w can be assumed equal to the integral value ($\varepsilon_{w\Delta\lambda} = \varepsilon_w$).

For a case of steady-state radiant-heat interchange the net flux is determined from the relationship:

$$q_{g,w} = -q_{w,g}$$

Then, taking into account the above considerations and relationships, Eq. (18-40) makes it possible to derive the following calculation formula for the radiant flux $Q_{g,w}$, W, emitted from the gas to the enclosure:

$$Q_{g,w} = \frac{c_0 F_w \left[\overset{\infty}{\varepsilon}_g \left(\frac{T_g}{100}\right)^4 - \overset{\infty}{\varepsilon}_{g,w} \left(\frac{T_w}{100}\right)^4\right]}{\frac{1}{\overset{\infty}{\varepsilon}_g/\varepsilon_g} + \frac{1}{\varepsilon_w} - 1} \tag{18-40}$$

As already stated above the integral values of the emissivity (absorptivity) of a mixture of gases are not equal in the general case to the sum of their separately calculated values for the individual components of the mixture. Thus for a gas mixture containing H_2O and CO_2 the emissivity and the coefficient of absorptivity are less than the sum of their values for H_2O and CO_2, it is explained by partial coincidence of their radiation spectrum

$$\varepsilon_g = \varepsilon_{CO_2} + \varepsilon_{H_2O} - \Delta\varepsilon_g \tag{18-41}$$

The values for water vapour and carbon dioxide are taken from the graphs in Figs. 18-4 and 18-5 for the gas temperature, and the appropriate products of

partial pressure and beam length (pl). The approximate mean length of the beam path is determined from the following relationship:

$$l = m\frac{4V}{F_w} \tag{18-42}$$

where V is the volume of the gas; m is the correction for gas emissivity, equal to 0.9.

The data on mean beam length for gaseous bodies of various geometrical shape are given in handbooks [18.13]. The correction for gas emissivity that accounts for the deviation from the additivity law for gas mixtures, due to mutual absorption by the components, is read from the graphs shown in Fig. 18-7.

The maximum emissivities of gases at T_g and T_w are determined with the aid of Fig. 18-6.

If the gas is a selective black medium and the wall is a black surface, Eq. (18-43) acquires a simpler form:

$$Q_{g,\,w} = c_0 F_w \left[\overset{\infty}{\varepsilon}_g \left(\frac{T_g}{100}\right)^4 - \overset{\infty}{\varepsilon}_{gw} \left(\frac{T_w}{100}\right)^4\right] \tag{18-43}$$

The above considered calculation method for radiant-heat interchange relates to gases containing no hard particles of fuel, which are present in the flue gases of furnaces. Fluxes of gases contain these hard particles in combustible chambers and furnaces. The radiant-heat interchange in furnaces is calculated by various methods given in the specialist literature [18.14-18.17].

18-7. Combined Modes of Heat Transfer

Radiation heat transfer can be accompanied by simultaneous transfer of heat by conduction and convection. Combined radiation and conduction or convection and also the processes involving all the three modes of heat transfer are referred to as the *combined modes of heat transfer*, including combined radiation-convection and radiation-conduction.

The *radiation-convection* mode of heat transfer is the most general case of combined modes of heat transfer, with heat being transferred not only by radiation, but also by conduction and convection.

In the event of combined radiation and conduction heat is being transferred in a stagnant, attenuating and conducting medium by radiation and conduction.

Combined modes of heat transfer are described by a system of equations consisting of the equations of energy, flow and continuity, and the conditions of unambiguity are added to this system. For a model of a continuous medium the equations of conservation of energy and momentum (Chapter 4) do not change. When applied to steady-state radiation-convection heat transfer in a single-component incompressible fluid, which absorbs, emits and scatters radiation, the equation of energy takes the following form:

$$\operatorname{div}\vec{q}_c + \operatorname{div}\vec{q}_{conv} + \operatorname{div}\vec{q}_r = 0 \tag{18-44}$$

where q_c, q_{conv} and q_r are respectively the radiant flux vectors due to conduction, convection and radiation.

In the general case these quantities vary in the space considered. Eq. (18-44) ignores possible inner heat sources and dissipation of mechanical energy.

Different boundary conditions are given depending on the formulation of the problem of combined heat transfer. The optical and physical parameters of the medium and boundary surface can be given in a different manner [18.4, 18.7].

Problems in combined energy transfer by conduction and radiation are generally rather complicated and they are therefore solved by means of numerical or approximation methods. However, when dealing with optically thin and optically thick layers, Sec. 18-2, the problems have simple solutions.

In the absence of convection, with account taken of the fact that according to Fourier's law $q_c = -\lambda \operatorname{grad} t$, relationship (18-44) takes the following appearance:

$$\operatorname{div}(\lambda \operatorname{grad} t) = \operatorname{div} q_r$$

For a one-dimensional problem this relationship reduces into

$$\lambda \frac{d^2t}{dx^2} = \frac{dq_r}{dx}$$

which is equivalent to the equality

$$q = -\lambda \frac{dt}{dx} + q_r = \text{const} \tag{18-45}$$

In accordance with Eq. (18-34), for an *optically thin layer* radiation heat transfer is determined by relationship

$$q_r = \frac{\sigma (T_1^4 - T_2^4)}{\frac{1}{A_1} + \frac{1}{A_2} - 1}$$

whence it follows that q_r does not depend on the position of a point.

Integrating relationship (18-45), we obtain:

$$q = \frac{\lambda (T_1 - T_2)}{l} + q_r \tag{18-46}$$

Just like q_r, the total flux q does not depend on the coordinates of a point and is determined by the sum of the heat fluxes transferred by conduction and radiation.

In accordance with Eq. (18-23), for an *optically thick layer*

$$q_r = -\frac{16\sigma T^3}{3\bar{\alpha}} \frac{dT}{dx}$$

Substitution of this quantity into Eq. (18-45) yields:

$$q = -\left(\lambda + \frac{16\sigma T^3}{3\bar{\alpha}}\right) \frac{dT}{dx}$$

Upon integration we obtain:

$$q = \frac{\lambda}{l}(T_1 - T_2) + \frac{4\sigma}{3\bar{\alpha} l}(T_1^4 - T_2^4) \tag{18-47}$$

Equation (18-47) shows that for an optically thick layer the heat fluxes q_c and q_r do not depend on each other and the total heat flux is the sum of the two fluxes. Combined radiation-conduction heat transfer in a flat layer under other initial conditions was treated in published works [18.4, 18.18, 18.19], and a cylindrical layer in [18.20].

Even simple problems in combined radiation-convection transfer are much more complicated than radiation-conduction problems. An approximate solution [18.6] for one common problem in combined radiation-convection heat transfer is given below, the final solution being possible due to essential simplifications.

Let us consider radiation-convection heat transfer with an emitting medium in turbulent flow inside a cylindrical channel having a diameter $d = 2r_0$, length l; surface temperature is constant and equal to T_w. The inlet temperature of the medium, T_1, is given as well as its physical properties that are independent of temperature, and the uniform distribution of the mean velocity w_x over the cross-section of the channel. The process of heat transfer is steady-state in time. It is necessary to determine the distribution of temperature in the emitting medium and the rate of heat transfer [18.6].

Let us use Eq. (18-44)

$$\operatorname{div} \vec{q}_{c,\,t} + \operatorname{div} \vec{q}_{conv} + \operatorname{div} \vec{q}_r = 0$$

but allowing for turbulent heat transfer.

In the case considered the vector $\vec{q}_{c,\,t}$ must account for heat transfer both by molecular and turbulent conduction. The transfer of heat due to molecular conduction, described by Fourier's law $q_t = -\lambda \nabla T$, is of importance only at the wall in the region of the viscous sublayer (here T is time-averaged temperature in the turbulent flow, as considered in Sec. 4-5).

Turbulent transfer of heat can be described by equation (for isotropic turbulence)

$$q_t = -\lambda_t \nabla T \tag{18-48}$$

where λ_T is the eddy diffusivity of heat transfer.

The transport of enthalpy by convection is

$$q_{conv} = \rho c_p w_x T \tag{18-49}$$

Radiation heat transfer is determined approximately by relationship

$$q_r = -\lambda_r \nabla T \tag{18-50}$$

where, in accordance with (16-38) the radiative conductivity

$$\lambda_r = \frac{16\sigma T^3}{3\bar{\alpha}}$$

With account taken of molecular conduction relationships (18-48) and (18-50) can be presented in the form

$$q_* = -\lambda_* \nabla T \tag{18-51}$$

where $\lambda_* = \lambda + \lambda_T + \lambda_r$ is the general heat-transfer coefficient counting for heat transfer by conduction, eddy diffusion and radiation.

As applied to axisymmetric flows, the temperature field in an emitting medium can be described by the following differential equation presented here in cylindrical coordinates:

$$w_* \frac{\partial \vartheta}{\partial x} = a_* \left(\frac{\partial^2 \vartheta}{\partial r^2} + \frac{1}{r} \frac{\partial \vartheta}{\partial r} + \frac{\partial^2 \vartheta}{\partial x^2} \right) \tag{18-52}$$

where $a_* = \lambda_*/\rho c_p$; $\vartheta = T - T_w$ = excess temperature of the medium;
r, x = respectively current radius and distance from channel inlet.

The problem of the temperature field was approached here in the same way as above in solving the problem of cooling of an infinite cylinder.

Let us solve differential equation (18-52) by separating the variables:

$$\vartheta(x, r) = \vartheta(x)\,\vartheta(r) \tag{18-53}$$

Substituting relationship (18-53) into Eq. (18-52), we obtain two ordinary differential equations

$$\vartheta''(r)+\frac{1}{r}\vartheta'(r)+\delta^2\vartheta(r)=0 \tag{18-54}$$

$$\vartheta''(x)+w_x/a_*\vartheta'(x)+\delta^2\vartheta(x)=0 \tag{18-55}$$

Allowing for w_x = const, the particular solutions of these equations have the following appearance:

$$\vartheta(r)=AJ_0(\delta r) \tag{18-56}$$

$$\vartheta(x)=c_1e^{\gamma_1 x}+c_2e^{\gamma_2 x} \tag{18-57}$$

where δ = arbitrary quantity, m^{-1};
$J_0(\delta r)$ = zero-order Bessel function of the first kind;

$$\gamma_1=\frac{1}{2}\frac{w_x}{a_*}+\sqrt{\left(\frac{1}{2}\frac{w_x}{a_*}\right)^2+\delta^2};\quad \gamma_2=\frac{1}{2}\frac{w_x}{a_*}-\sqrt{\left(\frac{1}{2}\frac{w_x}{a_*}\right)^2+\delta^2} \tag{18-58}$$

Hence, the solution of Eq. (18-53) can be presented by the function

$$\vartheta(x,\ r)=AJ_0(\delta r)(c_1e^{\gamma_1 x}+c_2e^{\gamma_2 x}) \tag{18-59}$$

Let us consider the boundary condition expressing heat interchange between the emitting medium and wall:

$$-\lambda_r\left(\frac{\partial\vartheta}{\partial r}\right)_{r=r_0}=\alpha_r(\vartheta)_{r=r_0} \tag{18-60}$$

Relationship (18-60) involves the assumption that the transfer of heat by molecular conduction is negligibly small compared with radiation heat transfer. Using Eqs. (18-56) and (18-60), we get:

$$\mu\frac{J_1(\mu)}{J_0(\mu)}=\frac{\bar{\alpha}_r r_0}{\lambda_r}=\mathrm{Nu}_r \tag{18-61}$$

where Nu_r = Nusselt number for radiation heat transfer;
$\mu=\delta r_0$; $J_1(\mu)$ = zero-order Bessel function of the first kind.
By definition the radiation heat-transfer coefficient is

$$\alpha_r=\frac{q_r}{T-T_w}$$

The radiant heat flux is expressed by relationship (18-30)

$$q_r=\frac{\sigma(T^4-T_w^4)}{\frac{1}{A_w}-\frac{1}{2}}$$

Substituting the values of α_r, λ_r and q_r into Eq. (18-61), we obtain the following transcendental equation with an infinite number of roots:

$$\mu\frac{J_1(\mu)}{J_0(\mu)}=\frac{\bar{\alpha}r_0}{\frac{1}{A_w}-\frac{1}{2}}\left[1+\frac{T_w}{T}+\left(\frac{T_w}{T}\right)^2+\left(\frac{T_w}{T}\right)^3\right] \tag{18-62}$$

It follows from Eq. (18-62) that the Nusselt number for radiation heat transfer depends on the optical thickness of the medium ($\bar{\alpha}r_0$), absorptivity of the wall (A_w) and changes but little with temperature ($T_w/T<1$). The boundary condition at the inlet into the channel ($x=0$) is

$$\vartheta(r)_{x=0}=\vartheta_1=T_1-T_w \tag{18-63}$$

Then, Eq. (18-56) reduces into:

$$\vartheta_1 = \sum_{n=1}^{\infty} A_n J_0(\mu_n r/r_0) \tag{18-64}$$

where

$$A_n = \vartheta_1 \frac{2J_1(\mu_n)}{\mu_n [J_1^2(\mu_n)+J_0^2(\mu_n)]} = \vartheta_1 D_n \tag{18-65}$$

With $x = 0$, for the inlet section of the channel, Eq. (18-57) acquires the following appearance:

$$\vartheta_1 = c_1 + c_2 \tag{18-66}$$

For the outlet section of the channel ($x = l$) the following condition is valid:

$$-\lambda_r \left(\frac{\partial \vartheta}{\partial x}\right)_{x=l} = \alpha_r \vartheta(x)_{x=l} \tag{18-67}$$

Substituting into Eq. (18-57) the value of ϑ from Eq. (18-57), we obtain:

$$c_1\gamma_1 e^{\gamma_1 l} + c_2\gamma_2 e^{\gamma_2 l} = -m(c_1 e^{\gamma_1 l} + c_2 e^{\gamma_2 l}) \tag{18-68}$$

where

$$m = \frac{\alpha_r}{\lambda_r} = \frac{\bar{\alpha}}{\frac{1}{A_w} - \frac{1}{2}} \left[1 + \frac{T_w}{T_2} + \left(\frac{T_w}{T_2}\right)^2 + \left(\frac{T_w}{T_2}\right)^3\right] \tag{18-69}$$

where T_2 is the channel outlet temperature.

Equations (18-68) and (18-69) permit determination of the constants c_1 and c_2. Substituting the values of c_1, c_2 and A_n into relationship (18-59), and performing some transformations, we obtain the equation expressing temperature distribution in the emitting medium:

$$\frac{\vartheta(x, r)}{\vartheta_1} = \sum_{n=1}^{\infty} \frac{2J_1(\mu_n)}{\mu_n [J_1^2(\mu_n)+J_0^2(\mu_n)]} J_0\left(\mu_n \frac{r}{r_0}\right) e^{\gamma_2 p x} \tag{18-70}$$

where

$$p = \frac{1 - e^{(\gamma_2-\gamma_1)(l-x)}\beta_2/\beta_1}{1 - e^{(\gamma_2-\gamma_1)l}\beta_2/\beta_1}; \quad \beta_1 = m + \gamma_1; \quad \beta_2 = m + \gamma_2$$

The mean outlet temperature of the emitting medium is determined by integrating relationship (18-70):

$$\frac{\bar{\vartheta}_2}{\vartheta_1} = \sum_{n=1}^{\infty} D_n \frac{2J_1(\mu_n)}{\mu_n} e^{\gamma_2 p x} \tag{18-71}$$

Relationship (18-71) can be presented in the dimensionless form:

$$\frac{\bar{\vartheta}_2}{\vartheta_1} = f(l/d,\ T_w/T_1,\ w_x d/a_*,\ \bar{\alpha}d,\ A_w) \tag{18-72}$$

where $\mathrm{Pe}_r = w_x d/a_*$—the Peclet number for radiation-convection heat transfer and d—the diameter of the channel.

The amount of heat transferred from the medium to the walls of the channel is determined by the mass flow rate, heat capacity of the wall and the temperature difference $(\vartheta_1 - \bar{\vartheta}_2)$; the heat-transfer coefficient is determined by Q, F and $(T_1 - T_w)$.

Consequently, the rate of heat transfer will also be determined by the system of dimensionless terms present in Eq. (18-72). Some criteria of combined and radiation heat transfer are considered below.

18-8. Criteria of Radiation Similarity

The criteria of radiation similarity are obtained by presenting in the dimensionless form the equations describing radiation and combined heat transfer and also the conditions of unambiguity.

The criteria of similarity for multimode heat-transfer processes are obtained from the equation of energy (18-44) and its particular cases.

The dimensionless terms obtained characterize the contribution of various modes of heat transfer. Dimensionless terms of this kind include the Boltzmann (Bo) and Kirpichev (Ki) numbers expressed by the following relationships

$$\mathrm{Bo} \equiv \frac{\rho c_p w}{\sigma_0 T^3}; \quad \mathrm{Ki} = \frac{\sigma_0 T^3}{\lambda k}$$

The Boltzmann number (Bo) characterizes radiation-convection heat transfer and the greater its magnitude, the greater the contribution of radiation heat transfer in the medium, compared with heat transfer by convection.

The Ki number characterizes radiation-conduction heat transfer and the quantities λ and k are respectively the thermal conductivity and attenuation coefficient of the medium.

A heat balance composed for a medium-solid wall interface is used to obtain the Stark number, $\mathrm{St} = \sigma_0 T^3 l/\lambda_w$, where T, l are the reference temperature and linear dimension and λ_w is the thermal conductivity of the solid.

The Stark number is an analogue of the Biot number and it characterizes the relation between the temperature field in a solid and the conditions for radiation heat transfer on the surface of the solid.

The radiation energy transfer equation permits determination of the Bouguer number

$$\mathrm{Bu} = \overline{k} l_0$$

which characterizes the optical density of the medium and, consequently, the passage of radiant energy through it, l_0 is the reference dimension of the attenuating medium and $\overline{k}$ is the mean attenuating coefficient.

There are a number of other criteria of similarity, dimensionless terms, finding application in processes involving radiant heating of materials in furnaces, in calculating boiler furnaces and in other special cases [18.2, 18.21].

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5 HEAT-EXCHANGE EQUIPMENT

CHAPTER
19
CALCULATION OF HEAT EXCHANGERS

19-1. Classification

Heat exchangers are devices in which heat is transferred from one medium to another.

Heat exchange between different heat carriers is one of the most important heat-transfer processes encountered in engineering. For instance, the generation of steam of given parameters in a modern steam generator is based on the process of transferring heat from one agent to another. The basic process in the condensers and cooling towers of thermal power stations, in the air heaters of blast furnaces, and in many apparatuses in the chemical industry is one of heat exchange between heat carriers. Heat exchangers can be classified according to this principle of operation as recuperators (heat exchangers without storage), regenerators (heat exchangers with storage), and direct-contact apparatus. There are, in addition, heat exchangers in which the heat carrier is heated or cooled by means of internal heat sources or sinks.

Recuperators are heat exchangers in which two fluids flow at different temperatures in a space separated by a solid partition. Heat is transferred by convection and conduction through the separating wall, and, if one of the fluids is a radiating gas, by thermal radiation as well. The recuperators include, for example, steam generators, heaters, condensers, evaporators, etc.

Regenerators are heat exchangers in which one heating surface is exposed at certain intervals of time first to a hot fluid and then to a cold one. The surface of the regenerator first removes heat from the hot fluid and is itself heated in the process; then the surface gives up this heat to the cold fluid. Thus the process of heat transfer taking place in a regenerator is always of a transient nature, while steady-state conditions are typical on the whole of recuperators. The air heaters of open-hearth furnaces and blast furnaces are typical regenerators.

Both the regenerators and recuperators are often also referred to as surface-type heat exchangers, because the process of heat transfer in them is invariably linked with the surface of a solid.

In *direct-contact apparatus* heat is transferred by direct contact between hot and cold fluids and their mixing. Typical of these heat exchangers are the cooling towers of thermal power stations, in which the water is cooled by atmospheric air. Air comes into direct contact with the water and mixes with the steam

forming in the tower owing to partial evaporation of the cooling water. This is a combined process of heat and mass transfer.

Heat exchangers with internal heat sources have a single heat-carrying agent, instead of the usual two, to remove the heat liberated in the apparatus. Examples of heat exchangers of this type are nuclear reactors, electrical heating appliances, and other devices. The heat exchangers used in various engineering applications are also given special names, quite irrespective of their principle of operation, names that are determined by their technological purpose and features of their design. However, from the heat engineering point of view all heat exchangers have one purpose, namely, to transfer heat from one heat-carrier to another, or to ensure the exchange of heat between the surface of a solid and a moving agent. This characteristic determines the general conditions for analysing any heat-exchange equipment.

19-2. Basic Concepts and Calculation Equations

Two kinds of analysis are distinguished for heat exchangers: design and checking calculations.

Design calculations are carried out when new equipment is being designed. The purpose of the calculations is to determine the heat-transfer surfaces.

Checking calculations are carried out where the heating surface of the heat exchanger is known, but it is necessary to determine the quantity of heat transferred and the final temperatures of the working fluids. Calculation of heat exchangers resolves itself into a simultaneous solution of heat balance and heat-transfer equations, and any calculation is based on these two equations.

Being the same in essence, *heat balance* and *heat-transfer equations* differ from each other only in details, depending on the type of the heat exchanger investigated (whether recuperative, regenerative, or direct contact). The equations given below refer to recuperators.

Let us consider the steady-state operation of a heat exchanger.

Heat balance equation. The change in enthalpy of the heat carrier is determined by the relationship

$$dQ = G\,di \tag{19-1}$$

where G is the rate of mass flow, kg/s, and i is the specific enthalpy, J/kg; dQ is measured in J/s or W.

Assuming a rate of mass flow as constant, we have for the final changes in enthalpy:

$$Q = G\int_{i'}^{i''} di = G\,(i'' - i') \tag{19-2}$$

where i' and i'' are the initial and final enthalpies of the heat carrier.

If the heat of the primary (hot) heat-carrying agent is taken up by the secondary (cold) agent, then, ignoring any loss of heat, the heat balance equation is written as:

$$dQ = -G_1\,di_1 = G_2\,di_2 \tag{19-3}$$

or, for the final change in enthalpy, as

$$Q = G_1\,(i_1' - i_1'') = G_2\,(i_2'' - i_2') \tag{19-4}$$

Here and further subscript "1" indicates that the value refers to the hot fluid, while subscript "2" indicates the cold one. The sign (′) indicates the value at the inlet, and (″) the value at the outlet of the apparatus.

Assuming the specific heat c_p = const and $di = c_p dt$, the preceding equations can be presented in the following forms:

$$dQ = Gc_p\, dt \tag{19-1'}$$

$$Q = Gc_p\,(t'' - t') \tag{19-2'}$$

$$Q = G_1 c_{p1}\,(t_1' - t_1'') = G_2 c_{p2}\,(t_2'' - t_2') \tag{19-4'}$$

The specific heat c_p depends on temperature. In practice, therefore, the isobaric mean specific heat is substituted in Eq. (19-4), within the $t' - t''$ temperature range.

The concept of the full specific heat or the so-called water equivalent often used in calculations is linked with the rate of mass flow of the heat carrier, and is expressed by

$$C = Gc_p \tag{19-5}$$

and is measured in W/K.

In the literature the quantity C is also referred to as the water equivalent.

From Eq. (19-4) it follows that

$$\frac{C_1}{C_2} = \frac{t_2'' - t_2'}{t_1' - t_1''} = \frac{\delta t_2}{\delta t_1} \tag{19-6}$$

which shows that the ratio of the differences in inlet and outlet temperatures of the colder and hotter fluids is inversely proportional to the ratio of the water equivalents. It is easily seen that the temperature of a heat carrier remains constant with a change in its aggregate state, and that δt is zero. Hence the water equivalent of such an agent is infinite, i.e. $C = \infty$.

Relationship (19-6) holds for both a finite heat-transfer surface F, and any elementary area dF, i.e.

$$\frac{C_1}{C_2} = \frac{dt_2}{dt_1} \tag{19-6'}$$

The *heat-transfer equation* is usually used to determine the heat-transfer surface and is written as

$$Q = k\,(t_1 - t_2)\,F \tag{19-7}$$

where k = overall heat-transfer coefficient;

t_1 and t_2 = temperature of the primary (hotter) and secondary (colder) agent respectively;

F = area of the heat-transfer surface; Q is measured in W

Equation (19-7) holds, provided that t_1 and t_2 remain constant over the entire heat-transfer surface, but this condition is satisfied only in individual cases. In general they vary over the surface and, consequently, the temperature difference $\Delta t = t_1 - t_2$ varies. The heat-transfer coefficient also changes over the heat-transfer surface. The temperature difference Δt and the overall heat-transfer coefficient k can be taken constant only within the limits of an elementary area of the heat-transfer surface dF. Hence, the heat-transfer equation is true for an element of the surface dF only in the differential form:

$$dQ = k\,\Delta t\, dF \tag{19-8}$$

The total heat flux through F is determined by the integral

$$Q = \int_0^F k\,\Delta t\, dF \tag{19-9}$$

To solve this last equation, it is necessary to know the law governing the change of Δt and k over the surface. The overall heat-transfer coefficient k, W/m²·K, usually changes only slightly, and can be taken constant; but where it varies considerably along individual sections of the heat-transfer surface it is averaged:

$$\bar{k} = \frac{F_1 k_1 + F_2 k_2 + \ldots + F_n k_n}{\sum_{i=1}^{i=n} F_i} \tag{19-10}$$

Having thus assumed a constant heat-transfer coefficient over the entire surface, Eq. (19-9) becomes:

$$Q = \bar{k} \int_0^F \Delta t \, dF \tag{19-11}$$

Multiplying and dividing Eq. (19-11) by F, we get:

$$Q = \bar{k} \left(\frac{1}{F} \int_0^F \Delta t \, dF \right) F = \bar{k} \, \overline{\Delta t} \, F \tag{19-12}$$

here Q is measured in W.

Expression (19-12) is the second basic equation for analysing heat exchangers, and is referred to as the heat-transfer equation.

In the design of heat exchangers, the heat capacity Q in watts is given, and it is necessary to calculate the heat-transfer surface F, which is found from Eq. (19-12):

$$F = \frac{Q}{\bar{k} \, \overline{\Delta t}} \, .$$

From that it follows that the problem of finding the heat-transfer surface comes down to calculating the mean heat-transfer coefficient $\bar{k}$ and the mean surface temperature difference $\overline{\Delta t}$.

For a plain wall, for instance, the heat-transfer coefficient is measured in W/m²·K and is found from the equation

$$\bar{k} = \frac{1}{\frac{1}{\alpha_1} + \sum_{i=1}^{i=n} \frac{\delta_i}{\lambda_i} + \frac{1}{\alpha_2}}$$

The local-transfer coefficients α_1 and α_2 can account not only for heat transfer by convection, but also by radiation. In that case, for instance, $\alpha_1 = \alpha_{c1} + \alpha_{r1}$.

The denominator of the summation $\sum_{i=1}^{i=n} \frac{\delta_i}{\lambda_i}$ is the full thermal resistance of the solid wall of conduction separating the hot and cold fluids. The separating wall may be either composite or homogeneous.

In studying the temperature profiles of the heat carriers along a heat-transfer surface, cases can be encountered in which the temperatures of the hot and cold

fluids vary in a monotonous way (in particular, the temperature of one of the heat-carrying agents may remain constant). Cases may also be encountered in which the temperature of one of the fluids changes monotonously, and that of the second stepwise; this is the case, for instance, in the steaming economizers of steam generators. The temperature of the hot flue gas fluctuates continuously, while that of the heated water changes monotonously in the preheating zone of the economizer and remains practically constant in the steaming zone.

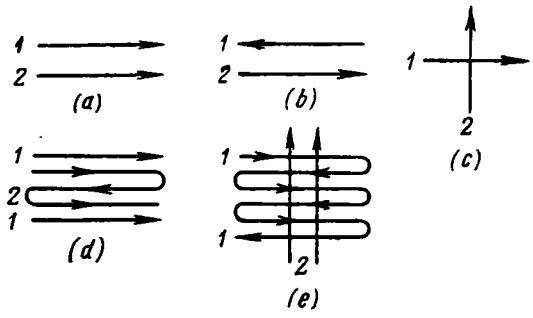

Fig. 19-1. Typical arrangements for fluid flow in heat exchangers:
a—parallel-flow; *b*—counter-flow; *c*—cross-flow; *d*—compound scheme; *e*—multiple mixed-flow

Heat exchangers in which the temperatures of the hot and cold fluids vary continuously, may have the following patterns of flow arrangement: (1) parallel-flow; (2) counter-flow; (3) cross-flow; and (4) mixed-flow (with complex flow of the heat carriers).

A heat exchanger in which the primary (hot) and secondary (cold) fluids flow in one direction, parallel to each other, is referred to as a *parallel-flow apparatus* (Fig. 19-1*a*). If the two fluids flow parallel to one another, but in opposite directions, the heat exchanger is known as a *counter-flow device* (Fig. 19-1*b*). If the fluids cross each other, the pattern is referred to as *cross-flow* (Fig. 19-1*c*). In addition to these simple flow arrangements, more complex ones are encountered in practice: simultaneous parallel- and counter-flow (Fig. 19-1*d*), multiple mixed-flow (Fig. 19-1*e*), etc.

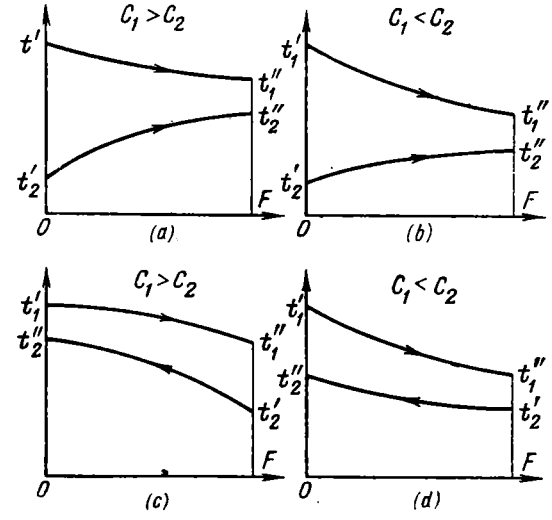

Fig. 19-2. Fluid temperature variation in parallel- and counter-flow heat exchangers in relation to C_1 and C_2

The temperature profiles of the two fluids along a heat-transfer surface are determined by the type of flow arrangement and the ratio of the water equivalents C_1 and C_2. Depending on the flow arrangement and this ratio, the temperature profiles of the heat carriers are illustrated in Fig. 19-2, in which the heat-transfer surface F is plotted along the x-axis and temperatures of the fluids along the y-axis. It is clear from the figure that the heat carrier of smaller water equivalent suffers a larger variation in temperature, in accordance with Eq. (19-6).

19-3. Mean Temperature Difference and Its Calculation Methods

For the simplest cases the variation in temperature of the hot and cold fluids can be determined analytically. Let us consider the simplest parallel-flow heat exchanger (Fig. 19-3). The heat-transfer equation for an element dF of the heat-

transfer surface is written as

$$dQ = k\,(t_1 - t_2)\,dF = k\,\Delta t\,dF \qquad \text{(a)}$$

The temperature of the primary agent will drop by dt_1, and the temperature of the cold fluid will increase by dt_2. Consequently,

$$dQ = -C_1\,dt_1 = C_2\,dt_2 \qquad \text{(b)}$$

from which

$$dt_1 = -\frac{dQ}{C_1} \quad \text{and} \quad dt_2 = \frac{dQ}{C_2} \qquad \text{(c)}$$

The change in temperature difference is then

$$d\,(t_1 - t_2) = dt_1 - dt_2 = -\left(\frac{1}{C_1} + \frac{1}{C_2}\right) dQ = -m\,dQ \qquad \text{(d)}$$

where

$$m = \left(\frac{1}{C_1} + \frac{1}{C_2}\right)$$

Substituting the value of dQ from (a) in (d), we find that

$$d\,(t_1 - t_2) = -mk\,(t_1 - t_2)\,dF$$

Denoting $(t_1 - t_2)$ as Δt, this last equation becomes:

$$\frac{d\,(\Delta t)}{\Delta t} = -mk\,dF \qquad \text{(e)}$$

If m and k are constant, then Eq. (e) can be integrated from 0 to F and from $\Delta t'$ to Δt:

$$\int_{\Delta t'}^{\Delta t} \frac{d\,(\Delta t)}{\Delta t} = -mk \int_0^F dF$$

Integration gives:

$$\ln \frac{\Delta t}{\Delta t'} = -mkF \qquad \text{(19-13)}$$

or

$$\Delta t = \Delta t' e^{-mkF} \qquad \text{(19-14)}$$

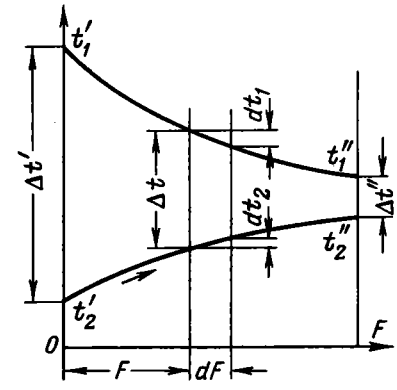

Fig. 19-3. Deriving the equation for variation in fluid temperature difference along a heat transfer surface

From Eq. (19-14) it follows that the variation in the temperature difference along the heat-transfer surface is exponential. Hence, in parallel-flow heat exchangers the difference between the temperatures of the hot and cold fluids drops continuously. With counter-flow the temperatures of the two heat carriers drop along the heat-transfer surface (Fig. 19-2*c* and *d*); the heat balance equation then becomes

$$dQ = -C_1\,dt_1 = -C_2\,dt_2 \qquad \text{(f)}$$

and the variation in temperature difference is

$$d\,(t_1 - t_2) = -m\,dQ$$

and

$$m = \left(\frac{1}{C_1} - \frac{1}{C_2}\right)$$

Therefore, in counter-flow heat exchangers the temperature difference Δt of the hot fluid drops in the direction of flow only when $C_1 < C_2$ ($m > 0$), but increases if $C_1 > C_2$ ($m < 0$).

The mean difference between the temperatures of the hot and cold fluids along a section of surface F is determined by the relationship

$$\overline{\Delta t} = \frac{1}{F} \int_0^F \Delta t \, dF \qquad (g)$$

where Δt is the local temperature difference ($t_1 - t_2$), relating to surface element dF and expressed by Eq. (19-14).

Substituting the value of Δt from Eq. (19-14) in Eq. (g), we obtain:

$$\overline{\Delta t} = \frac{\Delta t'}{F} \int_0^F e^{-mkF} \, dF = \frac{\Delta t'}{-mkF} (e^{-mkF} - 1) \qquad (19\text{-}15)$$

Substituting the values of mkF and e^{-mkF} from Eqs. (19-13) and (19-14) in Eq. (19-15), we get:

$$\overline{\Delta t} = \frac{\Delta t'}{\ln \frac{\Delta t}{\Delta t'}} \left(\frac{\Delta t}{\Delta t'} - 1 \right) = \frac{\Delta t - \Delta t'}{\ln \frac{\Delta t}{\Delta t'}} \qquad (19\text{-}16)$$

If the temperature drop is averaged over the whole surface of heat exchange, then $\Delta t = \Delta t''$, and Eq. (19-16) acquires the following form:

$$\overline{\Delta t} = \frac{\Delta t'' - \Delta t'}{\ln \frac{\Delta t''}{\Delta t'}} \qquad (19\text{-}17)$$

Formula (19-17) is frequently written as follows:

$$\overline{\Delta t} = \frac{\Delta t_l - \Delta t_s}{\ln \frac{\Delta t_l}{\Delta t_s}} = \frac{\Delta t_l - \Delta t_s}{2.3 \log \frac{\Delta t_l}{\Delta t_s}} \qquad (19\text{-}18)$$

where Δt_l is the larger temperature difference; Δt_s is the smaller temperature difference.

Formula (19-18) is suitable both for parallel-flow and counter-flow arrangements.

The mean temperature difference (19-18) obtained is called the logarithmic mean temperature drop. Formula (19-18) holds for heat exchangers of the simplest flow arrangements, provided that the mass rates of flow of the hot and cold fluids and the heat-transfer coefficients are constant along the entire surface.

With equal water equivalent ($m = 0$) in a counter-flow arrangement it follows from Eq. (19-14) that the temperature drop is constant along the entire surface, i.e. that $\Delta t = \text{const}$.

When the temperature of the two fluids changes only slightly along the heat-transfer surface, the mean temperature difference can be calculated as the arithmetic mean of the inlet and outlet differences:

$$\overline{\Delta t} = \frac{1}{2} (\Delta t_l + \Delta t_s) = \frac{\Delta t_l}{2} \left(1 + \frac{\Delta t_s}{\Delta t_l} \right) \qquad (19\text{-}19)$$

Since the arithmetic mean temperature drop is always larger than the logarithmic, the temperature difference can be calculated with sufficient accuracy by formula (19-19) when $\frac{\Delta t_l}{\Delta t_s} < 2$.

The mean temperature difference for intricate flow arrangements is calculated as follows:

(1) the temperature drop is determined by Eq. (19-18):

$$\overline{\Delta t}_{cf} = \frac{\Delta t_l - \Delta t_s}{2.3 \log \frac{\Delta t_l}{\Delta t_s}}$$

(2) the auxiliary values P and R are determined by formulae:

$$P = \frac{t_1'' - t_2'}{t_1' - t_2'} = \frac{\delta t_2}{\Delta t_{max}} \tag{19-20}$$

$$R = \frac{t_1' - t_1''}{t_2'' - t_2'} = \frac{\delta t_1}{\delta t_2} \tag{19-21}$$

From Eqs. (19-20) and (19-21) it follows that P is always smaller than unity ($P < 1$). The value of R may be larger or smaller than unity, depending on the ratio of heat content of mass carriers C_1 and C_2. According to the magnitudes of P and R a correction $\varepsilon_{\Delta t} = f(P, R)$ is found from an auxiliary chart. An example of such a chart for a cross-flow reversed-current heat exchanger is shown in Fig. 19-4 [19.1]. The temperature drop is found as

$$\overline{\Delta t} = \overline{\Delta t}_{cf} \varepsilon_{\Delta t} \tag{19-22}$$

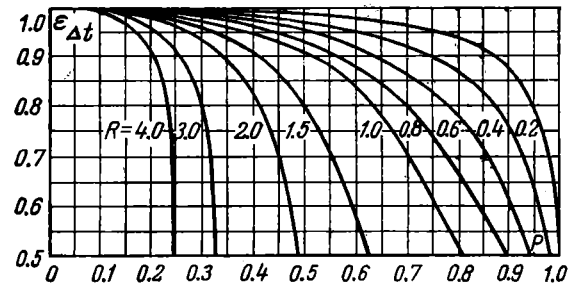

Fig. 19-4. Graph for determining the correction $\varepsilon_{\Delta t} = f(R, P)$

19-4. Calculation of Outlet Temperature of the Working Fluid

It is often necessary to use the given inlet temperatures t_1' and t_2' of the heat carriers, the known heat-transfer surface F and the overall heat-transfer coefficient k to calculate the outlet temperatures of the two fluids and the quantity of heat Q transferred in the heat exchanger. Problems of this kind have to be dealt with in check calculations, when a heat exchanger is already available or at least designed. The calculation is based on the known heat balance and heat-transfer formulae:

$$Q = C_1 \delta t_1 = C_2 \delta t_2$$

and

$$Q = k \overline{\Delta t} F$$

The following values are known: the heat-transfer surface F, the overall heat-transfer coefficient k, the water equivalents C_1 and C_2, and the inlet temperatures t_1' and t_2'. The unknown values to be determined are the outlet temperatures t_1'' and t_2'' and Q, the quantity of heat transferred.

Let us consider the case where temperature changes slightly along the heat-transfer surface $\left(\frac{\Delta t_l}{\Delta t_s}\right) < 2$ and the temperature distribution along it can be assumed to be linear. The temperature drop between the hot and cold fluids can then be calculated by Eq. (19-19), i.e.

$$\overline{\Delta t} = \frac{\Delta t_l + \Delta t_s}{2} = \left(\frac{t_1' + t_1''}{2} - \frac{t_2'' + t_2'}{2}\right) \tag{a}$$

The outlet temperatures t_1'' and t_2'' are determined from the heat balance equation, from which

$$\left.\begin{aligned} t_1'' &= t_1' - \frac{Q}{C_1} \\ \text{and} \\ t_2'' &= t_2' + \frac{Q}{C_2} \end{aligned}\right\} \quad \text{(b)}$$

Substituting these values in (a), we obtain:

$$\overline{\Delta t} = (t_1' - t_2') - \left(\frac{1}{2C_1} + \frac{1}{2C_2}\right) Q \quad \text{(c)}$$

Substituting (c) in the heat-transfer equation and solving the latter for Q, W, we obtain:

$$Q = \frac{t_1' - t_2'}{\frac{1}{kF} + \frac{1}{2C_1} + \frac{1}{2C_2}} \quad \text{(19-23)}$$

The unknown outlet temperatures of the heat carriers are found by calculating the magnitude of Q by Eq. (19-23) and substituting it into (b).

This method of calculation gives approximate results and is suitable only for preliminary calculations. In general the temperature distribution of the fluids is not linear, and depends on the water equivalents C_1 and C_2, the area of heat-transfer surface, and the flow arrangements chosen for the fluids. Different formulae are therefore used for parallel-flow and counter-flow calculations.

(*a*) *Parallel-flow Arrangement*

To derive the formulae, we use the exponential distribution of the temperature drop along the heat-transfer surface [Eq. (19-14)]:

$$\Delta t'' = \Delta t' e^{-mkF}$$

If this equation is written in the following form:

$$\frac{t_1'' - t_2''}{t_1' - t_2'} - 1 = e^{-\frac{kF}{C_1}\left(1 + \frac{C_1}{C_2}\right)} - 1$$

then, after simple transformations, we get:

$$(t_1' - t_1'') + (t_2'' - t_2') = (t_1' - t_2')\left[1 - e^{-\frac{kF}{C_1}\left(1 + \frac{C_1}{C_2}\right)}\right]$$

or

$$\delta t_1 + \delta t_2 = \Delta t' \left[1 - e^{-\frac{kF}{C_1}\left(1 + \frac{C_1}{C_2}\right)}\right] \quad \text{(d)}$$

The unknowns δt_1 and δt_2 are determined from the heat balance:

$$\frac{C_1}{C_2} = \frac{\delta t_2}{\delta t_1}$$

from which

$$\delta t_2 = \delta t_1 \frac{C_1}{C_2} \quad \text{(e)}$$

Substituting the expression for δt_2 in (d), we obtain:

$$\delta t_1 + \delta t_1 \frac{C_1}{C_2} = \Delta t' \left[1 - e^{-\frac{kF}{C_1}\left(1 + \frac{C_1}{C_2}\right)}\right]$$

or

$$\delta t_1 = \Delta t' \frac{1 - e^{-\frac{kF}{C_1}\left(1+\frac{C_1}{C_2}\right)}}{1+\frac{C_1}{C_2}} \tag{19-24}$$

From Eq. (19-24) it follows that the variation in temperature of the hot fluid δt_1 is a certain fraction of the inlet temperature drop between the two fluids $(t_1' - t_2')$.

The expression

$$\frac{1 - e^{-\frac{kF}{C_1}\left(1+\frac{C_1}{C_2}\right)}}{1+\frac{C_1}{C_2}}$$

is a function only of $\frac{kF}{C_1}$, $\frac{C_1}{C_2}$, and can be calculated in advance and tabulated.

Similarly, we derive formula for the temperature variation of the second heat carrier for parallel-flow apparatus:

$$\delta t_2 = \Delta t' \frac{C_1}{C_2} \frac{1 - e^{-\frac{kF}{C_1}\left(1+\frac{C_1}{C_2}\right)}}{1+\frac{C_1}{C_2}} \tag{19-25}$$

Formulae (19-24) and (19-25) can finally be presented in the form:

$$\delta t_1 = (t_1' - t_2') f_p\left(\frac{kF}{C_1}, \frac{C_1}{C_2}\right) \tag{19-24'}$$

and

$$\delta t_2 = (t_1' - t_2') \frac{C_1}{C_2} f_p\left(\frac{kF}{C_1}, \frac{C_1}{C_2}\right) \tag{19-25'}$$

(b) Counter-flow Arrangement

To derive a formula describing the temperature profiles of the hot and cold fluids along the surface, we start from Eq. (19-14), applying it to a counter-flow arrangement:

$$\frac{t_1'' - t_2'}{t_1' - t_2''} = e^{-\frac{kF}{C_1}\left(1-\frac{C_1}{C_2}\right)}$$

The further calculations are like those for parallel flow. The final results have the form:

$$\delta t_1 = (t_1' - t_2') \frac{1 - e^{-\frac{kF}{C_1}\left(1-\frac{C_1}{C_2}\right)}}{1 - \frac{C_1}{C_2} e^{-\frac{kF}{C_1}\left(1-\frac{C_1}{C_2}\right)}} \tag{19-26}$$

and

$$\delta t_2 = (t_1' - t_2') \frac{C_1}{C_2} \frac{1 - e^{-\frac{kF}{C_1}\left(1-\frac{C_1}{C_2}\right)}}{1 - \frac{C_1}{C_2} e^{-\frac{kF}{C_1}\left(1-\frac{C_1}{C_2}\right)}} \tag{19-27}$$

If we denote

$$\frac{1-e^{-\frac{kF}{C_1}\left(1-\frac{C_1}{C_2}\right)}}{1-\frac{C_1}{C_2}e^{-\frac{kF}{C_1}\left(1-\frac{C_1}{C_2}\right)}}=f_z\left(\frac{kF}{C_1},\ \frac{C_1}{C_2}\right)$$

formulae (19-26) and (19-27) become:

$$\delta t_1=(t_1'-t_2')\,f_z\left(\frac{kF}{C_1},\ \frac{C_1}{C_2}\right) \tag{19-26'}$$

and

$$\delta t_2=(t_1'-t_2')\frac{C_1}{C_2}\,f_z\left(\frac{kF}{C_1},\ \frac{C_1}{C_2}\right) \tag{19-27'}$$

The quantity of heat transferred with counter-flow is

$$Q=C_1\delta t_1=C_1(t_1'-t_2')\,f_z\left(\frac{kF}{C_1},\ \frac{C_1}{C_2}\right) \tag{19-28}$$

and with parallel-flow is:

$$Q=C_1\delta t_1=C_1(t_1'-t_2')\,f_p\left(\frac{kF}{C_1},\ \frac{C_1}{C_2}\right) \tag{19-29}$$

Let us consider some special cases.

The water equivalents of the two fluids are numerically equal, i.e. $C_1 = C_2 = C$. Equations (19-24) and (19-25) are simplified and acquire the following form:

$$\delta t_1=\delta t_2=(t_1'-t_2')\frac{1-e^{-2\frac{kF}{C}}}{2} \tag{19-30}$$

For counter-flow, when the water equivalents $C_1 = C_2 = C$, it follows from Eq. (19-14) that $\Delta t' = \Delta t'' = \text{const}$.

From the heat balance equation it follows that

$$t_1'-t_1''=t_2''-t_2'$$

and this leads to:

$$t_2''=t_1'-t_2''+t_2' \tag{f}$$

From the heat-transfer and heat-balance equations we obtain:

$$t_2''=t_1'-\frac{C}{kF}(t_1'-t_1'') \tag{g}$$

Equating (f) and (g) and solving for δt_1, we find that

$$\delta t_1=\frac{t_1'-t_2'}{1+\frac{C}{kF}}=\Delta t_{max}\frac{kF}{kF+C} \tag{19-31}$$

$$Q=(t_1'-t_1'')\,C=(t_1'-t_2')\,C\frac{kF}{kF+C} \tag{19-32}$$

Practical cases of this kind are frequently encountered, e.g. heat transfer of mass consumption of the first and second carriers C_1 and C_2 is infinite (in steam generators, evaporators, condensers of various types, etc.). The temperature of one of the heat carriers here remains constant over entire heat-transfer surface, and the parallel-flow and counter-flow arrangements become equivalent.

If the water equivalent of the hot fluid $C_1 = \infty$, and of the cold fluid is C_2, then we derive a calculation formula for the temperature variation of the cold fluid along the heat-transfer surface from Eq. (19-14):

$$t_2'' = t_1 - (t_1 - t_2')\, e^{-\frac{kF}{C_2}} \tag{19-33}$$

Accordingly, if $C_2 = \infty$, and C_1 is finite, then at $C_2 = \infty$ $t_2' = t_2'' = t_2 =$ $=$ const (boiling liquid), and from Eq. (19-14) we obtain the following formula suitable for calculating the temperature variation of the hot fluid along the heat-transfer surface:

$$t_1'' = t_2 + (t_1' - t_2)\, e^{-\frac{kF}{C_1}} \tag{19-34}$$

To calculate the quantity of heat transferred through the surface of heat exchange, we obtain the following formulae:

$$C_1 = \infty; \quad t_1' = t_1'' = t_1$$

$$Q = C_2 (t_1 - t_2') \left(1 - e^{-\frac{kF}{C_2}}\right) \tag{19-35}$$

$$C_2 = \infty; \quad t_2' = t_2'' = t_2$$

$$Q = C_1 (t_1' - t_2) \left(1 - e^{-\frac{kF}{C_1}}\right) \tag{19-36}$$

The variation in temperatures t_1 and t_2 for intermediate sections of the surface in parallel-flow is readily determined from Eqs. (19-24) and (19-25), if we replace the total heat-transfer surface F in these equations by F_x, measured from the inlet side of the heat exchanger.

When determining intermediate temperatures in counter-flow heat exchangers, the total heat-transfer surface F cannot be replaced by a section of it, F_x, in Eqs. (19-26) and (19-27), because with a counter-flow arrangement the entire surface is involved in heat transfer. Therefore, when calculating the temperature of the hot fluid, F is replaced by F_x in Eq. (19-26) only in the numerator, i.e.

$$\delta t_{1x} = (t_1' - t_2') \frac{1 - e^{-\frac{kF_x}{C_1}\left(1 - \frac{C_1}{C_2}\right)}}{1 - \frac{C_1}{C_2} e^{-\frac{kF}{C_1}\left(1 - \frac{C_1}{C_2}\right)}} \tag{19-37}$$

The variation in temperature of the cold fluid in a counter-flow exchanger is described by formula:

$$\delta t_{2x} = (t_1' - t_2') \frac{C_1}{C_2} \frac{1 - e^{-\frac{kF_x}{C_1}\left(1 - \frac{C_1}{C_2}\right)}}{1 - \frac{C_1}{C_2} e^{-\frac{kF}{C_1}\left(1 - \frac{C_1}{C_2}\right)}} \tag{19-38}$$

(c) *Comparison of Parallel-flow and Counter-flow*

To evaluate the merits of one flow arrangement over another, it is sufficient to compare the rates of heat-transfer in a parallel- and counter-flow heat exchangers, other conditions being equal. In Fig. 19-5 [19.2] the ratio of Q_p, the quantity of heat transferred in a parallel-flow exchanger, to Q_z, the quantity of heat

transferred in a counter-flow apparatus, is plotted as a function of $\frac{C_1}{C_2}$ and $\frac{kF}{C_1}$, namely:

$$\frac{Q_p}{Q_z} = \varphi\left(\frac{C_1}{C_2}, \frac{kF}{C_1}\right)$$

From the graph (Fig. 19-5) it follows that counter- and parallel-flow can be equivalent only with very large or very small values of $\frac{C_1}{C_2}$, or at small values of the parameter $\frac{kF}{C_1}$.

The first condition is characteristic of cases in which the temperature of one of the heat carriers changes little.

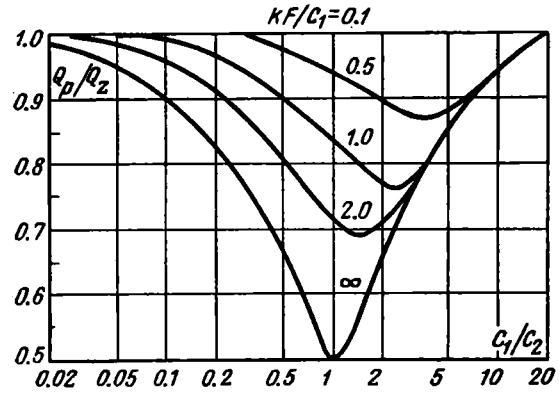

Fig. 19-5. $\frac{Q_p}{Q_z} = f\left(\frac{C_1}{C_2}, \frac{kF}{C_1}\right)$ — comparison of parallel- and counter-flow

The second case is that in which the temperature drop is large in comparison with the variation in temperature of the working fluid. This follows from the relationship:

$$\frac{kF}{C_1} = \frac{\delta t_1}{\Delta t}$$

or

$$\frac{kF}{C_2} = \frac{\delta t_2}{\Delta t}$$

In all other cases less heat is transferred in a parallel-flow than in a counter-flow, other conditions being equal, as is clear from Fig. 19-5. Preference therefore should be given to counter-flow over parallel-flow exchangers.

It must be borne in mind, however, that if one of the fluids is at a high temperature, then the heat-exchange surface of a counter-flow exchanger will be subject to heavier temperature conditions than that of parallel-flow equipment.

19-5. Calculation Methods of Heat-transfer Surface Temperature

When the distribution of heat flux over the heat-transfer surface is known, heating-surface temperature can be calculated by formulae (2-29), (2-33), (2-47), and (2-57). In most practical applications however, it is more convenient to express them in different form.

Proceeding from equations

$$q = \alpha_1 (t_1 - t_{w1}) \tag{a}$$

$$q = \frac{\lambda}{\delta} (t_{w1} - t_{w2}) \tag{b}$$

$$q = \alpha_2 (t_{w2} - t_2) \tag{c}$$

for a plain wall we can obtain:

$$\frac{t_1 - t_{w1}}{t_{w2} - t_2} = \frac{\alpha_2}{\alpha_1} \tag{d}$$

Simultaneous solution of equations (a) and (b) gives:

$$t_{w2} - t_{w1} = \frac{\alpha_1 \delta}{\lambda} (t_1 - t_{w1}) \tag{e}$$

Solving the simultaneous equations (d) and (e) for the unknown temperature t_{w1} or t_{w2}, we obtain:

$$t_{w1} = \frac{t_1\left(\frac{\alpha_1\delta}{\lambda} + \frac{\alpha_1}{\alpha_2}\right) + t_2}{1 + \frac{\alpha_1}{\alpha_2} + \frac{\alpha_1\delta}{\lambda}} \tag{19-39}$$

$$t_{w2} = \frac{t_2\left(\frac{\alpha_2\delta}{\lambda} + \frac{\alpha_2}{\alpha_1}\right) + t_1}{1 + \frac{\alpha_2}{\alpha_1} + \frac{\alpha_2\delta}{\lambda}} \tag{19-40}$$

The formulae derived are also suitable for calculating the surface temperature of a composite heat-transfer wall. For a flat composite wall, substituted in Eqs. (19-39) and (19-40) are δ, the total thickness, and λ, the equivalent thermal conductivity of the composite wall. If the thermal resistance of the composite wall $\frac{\delta}{\lambda}$ can be ignored ($\delta \to 0$ or $\lambda \to \infty$), then Eqs. (19-39) and (19-40) acquire the form:

$$t_{w1} = t_{w2} = \frac{\alpha_1 t_1 + \alpha_2 t_2}{\alpha_1 + \alpha_2} \tag{19-41}$$

When applied to walls of any thermal resistance, calculation according to Eq. (19-40) gives the mean temperature of the wall.

For thin cylindrical walls ($d_2/d_1 < 2$) the following relationships hold:

$$Q = \alpha_1 F_1 (t_1 - t_{w1})$$

$$Q = \frac{\lambda}{\delta}(t_{w1} - t_{w2}) F_m$$

$$Q = \alpha_2 F_2 (t_{w2} - t_2)$$

where F_1 = surface on the side of the hot medium;

F_m = mean wall temperature, equal to $\frac{F_1 + F_2}{2}$;

F_2 = surface on the side of the cold medium.

As with a plain wall, we find

$$t_{w1} = \frac{\left(\frac{\alpha_1 F_1}{\alpha_2 F_2} + \frac{\alpha_1 F_1 \delta}{\lambda F_m}\right) t_1 + t_2}{1 + \frac{\alpha_1 F_1}{\alpha_2 F_2} + \frac{\alpha_1 F_1 \delta}{\lambda F_m}} \tag{19-42}$$

$$t_{w2} = \frac{\left(\frac{\alpha_2 F_2}{\alpha_1 F_1} + \frac{\alpha_2 F_2 \delta}{\lambda F_m}\right) t_2 + t_1}{1 + \frac{\alpha_2 F_2}{\alpha_1 F_1} + \frac{\alpha_2 F_2 \delta}{\lambda F_m}} \tag{19-43}$$

If the wall is of composite construction, its total thickness δ and the equivalent thermal conductivity λ are substituted in Eqs. (19-42) and (19-43).

In general the surface temperature of a cylindrical wall is calculated by the following formulae:

$$t_{w1} = \frac{\left(\frac{\alpha_1 F_1}{\alpha_2 F_2} + \frac{\alpha_1 F_1}{2\pi l\lambda} \ln \frac{d_2}{d_1}\right) t_1 + t_2}{1 + \frac{\alpha_1 F_1}{\alpha_2 F_2} + \frac{\alpha_1 F_1}{2\pi l\lambda} \ln \frac{d_2}{d_1}} \tag{19-44}$$

$$t_{w2} = \frac{\left(\frac{\alpha_2 F_2}{\alpha_1 F_1} + \frac{\alpha_2 F_2}{2\pi l\lambda} \ln \frac{d_2}{d_1}\right) t_2 + t_1}{1 + \frac{\alpha_2 F_2}{\alpha_1 F_1} + \frac{\alpha_2 F_2}{2\pi l\lambda} \ln \frac{d_2}{d_1}} \tag{19-45}$$

In dealing with a tube with a compositive wall, λ must be replaced in Eqs. (19-44) to (19-45) by the equivalent thermal conductivity; F_1, F_2 are respectively the surfaces which are in direct contact with the heat-carriers.

19-6. Calculation of Regenerators

The heating surface of regenerative heat exchangers is alternatively exposed to the primary ("hot") and to the secondary ("cold") heat-carrier. Hence, it serves alternately as a heat-absorbing and a heat-supplying surface. The interval of time τ_1 during which the packing is heated and the hot fluid is cooled is called the heating period, and the time τ_2 during which it cools and the cold fluid heats, the cooling period. The interval between the beginning of one heating period and the start of the next, i.e. the duration of heating and cooling of the regenerator packing is known as the total period or cycle, which we denote as τ_{tp}

$$\tau_{tp} = \tau_1 + \tau_2$$

In general the two periods may be of different duration. Unlike recuperative exchangers, regenerators have a transient heat transfer process, i.e. both the temperatures of the wall and of the fluids vary in time during the periods of heating and cooling. This is one of the difficulties complicating analytical calculation of regenerators. Practical calculation of regenerators is presently based on approximate methods.

The peculiarities of the calculations consist in their being based on mean characteristics, averaged over the total period (cycle). The heat flux is reduced not to unit time, but to the period. Hence Q is the quantity of heat transferred during the heating or cooling period per unit area, in joules per square metre per period.

The calculations are based on the equations used for recuperators. The heat balance equation is practically the same as the equations used for recuperators, and the heat-transfer equation has the form

$$Q = k_{per}(\bar{t}_1 - \bar{t}_2) \tag{19-46}$$

where k_{per} = overall heat-transfer coefficient for the heating and cooling period, J/m^2·K·period;

$\bar{t}_1$ = mean temperature of hot fluid during the heating period;

$\bar{t}_2$ = mean temperature of cold fluid during the cooling period; Q is in J/m^2·period.

From Eq. (19-46) it follows that the main difficulty encountered in finding the heat-transfer surface resolves into the calculation of the overall heat-transfer coefficient.

Calculation of the heat-transfer coefficient. In calculation of the heat-transfer coefficient we shall follow the method described in [19.3]. The quantity of heat, J/m^2·period, to be transferred to unit area during the period of heating the packing is

$$Q_1 = \alpha_1 \tau_1 (\bar{t}_1 - \bar{t}_{w1}) \tag{19-47}$$

where α_1 = overall heat-transfer coefficient during the heating period;

τ_1 = duration of the heating period;

$\bar{t}_1$ = mean temperature of the hot fluid at the given cross-section during the heating period;

$\bar{t}_{w1}$ = mean surface temperature at the given cross-section during the heating period.

The quantity of heat imparted to the cold fluid during the cooling period is

$$Q_2 = \alpha_2 \tau_2 (\bar{t}_{w2} - \bar{t}_2) \tag{19-48}$$

where α_2 = overall heat-transfer coefficient during the cooling period;

$\bar{t}_{w2}$ = mean surface temperature at the given cross-section during the cooling period;

$\bar{t}_2$ = mean temperature of the cold fluid during the cooling period;

τ_2 = duration of cooling period.

Under steady-state operation (cycle of operation) and in the absence of heat losses to the surroundings, the quantity of heat transferred during the heating period must be equal to the quantity of heat imparted to the cold fluid during the cooling period. For that reason, we can write:

$$\alpha_1 \tau_1 (\bar{t}_1 - \bar{t}_{w1}) = \alpha_2 \tau_2 (\bar{t}_{w2} - \bar{t}_1) \tag{19-49}$$

The difference between the mean temperature of the heating surface during the two periods $\bar{t}_{w1} - \bar{t}_{w2}$ is denoted by $\Delta\bar{t}_w$. We can then write:

$$\bar{t}_{w2} = \bar{t}_{w1} - \Delta\bar{t}_w \tag{19-50}$$

Substituting Eq. (19-50) in Eq. (19-49), and solving for $\bar{t}_{w1}$, we obtain:

$$\bar{t}_{w1} = \frac{\alpha_1 \tau_1 \bar{t}_1 + \alpha_2 \tau_2 \bar{t}_2 + \alpha_2 \tau_2 \Delta\bar{t}_w}{\alpha_1 \tau_1 + \alpha_2 \tau_2} \tag{19-51}$$

Equating (19-46) and (19-47), and substituting $\bar{t}_{w1}$ from (19-51) in the result, we have:

$$k_{per} = \frac{1}{\frac{1}{\alpha_1 \tau_1} + \frac{1}{\alpha_2 \tau_2}} \left(1 - \frac{\bar{t}_{w1} - \bar{t}_{w2}}{\bar{t}_1 - \bar{t}_2}\right) \tag{19-52}$$

where k_{per} is in J/m^2·K·period.

If we assume the durations of the two periods to be unity, and assume that the difference in surface temperatures $\bar{t}_{w1} - \bar{t}_{w2} = \Delta\bar{t}_w = 0$, we get from Eq. (19-52):

$$k_{per} = \frac{1}{\frac{1}{\alpha_1} + \frac{1}{\alpha_2}} \tag{19-53}$$

Equation (19-53) is analogous to the equation of the heat-transfer coefficient for a recuperator. Therefore, in the case considered, the formulae for the period-averaged temperatures and period-averaged rates of heat transfer of recuperators also hold for regenerators.

If, in Eq. (19-52), we assume that $\Delta\bar{t}_w = 0$ and allow for arbitrary durations of the heating and cooling periods τ_1 and τ_2, we obtain the formula for the heat-transfer coefficient of so-called "ideal regenerators":

$$k_{id} = \frac{1}{\frac{1}{\alpha_1 \tau_1} + \frac{1}{\alpha_2 \tau_2}} \tag{19-54}$$

An ideal regenerator is one in which the mean surface temperature of the packing remains constant during both periods.

From Eq. (19-52) it follows that, if the local heat-transfer coefficients α_1 and α_2 are known (they can be calculated by the formulae already presented), calculation of a regenerator reduces itself to calculation of the mean surface temperature of the packing during the heating period $\bar{t}_{w1}$ and during the cooling period $\bar{t}_{w2}$.

Thus, the calculation of the heat-transfer coefficient for any regenerator can be based on Eq. (19-54), and the performance of the heat exchanger can be judged by the difference between its heat-transfer coefficient and that of an ideal regenerator.

The following formula can be recommended for practical calculations:

$$k = k_{id}\frac{u}{v} \tag{19-55}$$

Here

$$u = \frac{c\delta\rho}{2\alpha_1\tau_1}\left(1-e^{-\frac{2\alpha_1\tau_1}{c\delta\rho}}\right)+9.82\frac{\Delta\vartheta_1+\Delta\vartheta_2}{\bar{t}_1-\bar{t}_2}\times$$

$$\times\left\{1.05\left[1-\frac{c\delta\rho}{2\alpha_1\tau_1}\left(1-e^{-\frac{2\alpha_1\tau_1}{c\delta\rho}}\right)\right]-\frac{1.05-\frac{c\delta\rho}{2\alpha_1\tau_1}\left(1-e^{-\frac{2\alpha_1\tau_1}{c\delta\rho}}\right)}{1+0.1\frac{c\delta\rho}{2\alpha_1\tau_1}}\right\} \tag{19-56}$$

$$v = 1-\frac{\alpha_1\tau_1}{c\delta\rho}\left[\frac{c\delta\rho}{2\alpha_1\tau_1}\left(1-e^{-\frac{2\alpha_1\tau_1}{c\delta\rho}}\right)-\frac{c\delta\rho}{2\alpha_1\tau_1}\left(1-e^{-\frac{2\alpha_1\tau_1}{c\delta\rho}}\right)\times\right.$$

$$\left.\times\frac{1-\eta}{\eta}\left(1+\frac{2}{\frac{4c\delta\rho}{\alpha_1\tau_1}-1}\right)+\frac{1-\eta}{4\eta}\left(1+\frac{1}{\frac{4c\delta\rho}{\alpha_1\tau_1}-1}\right)\right] \tag{19-57}$$

k is in J/m^2·K·period.

The factor $\frac{u}{v}$ in Eq. (19-55) indicates a deterioration of the ideal heat-transfer coefficient owing to insufficient thermal conductivity and accumulating capacity in the packing. This ratio characterizes the efficiency of the heating surface. Consequently,

$$\eta_{hs} = \frac{k}{k_{id}} \tag{19-58}$$

or

$$\eta_{hs} = \frac{u}{v} \tag{19-59}$$

The value u is a function only of the dimensionless terms $\frac{2\alpha_1\tau_1}{c\delta\rho}$ and $\frac{\Delta\vartheta_1+\Delta\vartheta_2}{\bar{t}_1-\bar{t}_2}$, and the value v is a function of $\frac{2\alpha_1\tau_1}{c\delta\rho}$ and η.

In the dimensionless terms used:

α_1 = local heat-transfer coefficient for the heating period;
τ_1 = duration of the heating period;
c = specific heat;
δ = thickness of the regenerator packing;
ρ = density of the regenerator packing;
$\Delta\vartheta_1 = t'_1 - t'_{10}$ = variation in inlet temperature of the hot fluid;
$\Delta\vartheta_2 = t'_2 - t'_{20}$ = variation in inlet temperature of the cold fluid;
η = packing utilization factor.

Thus, we can write:

$$u = f_1\left(\frac{2\alpha_1\tau_1}{c\delta\rho};\ \frac{\Delta\vartheta_1+\Delta\vartheta_2}{\bar{t}_1-\bar{t}_2}\right) \tag{19-60}$$

and

$$v = f_2\left(\frac{2\alpha_1\tau_1}{c\delta\rho};\ \eta\right) \tag{19-61}$$

Relationships (19-60) and (19-61) are shown graphically in Figs. 19-6 and 19-7 as a family of curves.

Following the treated procedure, therefore, equations (19-55), (19-56), and

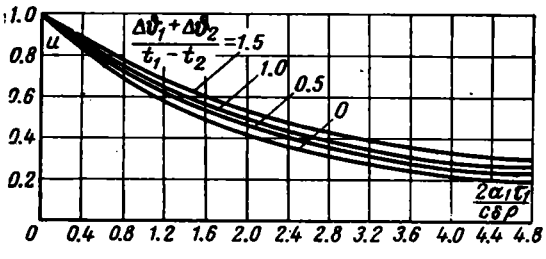

Fig. 19-6. Graph for determining u

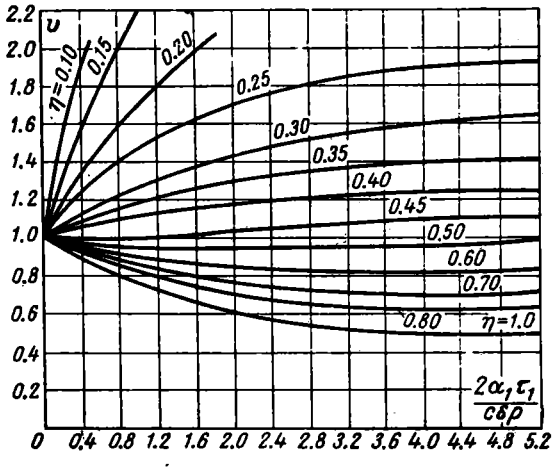

Fig. 19-7. Graph for determining v

(19-57) make it possible to calculate the heat-transfer coefficient for any regenerator.

Further calculation of regenerators is carried out by the formulae previously derived for recuperators.

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CHAPTER

20
HYDROMECHANICAL DESIGN OF HEAT EXCHANGERS

20-1. Object of Hydromechanical Design of Heat Exchangers

There is a close physical and economic link between heat transfer and pressure drop. The higher the velocity of the heat carrier, the larger is the heat-transfer coefficient, and the more compact the heat exchanger for a given heat capacity, and, hence, the smaller the capital outlay. Higher flow velocities, however, are accompanied by an increase in the resistance to flow, and a rise in operating expenditure. In designing heat exchangers, therefore, the problems of heat transfer and pressure drop, or hydraulic resistance to flow, must be solved together, so as to find the most advantageous characteristics.

The main problem of hydromechanical design is to determine the pressure drop of the heat carrier, as it flows through the apparatus. Since heat transfer and pressure drop are invariably linked with the velocity of the heat carrier, the last must be chosen within a certain optimum range, determined, on the one hand, by the cost of the heating surface, and, on the other hand, by the cost of the power consumed by the heat exchanger in operation.

Hydraulic pressure in heat exchangers is determined by the conditions of heat carriers and by the peculiarities of apparatus design.

The drop in pressure across a heat exchanger is determined by the flow arrangements chosen, and the design features of the apparatus.

From the foregoing it follows that the data of hydromechanical calculation are an important factor in deciding whether the design of the heat-exchange equipment is rational.

20-2. Hydraulic Pressure of Elements of a Heat Exchanger

The pattern of flow of a heat carrier has been shown experimentally to be very complex, even in the simplest heat exchangers. Owing to this, only approximate calculation of pressure drop is usually possible in most cases.

Depending on how the flow originates, the hydraulic resistance offered to the flowing heat carriers is considered to be made up of *flow friction*, determined by the viscosity of fluid and manifesting itself only in places where continuous flow occurs, and *local friction* (or end effects). The latter is brought about by various local obstacles to flow, such as sudden contraction and enlargement of the passage, turns, etc. This is true for isothermal flow; but if flow of the heat carrier is accompanied by heat transfer and the apparatus is in free communication with the surroundings, additional resistances appear, connected with *acceleration* of flow due to its non-isothermal nature, and from the presence of the self-draught effect. *Self-draught resistance* to flow appears because buoyancy, directed upwards, interferes with fluid flow in the descending sections of the channel.

Thus, the total drop in pressure experienced as a liquid or gas flow through a heat exchanger is determined by formula

$$\Delta p = \sum \Delta p_f + \sum \Delta p_{eef} + \sum \Delta p_a + \sum \Delta p_{sd} \qquad (20\text{-}1)$$

where $\sum\Delta p_f$ = total friction on all sections of the heat transfer surface (channels, banks of tubes, walls, etc.);

$\sum\Delta p_{eef}$ = total pressure drop due to end effects;

$\sum\Delta p_a$ = total pressure drop due to acceleration of flow;

$\sum\Delta p_{sd}$ = summary pressure drop due to self-draught. Δp is in N/m^2.

Since the pressure-drop components present in Eq. (20-1) are of different origin, they are calculated individually. The pressure drop through channels along the sections of non-separated and continuous flow is calculated in the general case by the following formula:

$$\Delta p_f = \xi \frac{l}{d} \frac{\rho w^2}{2} \tag{20-2}$$

where l = total length of the channel;

d = hydraulic or equivalent diameter, which in the general case is determined as $d = \frac{4f}{u}$ (f is the cross-section of the channel; u is the perimeter of the cross-section);

ρ = mean viscosity of the liquid or gas filling the channel, kg/m^3;

w = mean velocity of flow, m/s;

ξ = friction factor; it is dimensionless, and characterizes the ratio of friction to flow inertia.

The *friction factor* remains constant in channels with $l > 30d$ and when $l < 30d$ it is necessary to take into account its change at the inlet of the channel; Δp_f is measured in Pa. It depends on the pattern of fluid flow, and is therefore determined differently for turbulent and laminar flow. The regularities of flow are treated in various textbooks on hydraulics and hydroaerodynamics [20.1; 20.2]. In this chapter we shall restrict ourselves to the brief data required for pressure-drop and friction calculations.

For viscous non-isothermal flow in tubes and channels the formula suggested by Petukhov [20.3] can be recommended, by which

$$\xi = \frac{64\varphi}{\mathrm{Re}_1} \left(\frac{\mu_w}{\mu_1}\right)^n = \xi_{is} \left(\frac{\mu_w}{\mu_1}\right)^n \tag{20-3}$$

where φ is the coefficient accounting for the geometrical shape of the conduit, chosen in accordance with data for isothermal flow (for a round tube $\varphi = 1.0$; for a flat channel $\varphi = 1.5$); for other cross-sectional shapes φ can be found in [20.4].

The values for the exponent n in Eq. (20-3) are selected from the table below, or calculated by formula:

$$n = c \left(\mathrm{Pe}_1 \frac{d}{l}\right)^{-m} \left(\frac{\mu_w}{\mu_1}\right)^{-0.062} \tag{20-4}$$

where $c = 2.3$, and $m = 0.3$ at $60 < \mathrm{Pe}_1 \frac{d}{l} < 1500$;

$c = 0.535$, and $m = 0.1$ at $1500 < \mathrm{Pe}_1 \frac{d}{l} < 3 \times 10^4$.

The Reynolds and Peclet numbers Re_1 and Pe_1 are calculated by the equivalent diameter, and all physical parameters are based on passage inlet temperature. The viscosity coefficients μ_w and μ_1 are based respectively on wall temperature and fluid inlet temperature.

$Pe_1 \frac{d}{l}$	n at μ_c/μ_1 equal to					$Pe_1 \frac{d}{l}$	n at μ_c/μ_1 equal to				
	0.1	1	10	100	1000		0.1	1	10	100	1000
60	0.776	0.673	0.583	0.505	0.440	1 000	0.334	0.290	0.251	0.218	0.189
100	0.667	0.578	0.502	0.435	0.377	1 500	0.297	0.258	0.224	0.194	0.168
150	0.590	0.512	0.446	0.385	0.334	2 500	0.283	0.245	0.212	0.184	0.160
200	0.541	0.470	0.406	0.352	0.306	5 000	0.264	0.228	0.198	0.172	0.149
400	0.440	0.382	0.330	0.286	0.248	10 000	0.246	0.213	0.185	0.160	0.139
600	0.390	0.338	0.292	0.254	0.220	30 000	0.220	0.191	0.165	0.144	0.125

Formula (20-3) holds for liquids at $0.08 < \mu_w/\mu_1 < 1200$ and $60 < Pe_1 \frac{d}{l} < 3 \times 10^4$. Since the exponent n is always positive, it follows from it that, other things being equal, the friction factor ξ will be different during heating and cooling and will vary from the values suitable for isothermal flow. The friction factor of a cooled flow is always larger than that of a heated one.

With turbulent non-isothermal flow of an incompressible fluid in tubes and channels the pressure drop is determined by (20-2).

Calculation of the friction factor is based on formulae derived for isothermal flow. The effect of non-isothermal flow can be accounted for by means of the correction $(Pr_w/Pr_f)^{1/3}$ suggested by Mikheev [20.4]. Then, the formula for calculation of the friction factor of non-isothermal flow will be written as

$$\xi = \xi_{is} (Pr_w/Pr_f)^{1/3} \tag{20-5}$$

The friction factor for isothermal flow, ξ_{is}, in Eq. (20-5) can be calculated by the known formula:

$$\xi_{is} = \frac{1}{(1.82 \log Re_f - 1.64)^2} \tag{20-6}$$

Taking this into account, Eq. (20-5) acquires the following form:

$$\xi = \frac{1}{(1.82 \log Re_f - 1.64)^2} (Pr_w/Pr_f)^{1/3} \tag{20-5'}$$

All the physical parameters in Eq. (20-5') relate to the mean temperature of fluids, except the Pr_w term which is chosen for wall temperature. The equivalent diameter of the channel d_{eq} is taken as the linear reference or characteristic dimension.

The results of many experimental studies of friction resistance to gas flow in tubes and channels indicate that if the physical parameters are based on the length-averaged gas temperature, the friction resistance to non-isothermal flow can be calculated by the same formulae as for isothermal flow:

for laminar flow, in accordance with the Poiseuille law:

$$\xi = \frac{64\varphi}{Re_f} \tag{20-7}$$

and for turbulent flow in tubes and channels, by formula (20-6).

As already mentioned, *end effects* (*local friction*) are caused by components of heat exchangers that change the direction or shape of flow (turns, sudden contractions, enlargements of cross-section, etc.).

End effects are calculated by the formula:

$$\Delta p_{eef} = \zeta \frac{\rho w^2}{2} \qquad (20\text{-}8)$$

where ζ is the local friction factor; Δp_{eef} is in N/m².

The local friction factor depends on the nature of the obstacle that causes the end effects. Formulae and numerical data for the calculation of end effects in various pressure-flow systems can be found in handbooks [20.4, Chapter 5].

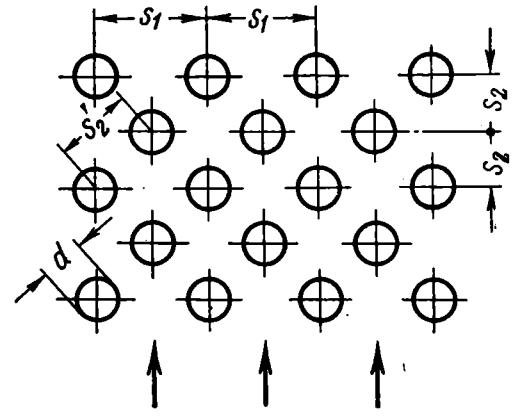

Fig. 20-1. Determining pressure drop through a staggered bank of tubes

Pressure drop across banks of tubes in cross flow must be considered as the sum of drops due to friction and to end effects. The total pressure drop through a bank of tubes is determined by Eq. (20-8), since the drop due to friction is only a small fraction of the end effects. The local friction factor is then determined by formulae suggested by the Moscow Heat Engineering Research Institute [20.5].

For banks of staggered tubes:

at $\dfrac{1-\dfrac{d}{s_2'}}{\dfrac{s_1}{d}-1} \leqslant 0.53$

$$\mathrm{Eu} = 1.4\,(z+1)\,\mathrm{Re}^{-0.25} \qquad (20\text{-}9)$$

at $\dfrac{1-\dfrac{d}{s_2'}}{\dfrac{s_1}{d}-1} > 0.53$

$$\mathrm{Eu} = 1.93\,(z+1)\sqrt{\frac{1-\dfrac{d}{s_2'}}{\dfrac{s_1}{d}-1}}\,\mathrm{Re}^{-0.25} \qquad (20\text{-}10)$$

In formulae (20-9) and (20-10) s_1 is the transverse pitch and s_2' the diagonal pitch of the bank (Fig. 20-1).

For banks of in-line tubes:

at $\dfrac{\dfrac{s_2}{d}-0.8}{\dfrac{s_1}{d}-1} \leqslant 1$

$$\mathrm{Eu} = 0.265\left(\frac{\dfrac{s_2}{d}-0.8}{\dfrac{s_1}{d}-1}\right)^{2.5} z\mathrm{Re}^{m} \qquad (20\text{-}11)$$

at $\dfrac{\frac{s_2}{d}-0.8}{\frac{s_1}{d}-1}>1$

$$\mathrm{Eu}=0.265\left(\frac{\frac{s_2}{d}-0.8}{\frac{s_1}{d}-1}\right)^2 z\mathrm{Re}^m \qquad (20\text{-}12)$$

The exponent m of formulae (20-11) and (20-12) can be calculated as follows:

at $\frac{s_2}{d}\geqslant 1.24$

$$m=0.88\left(\frac{\frac{s_1}{d}-1}{\frac{s_2}{d}-1}-0.1\right)^{0.138}-1$$

at $\frac{s_2}{d}\leqslant 1.24$

$$m=0.88\left(\frac{\frac{s_2}{d}}{1.24}\right)^{0.7}\left(\frac{\frac{s_1}{d}-1}{\frac{s_2}{d}-1}-0.1\right)^{0.138}-1$$

In these equations s_1 and s_2 are respectively the transverse and longitudinal pitches of in-line banks of tubes (Fig. 20-2).

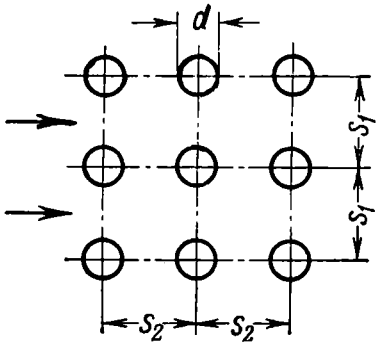

Fig. 20-2. Determining pressure drop through an in-line bank of tubes

The Euler number in formulae (20-9) to (20-12) is

$$\mathrm{Eu}=\frac{\Delta p}{\rho w^2}$$

where Δp = pressure drop across the bank of tubes, N/m²;
ρ = gas density at the mean temperature of the gas in the bank, kg/m³;
w = mean flow velocity at the narrow passage of the bank, m/s.

The formulae are true at $6\times10^3<\mathrm{Re}<6\times10^4$, and with the parameter $\frac{1-d/s_2'}{s_1/d-1}$ ranging from 0.25 to 2.5 for staggered banks of tubes and the $\frac{s_2/d-0.8}{s_1/d-1}$ term varying from 0.2 to 6.5 for in-line banks.

The drop in pressure caused by acceleration of flow due to a change in the volume of the heat carrier, with a constant cross-section of the channel, is

$$\Delta p_a=\rho_2 w_2^2-\rho_1 w_1^2 \qquad (20\text{-}13)$$

where w_1, ρ_1, and w_2, ρ_2 are the velocity, in m/s, and gas density, in kg/m³, respectively at the inlet and outlet cross-section of flow.

For liquids Δp_a, Pa, is small, compared with the total resistance to flow, and can be ignored.

If the heat exchanger is open to the surroundings, the effect of *self-draught* must be taken into account. The drop in pressure due to self-draught can be calculated by formula

$$\Delta p_{sd} = \pm g(\rho_0 - \rho) h \tag{20-14}$$

where h = distance along the vertical between the inlet and outlet of the heat carrier, m;
ρ and ρ_0 = mean densities of the heat carrier and ambient air, in kg/m^3; Δp_{sd} is in N/m^2.

The pressure drop is positive for fluid in descending flow, and negative for fluid in ascending flow. Therefore for fluid in descending flow the total pressure drop is augmented by Δp_{sd} and in ascending flow is reduced by Δp_{sd}. The pressure drop due to self-draught is zero, if the heat exchanger is not exposed to ambient air (connected in a closed system).

To calculate the complete pressure drop through a heat exchanger of chosen construction, the components $\sum\Delta p_f$, $\sum\Delta p_{eef}$, $\sum\Delta p_a$ and $\sum\Delta p_{sd}$ must be substituted in Eq. (20-1).

20-3. Calculation of Power Required to Originate Fluid Flow

The pressure drop through a given heat exchanger Δp calculated by Eq. (20-1) determines the power required to originate flow of the heat carrier.

The power, N, W, to be developed by the motor of the pump or fan is determined by formula

$$N = \frac{V\Delta p}{\eta} = \frac{G\Delta p}{\rho\eta} \tag{20-15}$$

where V = volume rate of fluid flow, m^3/s;
G = mass rate of fluid flow, kg/s;
Δp = total pressure drop, N/m^2;
ρ = density of the liquid or gas, kg/m^3;
η = efficiency of the pump or fan.

In selecting the optimum shape and size of the heating surface of a heat exchanger, preference is given to the most advantageous relation between the area of the surface and the power consumption, i.e. the power needed to originate flow. The tendency is to aim for the optimum relationship, which is established by technical and economic calculations.

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20.5. Kuznetsov, N. V., et al. *Teploenergetika*, No. 9, 1954.

APPENDICES

TABLE 1

International System of Units (SI)

Quantity	Unit of measurement	Reduced designation of unit
	Fundamental units	
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Current	Ampere	A
Thermodynamic temperature	Kelvin	K
Candle power	Candle	cd
	Some derived units	
Area	Square meter	m^2
Volume	Cubic meter	m^3
Velocity	Meter per second	m/s
Acceleration	Meter per second per second	m/s^2
Density	Kilogram per cubic meter	kg/m^3
Force	Newton	N (kgm/s^2)
Pressure	Newton per square meter	N/m^2
Dynamic viscosity	Newton-second per square meter	$N \cdot s/m^2$
Kinematic viscosity	Square meter per second	m^2/s
Work, energy, and heat	Joule	J (Nm)
Power, and heat flux	Watt	W (J/s)
Specific heat	Joule per kilogram-degree	J/kgK
Enthalpy	Joule per kilogram	J/kg
Rate of heat flow	Watt per square meter	W/m^2
Thermal conductivity	Watt per meter-degree K	$W/m \cdot K$
Heat-transfer coefficient	Watt per square meter-degree K	$W/m^2 \cdot K$
Radiation coefficient	To biquadratic power	$W/m^2 \cdot K^4$

TABLE 2

Conversion of MKGCS to SI

Energy	1 kcal = 4.187 kJ
Force	1 kgf = 9.81 N
Specific gravity	1 $kgf/m^3 = 9.81$ N/m^3
Density	1 $kgf \cdot s^2/m^4 = 9.81$ kg/m^3
Pressure	1 $kgf/cm^2 = 9.81$ N/cm^2
Dynamic viscosity	1 $kgf \cdot s/cm^2 = 9.81$ N/m^2
Heat capacity	1 kcal/kg °C = 4.187 kJ/kg·K
Enthalpy	1 kcal/kg = 4.187 kJ/kg
Heat flux	1 kcal/h = 1.163 W
Rate of heat flow	1 $kcal/m^2 \cdot h = 1.163$ W/m^2
Volume rate of heat flow	1 $kcal/m^3 \cdot h = 1.163$ W/m^3
Thermal conductivity	1 kcal/m·h °C = 1.163 W/m^3
Heat-transfer coefficient	1 $kcal/m^2 \cdot h$ °C = 1.163 $W/m^2 \cdot K$
Radiation coefficient	1 $kcal/m^2 \cdot h \cdot K = 1.163$ $W/m^2 \cdot K^4$

TABLE 3

Density ρ, Thermal Conductivity λ and Heat Capacity c of Building, Heat-insulating and Some Other Materials

Material	ρ, kg/m³	t *, °C	λ, W/m·K	c, kJ/kg·K
Aluminium foil at an air layer thickness of 10 mm	—	—	$0.0302+0.85\cdot10^{-4}$	—
Asbestos fibre:				
3d grade	340	—	$0.087+0.24\cdot10^{-3}$	0.816
6th grade	650	—	$0.11+0.19\cdot10^{-3}$	0.816
Asbestos board	900	—	$0.16-0.17\cdot10^{-3}$	0.816
Asbestos cord	800	—	$0.13-0.15\cdot10^{-3}$	0.816
Asbestos slate:				
with high asbestos content	1800	20	0.17-0.35	—
with 10-45% dry asbestos content	1800	20	0.64-0.52	—
Asphalt	2120	0-30	0.60-0.74	1.67
Concrete, crushed stone	2000	0	1.28	0.84
Ditto, dry	1600	0	0.84	—
Reinforced concrete, compacted	2200	0	1.55	0.84
Slag concrete	1500	0	0.70	0.80
Paper, ordinary	—	20	0.14	1.51
Cotton wool	80	30	0.042	—
Gypsum (formed, dry)	1250	20	0.43	0.8-0.92
Clay	2000-1600	20	0.9-0.7	0.84
Fire clay	1845	450	1.04	1.09
Gravel	1840	20	0.36	—
Wood laminate	—	35-70	0.21	—
Wood:				
oak across grain	825	0-15	0.20-0.21	2.39
oak along grain	819	12-50	0.35-0.43	2.39
pine across grain	546	0-50	0.14-0.16	2.72
pine along grain	—	20-25	0.35-0.72	2.72
Coal:				
gas	1420	20-100	3.6-4.0	—
ordinary, solid	1200-1350	20	0.24-0.27	—
Pulverized coal	730	30-150	0.12-0.13	—
Cardboard	—	20	0.14-0.35	1.51
Cambric (varnished)	—	38	0.157	—
Brick:				
common, machine moulded	1800	0	0.77	0.88
common, hand moulded	1700	0	0.70	0.88
lime masonry	1900	0	0.81	0.84

Table 3 (continued)

Material	ρ, kg/m³	t*, °C	λ, W/m·K	c, kJ/kg·K
Common brickwork:				
with cold mortar	1700	0	0.81	0.88
with warm mortar	1600	0	0.67	0.84
Lime brick masonry:				
cold mortar	1900	0	0.87	0.84
warm mortar	1700	0	0.76	0.80
Rubble masonry of medium-density stone	2000	0	1.28	0.88
Carbolite, black			0.231	—
Leather	1150	50	0.14-0.16	—
Coke, powdery	—	20	0.191	1.21
Scale, boiler:	449	100		
gypsum-rich			0.7-2.3	—
lime-rich	2000-2700	100	0.15-2.3	—
silicate-rich	1000-2500	100	0.08-0.23	—
Guartz, crystalline:	300-1200	100		
across axis			0.72	—
along axis			1.94	—
Lampblack	—	0	0.07-0.12	2.26
Ice	—	0	2.2	1.17
Linen fabric	165	40	3.5	—
Magnesium, segment-shaped for pipe insulation	917	0	0.088	—
Chalk	928	—100	0.073-0.084	0.88
Micanite	—	—	0.9	—
Marble	266	50-200	0.21-0.41	0.92
Paraffin			3.5	—
Sand, river, fine (dry)	2000	50	0.27	0.80
Sand, river, fine (damp)			0.30-0.38	2.09
Pressboard	—	20	1.13	—
Plexiglass			0.26-0.22	—
Cork, plate	2800	0	0.184	1.76
Cork fines, 4-5 mm piece si	920	20	0.042-0.053	1.76
Rubber:	1520	0-160	0.044-0.058	
hard, ordinary				1.38
soft	1650	20		1.38
Sugar, gra			.157-0.160	1.26
		20-50	.0.16	

Table 3 (concluded)

Material	ρ, kg/m³	t*, °C	λ, W/m·K	c, kJ/kg·K
Sulphur, rhombic	—	21	0.28	0.762
Slate	—	94	1.49	—
Mica (across layers)	2600-3200	20	0.46-0.58	—
Snow:				
loose	200	—	0.10	2.09
compacted	400	—	0.46	2.09
Glass:				
mirror	2550	0-100	0.78-0.88	0.779
ordinary	2500	20	0.74	0.67
thermometric	2590	20	0.96	—
pyrex	—	0	1.04	—
pyrex	—	400	1.55	—
quartz	—	400	1.76	—
ditto	—	800	2.40	—
ditto	—	1200	3.05	—
Glass wool	154-206	88	0.051-0.059	—
Textolite	1300-1400	20	0.23-0.34	1.46-1.51
Porcelain	2400	95	1.04	1.09
Ditto	2400	1055	1.96	1.09
Fibre, red	1290	20-100	0.46-0.50	—
Fibrolite	360-440	80	0.073-0.128	—
Celluloid	1400	30	0.21	—
Silk	100	0-93	0.043-0.06	—
Ebonite	1200	20	0.157-0.17	—
Slag:				
boiler	1000	0	0.29	0.75
blast-furnace, granulated	500	0	0.15	0.75
Plaster:				
lime	1600	0	0.70	0.84
cement-sand	1800	0	1.2	0.84
Plywood, bonded	600	0	0.15	2.51
Charcoal, lump	190	80	0.074	—

* Temperature for which the properties are given.

TABLE 4

Physical Properties of Dry Air at $p_B = 760$ mm Hg

t, °C	ρ, kg/m³	c_p, kJ/kg·K	$\lambda \cdot 10^2$ W/m·K	$a \cdot 10^6$ m²/s	$\mu \cdot 10^6$ N·s/m²	$\nu \cdot 10^6$ m²/s	Pr
—50	1.584	1.013	2.04	12.7	14.6	9.23	0.728
—40	1.515	1.013	2.12	13.8	15.2	10.04	0.728
—30	1.453	1.013	2.20	14.9	15.7	10.80	0.723
—20	1.395	1.009	2.28	16.2	16.2	12.79	0.716
—10	1.342	1.009	2.36	17.4	16.7	12.43	0.712
0	1.293	1.005	2.44	18.8	17.2	13.28	0.707
10	1.247	1.005	2.51	20.0	17.6	14.16	0.705
20	1.205	1.005	2.59	21.4	18.1	15.06	0.703
30	1.165	1.005	2.67	22.9	18.6	16.00	0.701
40	1.128	1.005	2.76	24.3	19.1	16.96	0.699
50	1.093	1.005	2.83	25.7	19.6	17.95	0.698
60	1.060	1.005	2.90	27.2	20.1	18.97	0.696
70	1.029	1.009	2.96	28.6	20.6	20.02	0.694
80	1.000	1.009	3.05	30.2	21.1	21.09	0.692
90	0.972	1.009	3.13	31.9	21.5	22.10	0.690
100	0.946	1.009	3.21	33.6	21.9	23.13	0.688
120	0.898	1.009	3.34	36.8	22.8	25.45	0.686
140	0.854	1.013	3.49	40.3	23.7	27.80	0.684
160	0.815	1.017	3.64	43.9	24.5	30.09	0.682
180	0.779	1.022	3.78	47.5	25.3	32.49	0.681
200	0.746	1.026	3.93	51.4	26.0	34.85	0.680
250	0.674	1.038	4.27	61.0	27.4	40.61	0.677
300	0.615	1.047	4.60	71.6	29.7	48.33	0.674
350	0.566	1.059	4.91	81.9	31.4	55.46	0.676
400	0.524	1.068	5.21	93.1	33.0	63.09	0.678
500	0.456	1.093	5.74	115.3	36.2	79.38	0.687
600	0.404	1.114	6.22	138.3	39.1	96.89	0.699
700	0.362	1.135	6.71	163.4	41.8	115.4	0.706
800	0.329	1.156	7.18	188.8	44.3	134.8	0.713
900	0.301	1.172	7.63	216.2	46.7	155.1	0.717
1000	0.277	1.185	8.07	245.9	49.0	177.1	0.719
1100	0.257	1.197	8.50	276.2	51.2	199.3	0.722
1200	0.239	1.210	9.15	316.5	53.5	233.7	0.724

TABLE 5

Physical Properties of Water on the Saturation Line

t, °C	$p\times10^{-5}$, Pa	ρ, kg/m³	i, kJ/kg	c_p, kJ/kg·K	$\lambda\times10^2$, W/m·K	$a\times10^8$, m²/s	$\mu\times10^6$, N·s/m²	$\nu\times10^6$, m²/s	$\beta\times10^4$, 1/K	$\sigma\times10^4$, N/m	Pr
0	1.013	999.9	0	4.212	55.1	13.1	1788	1.789	—0.63	756.4	13.67
10	1.013	999.7	42.04	4.191	57.4	13.7	1306	1.306	+0.70	741.6	9.52
20	1.013	998.2	83.91	4.183	59.9	14.3	1004	1.006	1.82	726.9	7.02
30	1.013	995.7	125.7	4.174	61.8	14.9	801.5	0.805	3.21	712.2	5.42
40	1.013	992.2	167.5	4.174	63.5	15.3	653.3	0.659	3.87	696.5	4.31
50	1.013	988.1	209.3	4.174	64.8	15.7	549.4	0.556	4.49	676.9	3.54
60	1.013	983.1	251.1	4.179	65.9	16.0	469.9	0.478	5.11	662.2	2.98
70	1.013	977.8	293.0	4.187	66.8	16.3	406.1	0.415	5.70	643.5	2.55
80	1.013	971.8	355.0	4.195	67.4	16.6	355.1	0.365	6.32	625.9	2.21
90	1.013	965.3	377.0	4.208	68.0	16.8	314.9	0.326	6.95	607.2	1.95
100	1.013	958.4	419.1	4.220	68.3	16.9	282.5	0.295	7.52	588.6	1.75
110	1.43	951.0	461.4	4.233	68.5	17.0	259.0	0.272	8.08	569.0	1.60
120	1.98	943.1	503.7	4.250	68.6	17.1	237.4	0.252	8.64	548.4	1.47
130	2.70	934.8	546.4	4.266	68.6	17.2	217.8	0.233	9.19	528.8	1.36
140	3.61	926.1	589.1	4.287	68.5	17.2	201.1	0.217	9.72	507.2	1.26
150	4.76	917.0	632.2	4.313	68.4	17.3	186.4	0.203	10.3	486.6	1.17
160	6.18	907.0	675.4	4.346	68.3	17.3	173.6	0.191	10.7	466.0	1.10
170	7.92	897.3	719.3	4.380	67.9	17.3	162.8	0.181	11.3	443.4	1.05
180	10.03	886.9	763.3	4.417	67.4	17.2	153.0	0.173	11.9	422.8	1.00
190	12.55	876.0	807.8	4.459	67.0	17.1	144.2	0.165	12.6	400.2	0.96
200	15.55	863.0	852.5	4.505	66.3	17.0	136.4	0.158	13.3	376.7	0.93
210	19.08	852.8	897.7	4.555	65.5	16.9	130.5	0.153	14.1	354.1	0.91
220	23.20	840.3	943.7	4.614	64.5	16.6	124.6	0.148	14.8	331.6	0.89
230	27.98	827.3	990.2	4.681	63.7	16.4	119.7	0.145	15.9	310.0	0.88
240	33.48	813.6	1037.5	4.756	62.8	16.2	114.8	0.141	16.8	285.5	0.87
250	39.78	799.0	1085.7	4.844	61.8	15.9	109.9	0.137	18.1	261.9	0.86
260	46.94	784.0	1135.7	4.949	60.5	15.6	105.9	0.135	19.7	237.4	0.87
270	55.05	767.9	1185.7	5.070	59.0	15.1	102.0	0.133	21.6	214.8	0.88
280	64.19	750.7	1236.8	5.230	57.4	14.6	98.1	0.131	23.7	191.3	0.90
290	74.45	732.3	1290.0	5.485	55.8	13.9	94.2	0.129	26.2	168.7	0.93
300	85.92	712.5	1344.9	5.736	54.0	13.2	91.2	0.128	29.2	144.2	0.97
310	98.70	691.1	1402.2	6.071	52.3	12.5	88.3	0.128	32.9	120.7	1.03
320	112.90	667.1	1462.1	6.574	50.6	11.5	85.3	0.128	38.2	98.10	1.11
330	128.65	640.2	1526.2	7.244	48.4	10.4	81.4	0.127	43.3	76.71	1.22
340	146.08	610.1	1594.8	8.165	45.7	9.17	77.5	0.127	53.4	56.70	1.39
350	165.37	574.4	1671.4	9.504	43.0	7.88	72.6	0.126	66.8	38.16	1.60
360	186.74	528.0	1761.5	13.984	39.5	5.36	66.7	0.126	109	20.21	2.35
370	210.53	450.5	1892.5	40.321	33.7	1.86	56.9	0.126	164	4.709	6.79

TABLE 6

Physical Properties of Flue Gases

($B = 760$ mm Hg $\approx 1.01 \times 10^5$ Pa; $\overline{P}_{CO_2} = 0.13$; $\overline{P}_{H_2O} = 0.11$; $\overline{P}_{N_2} = 0.76$) [A.1]

t, °C	ρ, kg/m³	c_p, kJ/kg·C	$\lambda \times 10$, W/m·C	$a \times 10^6$, m²/s	$\mu \times 10^6$, Pa·s	$\nu \times 10^6$, m/s	Pr
0	1.295	1.042	2.28	16.9	15.8	12.20	0.72
100	0.950	1.068	3.13	30.8	20.4	21.54	0.69
200	0.748	1.097	4.01	48.9	24.5	32.80	0.67
300	0.617	1.122	4.84	69.9	28.2	45.81	0.65
400	0.525	1.151	5.70	94.3	31.7	60.38	0.64
500	0.457	1.185	6.56	121.1	34.8	76.30	0.63
600	0.405	1.214	7.42	150.9	37.9	93.61	0.62
700	0.363	1.239	8.27	183.8	40.7	112.1	0.61
800	0.330	1.264	9.15	219.7	43.4	131.8	0.60
900	0.301	1.290	10.0	258.0	45.9	152.5	0.59
1000	0.275	1.306	10.90	303.4	48.4	174.3	0.58
1100	0.257	1.323	11.75	345.5	50.7	197.1	0.57
1200	0.240	1.340	12.62	392.4	53.0	221.0	0.56

TABLE 7

Physical Properties of Transformer Oil Depending on Temperature [A.2]

t, °C	ρ, kg/m³	c_p, kJ/kg·°C	λ, W/m·°C	$\mu \times 10^4$ Pa·s	$\nu \times 10^6$, m²/s	$a \times 10^8$, m²/s	$\beta \times 10^4$, K⁻¹	Pr
0.0	892.5	1.549	0.1123	629.8	70.5	8.14	6.80	866
10	886.4	1.620	0.1115	335.5	37.9	7.83	6.85	484
20	880.3	1.666	0.1106	198.2	22.5	7.56	6.90	298
30	874.2	1.729	0.1098	128.5	14.7	7.28	6.95	202
40	868.2	1.788	0.1090	89.4	10.3	7.03	7.00	145
50	862.1	1.846	0.1082	65.3	7.58	6.80	7.05	111
60	856.0	1.905	0.1072	49.5	5.78	6.58	7.10	87.8
70	850.0	1.964	0.1064	38.6	4.54	6.36	7.15	71.3
80	843.9	2.026	0.1056	30.8	3.66	6.17	7.20	59.3
90	837.8	2.085	0.1047	25.4	3.03	6.00	7.25	50.5
100	831.8	2.144	0.1038	21.3	2.56	5.83	7.30	43.9
110	825.7	2.202	0.1030	18.1	2.20	5.67	7.35	38.8
120	819.6	2.261	0.1022	15.7	1.92	5.50	7.40	34.9

TABLE 8

Physical Properties of Mercury and Some Liquid Metals [A.3]

Metal	t, °C	p, kg/m³	λ, W/m·°C	c_p, kJ/kg·°C	$a\times10^6$, m²/s	$\nu\times10^8$, m²/s	$Pr\times10^2$
Mercury Hg	20	13 550	7.90	0.1390	4.36	11.4	2.72
$t_f=38.9$ °C;	100	13 350	8.95	0.1373	4.89	9.4	1.92
$t_b=357$ °C;							
$r_f=11.72$ kJ/kg;	150	13 230	9.65	0.1373	5.30	8.6	1.62
$r_{ev}=291.8$ kJ/kg	200	13 120	10.3	0.1373	5.72	8.0	1.40
	300	12 880	11.7	0.1373	6.64	7.1	1.07
Tin Sn	250	6 980	34.1	0.255	19.2	27.0	1.41
$t_f=231.9$ °C;	300	6 940	33.7	0.255	19.0	24.0	1.26
$t_b=2270$ °C;							
$r_f=58.2$ kJ/kg;	400	6 865	33.1	0.255	18.9	20.0	1.06
$r_{ev}=3015$ kJ/kg	500	6 790	32.6	0.255	18.8	17.3	0.92
Bismuth Bi	300	10 030	13.0	0.151	8.61	17.1	1.98
$t_f=271$ °C;	400	9 910	14.4	0.151	9.72	14.2	1.46
$t_b=1477$ °C;							
$r_f=50.2$ kJ/kg;	500	9 785	15.8	0.151	10.8	12.2	1.13
$r_{ev}=855.4$ kJ/kg	600	9 660	17.2	0.151	11.9	10.8	0.91
Lithium Li	200	515	37.2	4.187	17.2	111.0	6.43
$t_f=179$ °C;	300	505	39.0	4.187	18.3	92.7	5.03
$t_b=1317$ °C;							
$r_f=661.5$ kJ/kg;	400	495	41.9	4.187	20.3	81.7	4.04
$r_{ev}=19\,595$ kJ/kg	500	484	45.3	4.187	22.3	73.4	3.28
Alloy	150	10 550	9.8	0.146	6.39	28.9	4.50
56.5% Bi+43.5% Pb	200	10 490	10.3	0.146	6.67	24.3	3.64
$t_f=123.5$ °C;	300	10 360	11.4	0.146	7.50	18.7	2.50
$t_b=1670$ °C	400	10 240	12.6	0.146	8.33	15.7	1.87
	500	10 120	14.0	0.146	9.44	13.6	1.44
Alloy	100	852	23.2	1.143	23.9	60.7	2.51
25% Na+75% K	200	828	24.5	1.072	27.6	45.2	1.64
$t_f=-11$ °C;	300	808	25.8	1.038	31.0	36.6	1.18
$t_b=784$ °C	400	778	27.1	1.005	34.7	30.8	0.89
	500	753	28.4	0.967	39.0	26.7	0.69
	600	729	29.6	0.934	43.6	23.7	0.54
	700	704	30.9	0.900	48.8	21.7	0.44
Sodium Na	150	916	84.9	1.356	68.3	59.4	0.87
$t_f=97.8$ °C;	200	903	81.4	1.327	67.8	50.6	0.75
$t_b=883$ °C;							
$r_f=113.26$ kJ/kg;	300	878	70.9	1.281	63.0	39.4	0.63
$r_{ev}=4208$ kJ/kg	400	854	63.9	1.273	58.9	33.0	0.56
	500	829	57.0	1.273	54.2	28.9	0.53

TABLE 9

Emissivity of Various Materials [A.4]

Material and kind of surface	t, °C	ε
Metals		
Aluminium:		
polished	225-575	0.039-0.057
rough	26	0.055
oxidized at 600 °C	200-600	0.11-0.19
Tungsten	230-2230	0.053-0.31
Tungsten filament	3300	0.39
Iron:		
electrolytic, thoroughly polished	175-225	0.052-0.064
welding, thoroughly polished	40-250	0.28
polished	425-1020	0.14-0.377
after treatment with emery (cloth)	20	0.242
cast iron, untreated	925-1115	0.87-0.95
Steel casting, treated, polished		
Steel:	770-1040	0.52-0.56
ground plate (sheet) steel	940-1100	0.52-0.61
oxidized at 600 °C	200-600	0.79
oxidized, rough	40-370	0.94-0.97
Cast iron:		
polished	200	0.21
turned (on a lathe)	830-990	0.60-0.70
oxidized at 600 °C	200-600	0.64-0.78
rough, heavily oxidized	40-250	0.96
Gold, thoroughly polished	225-625	0.018-0.035
Brass:		
thoroughly polished, composition 73.2% Cu, 26.7% Zn (by mass)	245-355	0.028-0.031
rolled, natural surface	22	0.06
dull brass plate (sheet)	50-350	0.22
oxidized when heated to 600 °C	200-600	0.61-0.59
Copper:		
thoroughly polished, electrolytic	80	0.018
polished	115	0.023
heated for a long time, coated with a thick oxide film	25	0.78
oxidized when heated to 600 °C	200-600	0.57-0.55
Platinum, pure, polished	225-625	0.054-0.105
Platinum strip	925-1115	0.12-0.17
Platinum filament	25-1230	0.036-0.192
Platinum wire	225-1375	0.073-0.182
Chromium	38-538	0.08-0.26
Refractory, building, heat insulating and other materials		
Asbestos, cardboard	24	0.96
Refractory materials:		
weak emitters	500-1000	0.65-0.75
strong emitters	500-1000	0.80-0.90

TABLE 9 (*concluded*

Material and kind of surface	t, °C	ε
Dinas brick, rough:		
non-glazed	1000	0.8
glazed	1100	0.85
Fireclay brick, glazed	1100	0.75
Red brick, rough	20	0.93
Porcelain, glazed	22	0.92
Plaster, rough lime	10-90	0.91
Lampblack, 0.075 mm layer and thicker	40-370	0.95

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TO THE READER

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